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ON STRICHARTZ AND EIGENFUNCTION ESTIMATES FOR LOW REGULARITY METRICS

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Abstract. We produce, for dimensions $n \geq 3$, examples of wave operators for which the Strichartz estimates fail. The examples include both Lipschitz and $C^{1,\alpha}$ metrics, for each $0 < \alpha < 1$, where by the latter we mean that the gradient satisfies a Hölder condition of order α. We thus conclude that, on the scale of Hölder regularity, an assumption of at least 2 bounded derivatives for the metric (i.e., $C^{1,1}$) is necessary in order to assure that the Strichartz estimates hold. The same construction also yields, for dimensions $n \geq 2$, second order elliptic operators with Lipschitz or $C^{1,\alpha}$ coefficients for which certain eigenfunction estimates, established by the second author for operators with C^∞ coefficients, fail to hold.

1. Strichartz Estimates

We recall here the global version of the Strichartz estimates on $\mathbb{R}^{1+n}$, for the Euclidean wave operator $\Box = \partial_t^2 - \sum_{j=1}^n \partial_j^2$. These state (see [St1,2]) that, if u solves the system of equations

$$\begin{cases} \Box u = F \in L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n}), \\ u(0,\,\cdot\,) = f \in H^{\frac{1}{2}}(\mathbb{R}^n), \\ \partial_t u(0,\,\cdot\,) = g \in H^{-\frac{1}{2}}(\mathbb{R}^n), \end{cases}$$

then the following estimate holds

(1.1)
$$\|u\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{1+n})} \leq C_n\Big(\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)} + \|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})}\Big).$$

Here, $H^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space of functions with s fractional derivatives in $L^2(\mathbb{R}^n)$. For variable coefficient wave equations with sufficiently regular coefficients, this estimate holds locally (that is,

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over compact regions of space and time that are consistent with causality); see [K] and [MSS]. We show that even the local estimate can fail for low regularity metrics if $n \geq 3$.

The counterexample consists of producing a traveling wave packet that is more highly concentrated than is possible for smooth metrics. Roughly speaking, the singularities of the metric produce a sharper focusing of the packet. To compare our example to the free case, we remark that the Strichartz estimate (1.1) is seen to be sharp by letting $u_\lambda(t,x,y) = \cos\bigl(t\sqrt{-\Delta}\bigr)\varphi_\lambda(x,y)$, where $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$ and

$$\varphi_\lambda(x) = \varphi(\lambda x, \lambda^{\frac{1}{2}} y)\,,$$

with φ being a Schwartz function with Fourier transform supported in a small ball about the point $(1,0,\dots,0)$. The function $u_\lambda(t,x,y)$ is, for small t, essentially a bump function concentrated in the set

$$|x-t| \leq \lambda^{-1}\,, \quad |y| \leq \lambda^{-\frac{1}{2}}\,.$$

The two sides of (1.1) are then comparable for all λ, and letting λ tend to ∞ shows that the estimates are sharp.

For examples of Lipschitz and $C^{1,\alpha}$ metrics, we produce a wave packet $u_\lambda(t,x,y)$, constructed of frequencies of size λ, concentrated in the region

$$|x-t| \leq \lambda^{2\delta-2}\,, \quad |y| \leq \lambda^{-\delta}\,,$$

where $\delta > \frac{1}{2}$. If $n \geq 4$, then the volume of this region is strictly less than in the free case. In case $n = 3$, we use the sharper fact that as $t \to 0$ the support of our packet becomes smaller, to a degree which still contradicts the Strichartz estimates, although the failure is only logarithmic in λ. Notice that for $n = 2$ the volume of this region is greater than in the free case, and this construction therefore does not contradict the Strichartz estimates.

For the example, consider the wave equation

$$\frac{1}{1-|y|^{1+\alpha}}\bigl(\Delta_y + \partial_x^2\bigr)u(t,x,y) = \partial_t^2 u(t,x,y)\,, \quad |y| < 1\,. \tag{1.2}$$

The metric is then $C^{1,\alpha}$ if $0 < \alpha < 1$, and is Lipschitz if $\alpha = 0$. We look for solutions to (1.2) of the form

$$u(t,x,y) = e^{ix\xi - it\tau}\, f(y)\,,$$

which leads to the differential equation

$$\Delta_y f(y) = \big(\tau^2 |y|^{1+\alpha} - (\tau^2 - \xi^2)\big) f(y)\,. \tag{1.3}$$

Let $A_\alpha(y)$ denote the normalized ground state for the Schrödinger equation with potential $|y|^{1+\alpha}$

$$-\Delta_y A_\alpha(y) + |y|^{1+\alpha} A_\alpha(y) = c_\alpha\, A_\alpha(y)\,.$$

By [RS], Theorems XIII.47, and [A], Theorems 4.1 and 5.1, the function $A_\alpha(y)$ is radial, strictly positive, and satisfies

$$\big|A_\alpha(y)\big| \le C_{N,\alpha}\, e^{-N\,|y|}$$

for all $N > 0$. By elliptic regularity, $A_\alpha(y) \in C^{3,\alpha}(\mathbb{R}^{n-1})$, provided that $0 < \alpha < 1$. In case $\alpha = 0$, then $A_0(y) \in C^{2,1-\varepsilon}(\mathbb{R}^{n-1})$ for all $\varepsilon > 0$.

Written as a function of the radial variable r, the following ordinary differential equation is satisfied

$$\Big((r\partial_r)^2 + (n-2)\, r\partial_r - r^{3+\alpha} + c_\alpha\, r^2\Big) A_\alpha(r) = 0\,.$$

An induction argument yields that

$$\big|(r\partial_r)^k A_\alpha(r)\big| \le C_k\, e^{-r}\,, \tag{1.4}$$

for all k, and $r > 0$.

Now suppose that

$$\tau^2 - \xi^2 = c_\alpha\, \tau^{2\delta}\,, \quad \delta = \frac{2}{3+\alpha}\,. \tag{1.5}$$

It follows that

$$f(y) = A_\alpha(\tau^\delta y)$$

is a solution to (1.3). Equation (1.5) determines τ as a function of ξ:

$$\tau = \rho(\xi) = \xi + \frac{c_\alpha}{2}\xi^{2\delta-1} + r(\xi)\,,$$

where

$$|\partial_\xi^j r(\xi)| \le C_j\, \xi^{4\delta-3-j}\,, \quad \forall j\,. \tag{1.6}$$

For λ large and positive, we obtain a solution $u_\lambda(t,x,y)$ to (1.2) by superposition:

$$u_\lambda(t,x,y) = \lambda^{(n\delta-\delta-1)/2} \int e^{-it\rho(\xi)+ix\xi} \beta(\lambda^{-1}\xi)\, A_\alpha\big(\rho(\xi)^\delta y\big)\, d\xi\,,$$

where β is a positive bump function that vanishes outside a small interval about 1.

Theorem 1. *The function $u_\lambda(t,x,y)$ satisfies*

$$
(1.7)\qquad \begin{cases} \Big(\big(1-|y|^{1+\alpha}\big)\partial_t^2 - \big(\Delta_y + \partial_x^2\big)\Big)u_\lambda(t,x,y) = 0\,, \\ \|u_\lambda(0,x,y)\|_{H^{\frac12}(\mathbb{R}^n)} + \|\partial_t u_\lambda(0,x,y)\|_{H^{-\frac12}(\mathbb{R}^n)} \le C\,\lambda^{\frac12}\,, \end{cases}
$$

where C depends only on n and α. On the other hand, if $n=3$,

$$
\int_0^{\frac14}\left(\int_{|(x,y)|\le\frac12}|u_\lambda(t,x,y)|^4\,dx\,dy\right)dt \ge C^{-1}\lambda^2\ln(\lambda^{2\delta-1})\,,
$$

and if $n\ge 4$,

$$
\int_0^{\frac14}\left(\int_{|(x,y)|\le\frac12}|u_\lambda(t,x,y)|^{\frac{2(n+1)}{n-1}}\,dx\,dy\right)dt \ge C^{-1}\lambda^{\frac{4+2\delta(n-3)}{n-1}}\,.
$$

The Strichartz estimates are thus seen to fail for the solution u_λ, since $\delta>\frac12$ for $\alpha<1$. We remark that for $n=3$ the failure is only logarithmic in the frequency λ, whereas for $n\ge4$ there is a strict power difference. We also remark that one can take either the homogeneous or nonhomogeneous Sobolev norms in (1.7).

Proof of Theorem 1. We begin by establishing (1.7). That u_λ is a solution of the wave equation follows from the above construction. We next observe that, by Plancherel's Theorem,

$$
(1.8)\qquad \int |u_\lambda(t,x,y)|^2\,dx\,dy \approx 1
$$

uniformly over t and λ. By equation (1.2) and the decay estimates on $A_\alpha(y)$, we similarly obtain

$$
\int \big|\big(\partial_x^2+\Delta_y\big)u_\lambda(t,x,y)\big|^2\,dx\,dy \approx \lambda^4\,.
$$

Interpolation yields that $\|u_\lambda(0,x,y)\|_{H^{\frac12}(\mathbb{R}^n)} \lesssim \lambda^{\frac12}$. To estimate the $-\frac12$ Sobolev norm of $\partial_t u_\lambda$, we write

$$
\partial_t u_\lambda(t,x,y) = \partial_x v_\lambda(t,x,y)\,,
$$

where $\|v_\lambda(0,x,y)\|_{H^{\frac12}(\mathbb{R}^n)} \lesssim \lambda^{\frac12}$.

To establish the remaining part of the theorem, we will show that, for $-\frac{1}{4} \le t \le \frac{1}{4}$,

$$\int_{|(x,y)|\le\frac{1}{2}} \left(1+\lambda^\delta|y| + \frac{\lambda\,|x-t|}{1+\lambda^{2\delta-1}|t|}\right)^{-2} \big|u_\lambda(t,x,y)\big|^2\,dx\,dy \ge C^{-1}\,, \tag{1.9}$$

where the constant is independent of λ and t. By Hölder's inequality, we conclude from (1.9) that

$$C^{-1} \le \Big[\lambda^{-\delta(n-1)-1}\big(1+\lambda^{2\delta-1}|t|\big)\Big]^{\frac{2}{n-1}} \int_{|(x,y)|\le\frac{1}{2}} \big|u(t,x,y)\big|^{\frac{2(n+1)}{n-1}}\,dx\,dy\,,$$

which immediately yields the desired estimates, after integrating in t.

We establish (1.9) by showing that

$$\int_{\mathbb{R}^n} \left(1+\lambda^\delta|y| + \frac{\lambda\,|x-t|}{1+\lambda^{2\delta-1}|t|}\right)^{2} |u_\lambda(t,x,y)|^2\,dx\,dy \le C\,. \tag{1.10}$$

An application of the Schwarz inequality, together with (1.8), shows that this implies (1.9) for large λ. Notice that this inequality quantifies the concentration mentioned before.

To establish (1.10), we write

$$u_\lambda(t,x,y) \\ = \lambda^{(n\delta-\delta-1)/2} \int e^{i(x-t)\xi - i\frac{1}{2}c_\alpha t\xi^{2\delta-1}} e^{-itr(\xi)} \beta(\lambda^{-1}\xi)\, A_\alpha\big(\rho(\xi)^\delta y\big)\,d\xi\,.$$

We make use of the following estimates:

$$\left|\partial_\xi^j \left(e^{-itr(\xi)}\beta(\lambda^{-1}\xi)\,A\big(\rho(\xi)^\delta y\big)\right)\right| \le \lambda^{-j}\,e^{-\lambda^\delta|y|}\,, \tag{1.11}$$

which follow by (1.4) and (1.6).

If $|t| \le \lambda^{1-2\delta}$, then in addition we have

$$\left|\partial_\xi^j e^{-i\frac{1}{2}c_\alpha t\,\xi^{2\delta-1}}\right| \le C_j \lambda^{-j}$$

on the support of $\beta(\lambda^{-1}\xi)$, and it follows easily that

$$|u_\lambda(t,x,y)| \le C_N\,\lambda^{(n\delta-\delta+1)/2}\big(1+\lambda^\delta|y| + \lambda\,|x-t|\big)^{-N}\,, \tag{1.12}$$

which implies (1.10) in this case. For $|t| \geq \lambda^{1-2\delta}$, (1.10) is a consequence of the following estimate
(1.13)

$$|u_\lambda(t,x,y)| \leq C_N\, \lambda^{(n\delta-\delta+1)/2}\big(\lambda^{2\delta-1}|t|\big)^{-1/2}\left(1+\lambda^\delta|y|+\frac{\lambda\,|x-t|}{\lambda^{2\delta-1}|t|}\right)^{-N}.$$

To establish (1.13), we write

$$u_\lambda(t,x,y) = \lambda^{(n\delta-\delta+1)/2}\int e^{i\lambda(x-t)\sigma - i\frac{1}{2}c_\alpha t\lambda^{2\delta-1}\sigma^{2\delta-1}}\,e^{-itr(\lambda\sigma)} \beta(\sigma)\,A_\alpha\big(\rho(\lambda\sigma)^\delta y\big)\,d\sigma\,.$$

Suppose that $\lambda\,|x-t| \leq C\,\lambda^{2\delta-1}|t|$. Then the phase takes the form

$$\big(t\lambda^{2\delta-1}\big)\left[\frac{c_\alpha}{2}\sigma^{2\delta-1}+\varepsilon\sigma\right],$$

where $|\varepsilon| \leq C$. The term in brackets has nonvanishing second derivative on the support of $\beta(\sigma)$, so that (1.13) follows by stationary phase and the estimates (1.11). On the other hand, if $\lambda\,|x-t| \geq C\,\lambda^{2\delta-1}|t|$, then we write the phase as

$$\lambda\,(x-t)\left[\sigma+\varepsilon\frac{c_\alpha}{2}\sigma^{2\delta-1}\right],$$

where $|\varepsilon| \leq C^{-1}$. For C sufficiently large, the term in brackets has nonvanishing derivative on the support of $\beta(\sigma)$. Integration by parts shows that the estimates (1.12) hold in this case, which implies (1.13) since $\lambda\,|x-t| \geq C\,\lambda^{2\delta-1}|t| \geq 1$. □

2. Eigenfunction Estimates

Suppose that P is a positive, second order elliptic differential operator on a compact n-dimensional manifold M^n, which is self-adjoint with respect to some smooth density, and suppose that $\{e_\mu\}$ is an orthonormal basis for $L^2(M^n)$ consisting of eigenfunctions for P:

$$Pe_\mu = \mu^2 e_\mu\,.$$

For $\lambda \in \mathbb{R}$, let $\chi_\lambda f$ denote the orthogonal projection of a function f onto the space of eigenfunctions with frequencies in the range $[\lambda, \lambda+1)$:

$$\chi_\lambda f = \sum_{\mu\in[\lambda,\lambda+1)} \big(e_\mu, f\big)\,e_\mu\,.$$

It was shown by the second author in [S] that, if the coefficients of P belong to $C^\infty(M^n)$, then the following estimates hold.

$$\|\chi_\lambda f\|_{L^q(M^n)} \le C\,\lambda^{n(\frac12-\frac1q)-\frac12}\|f\|_{L^2(M^n)}\,, \quad \frac{2(n+1)}{n-1} \le q \le \infty\,. \tag{2.1}$$

In [D], Davies showed that these estimates can fail for $q=\infty$ if the coefficients of P are bounded but not continuous. It is then natural to seek counterexamples for smoother coefficients. In his thesis [G], Grieser observed that, for eigenfunctions on the unit disc in $\mathbb{R}^2$ with Dirichlet conditions, the estimates (2.1) fail if $q<8$. (In [S] it is assumed that the manifold M^n does not have a boundary.) By attaching two copies of the disc along their boundaries, one obtains a compact manifold without boundary, where the coefficients of P are Lipschitz, such that (2.1) fails for $q<8$.

We extend these results by producing, for each $0\le\alpha<1$ and each $n\ge 2$, an operator P on the n-dimensional torus $\mathbb{T}^n$, whose coefficients belong to $C^{1,\alpha}(\mathbb{T}^n)$ if $0<\alpha<1$ and Lip $(\mathbb{T}^n)$ if $\alpha=0$, and such that the estimates (2.1) fail if

$$q < 2 + \frac{2}{(n-1)(1-\delta)}\,, \quad \delta = \frac{2}{3+\alpha}\,. \tag{2.2}$$

To describe the example, let (x,y) denote variables in $(-\pi,\pi]\times(-\pi,\pi]^{n-1}$, which we take as coordinates on $\mathbb{T}\times\mathbb{T}^{n-1}$. For the standard metric on $\mathbb{T}^n$, the estimates (2.1) are seen to be sharp by constructing, from frequencies in the range $[\lambda,\lambda{+}1)$, a function concentrated in the tubular region $|y|\le\lambda^{-\frac12}$. For metrics similar to those of the previous section, we construct approximate eigenfunctions concentrated in the tubular region $|y|\le\lambda^{-\delta}$. This yields counterexamples to (2.1) for all $n\ge2$, as opposed to the previous section where the spreading of the wave packet in the x direction, caused by superimposing a range of frequencies, produced counterexamples to the Strichartz estimates only for $n\ge3$.

Let $c(y)$ be a strictly positive function on $\mathbb{T}^{n-1}$ such that

$$c^2(y) = \frac{1}{1-|y|^{1+\alpha}}\,, \quad |y|\le\frac34\,,$$

and such that $c(y)$ belongs to $C^\infty(\mathbb{T}^{n-1})$ away from the point $y=0$. The operator

$$P(y,\partial_x,\partial_y) = -c(y)^2\left(\Delta_y+\partial_x^2\right),$$

with domain $H^2(\mathbb{T}^n)$, is then self adjoint on $L^2(\mathbb{T}^n)$ with respect to the measure $c(y)^{-2}\,dx\,dy$. The resolvent of P is easily seen to be compact, so that $L^2(\mathbb{T}^n)$ admits an orthonormal basis of eigenfunctions for P.

Fix $\psi(y) \in C^\infty(\mathbb{T}^{n-1})$, such that

$$\psi(y) = \begin{cases} 1\,, & |y| \le \frac{1}{2}\,, \\ 0\,, & |y| \ge \frac{3}{4}\,, \end{cases}$$

and for positive integers k consider the function

$$f_k(x,y) = k^{(n-1)\delta/2}\, e^{ikx}\, \psi(y)\, A_\alpha\big(\rho(k)^\delta y\big)\,.$$

We take k large so that $k \le \rho(k) \le k + \frac{1}{4}$. It is easily verified that

$$\|f_k\|_{L^q(\mathbb{T}^n)} \approx k^{(n-1)\delta(\frac{1}{2}-\frac{1}{q})}\,. \tag{2.3}$$

By the constructions of the previous section, we have

$$-c^2(y)\big(\Delta_y + \partial_x^2\big) f_k(x,y) = \rho(k)^2\, f_k(x,y) + r_k(x,y)\,,$$

where the error term $r_k(x,y)$ vanishes unless $\frac{1}{2} \le |y| \le \frac{3}{4}$, and where for all multi-indices γ the following holds

$$\big|\partial_{x,y}^\gamma r_k(x,y)\big| \le C_\gamma\, e^{-\frac{1}{2}k^\delta}\,. \tag{2.4}$$

If it were the case that $r_k \equiv 0$, then by taking $\lambda = k$ it would follow immediately from (2.3) that the estimates (2.1) fail if q is as in (2.2). That this failure occurs even though r_k need not vanish identically is a result of the following theorem.

Theorem 2. *Let $f_k(x,y)$ be as above. Then, if $|\lambda - k| \ge 2$, it follows that*

$$\|\chi_\lambda f_k\|_{L^2(\mathbb{T}^n)} \le C_N\, (1+\lambda^2)^{-N}\, e^{-\frac{1}{2}k^\delta}\,, \quad \forall N\,. \tag{2.5}$$

As a consequence, the estimate

$$\|\chi_\lambda f_k\|_{L^q(\mathbb{T}^n)} \le C\, \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \|\chi_\lambda f_k\|_{L^2(\mathbb{T}^n)} \tag{2.6}$$

cannot hold with C independent of λ and k if

$$q < 2 + \frac{2}{(n-1)(1-\delta)}\,, \quad \delta = \frac{2}{3+\alpha}\,.$$

Proof. Let us first show that (2.5) implies the desired consequence. It follows from (2.3) and Minkowski's inequality that

$$\sum_{j=1}^\infty \|\chi_j f_k\|_{L^q(\mathbb{T}^n)} \ge c\, k^{(n-1)\delta(\frac{1}{2}-\frac{1}{q})}\,.$$

If (2.6) holds for q, then by (2.5) the left hand side is dominated by $k^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}$, which yields a contradiction in case q is as in (2.2).

To establish (2.5), suppose that e_μ is an eigenfunction with eigenvalue μ^2, with $\mu \in [\lambda, \lambda+1)$ and $|\lambda - k| > 2$. We then have

$$\begin{aligned}
&\int f_k(x,y)\, e_\mu(x,y)\, c(y)^{-2}\, dx\, dy \\
&\quad = \frac{1}{\mu^2-\rho(k)^2}\int r_k(x,y)\, e_\mu(x,y)\, c(y)^{-2}\, dx\, dy \\
&\quad = \frac{(1+\mu^2)^{-N}}{\mu^2-\rho(k)^2}\int \Big(1 - c(y)^2\big(\Delta_y + \partial_x^2\big)\Big)^N r_k(x,y)\, e_\mu(x,y)\, c(y)^{-2}\, dx\, dy
\end{aligned}$$

where the integration by parts is valid since $r_k(x,y)$ is supported in the region where $c(y)$ is smooth. Notice that $\mu^2 - \rho(k)^2$ is bounded from below. Therefore, orthogonality and (2.4) yield

$$\begin{aligned}
\|\chi_\lambda f_k\|_{L^2} &\le C(1+\lambda^2)^{-N}\, \big\| \big(1 - c(y)^2(\Delta_y + \partial_x^2)\big)^N r_k(x,y)\, \big\|_{L^2} \\
&\le C_N(1+\lambda^2)^{-N} e^{-\frac{1}{2}k^\delta},
\end{aligned}$$

which finishes the proof. □

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