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TRANSLATIONS AND THE HOLONOMY OF COMPLETE AFFINE FLAT MANIFOLDS

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1. Introduction

Let $\Gamma \subset \mathbf{GL}(n) \ltimes \mathbb{V}$, where $\mathbb{V}$ is the group of translations which may be identified with $\mathbb{R}^n$. A complete affine flat manifold M can be written as $\mathbb{R}^n/\Gamma$, where Γ acts freely and properly discontinuously on $\mathbb{R}^n$.

$$\mathbb{H} : \pi_1(M) \to \mathbf{GL}(n) \ltimes \mathbb{V}$$

is the *affine holonomy* of M whose image, up to conjugation, is Γ.

$$\mathbb{L} \circ \mathbb{H} : \pi_1(M) \to \mathbf{GL}(n)$$

is the *linear holonomy* of M, where $\mathbb{L} : \mathbf{GL}(n) \ltimes \mathbb{V} \to \mathbf{GL}(n)$ denotes the usual projection.

Let M be any complete flat affine manifold. Milnor [9] conjectured that $\pi_1(M)$, which is isomorphic to the image of the holonomy of M, is virtually polycyclic. Margulis [5], [6] was the first to show the existence of complete flat Lorentz space-times with fundamental group not virtually polycyclic. These 3-dimensional manifolds are called *Margulis space-times.* Using methods from [7], one can construct complete affine flat manifolds whose linear holonomy is in $\mathbf{SO}(n+1, n)$ where n is odd and whose fundamental group is free of rank ≥ 2, i.e. not virtually polycyclic.

Margulis's counterexamples to Milnor's conjecture all have free fundamental group of rank ≥ 2. The related counterexamples in [3], [1], and [2] also have free fundamental groups. It will be shown that:

Theorem 1. *If M is a complete Lorentz space-time, then $\pi_1(M)$ is either virtually polycyclic or free. In particular, if $\pi_1(M)$ is not virtually polycyclic, then the holonomy of M contains no pure translations and the linear holonomy of M is torsion free.*

In the more general setting of manifolds whose linear holonomies lie in $\mathbf{SO}(n+1, n)$ where n is odd, it will be shown that:

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Theorem 2. *Let M be a $2n+1$-dimensional complete affine flat manifold with linear holonomy conjugate to a subgroup of $\mathbf{SO}(n+1,n)$ where n is odd. If the linear holonomy of M is Zariski dense in $\mathbf{SO}(n+1,n)$, then the linear holonomy of M is torsion free, i.e. there are no pure translations in the holonomy of M.*

2. The case of $\mathbb{R}^{n+1,n}$

$\mathbb{R}^{n+1,n}$ is the vector space diffeomorphic to $\mathbb{R}^{2n+1}$ and supplied with the $\mathbf{SO}(n+1,n)$-invariant inner product

$$\mathbb{B}(\mathsf{u},\mathsf{v}) = \sum_{i=0}^{n} u_i v_i - \sum_{i=n+1}^{2n} u_i v_i,$$

where $\mathsf{u} = [u_0, u_1, \ldots, u_{2n}]$ and $\mathsf{v} = [v_0, v_1, \ldots, v_{2n}]$ are elements of $\mathbb{R}^{n+1,n}$ written in terms of the standard basis $\{\mathsf{e}_0, \mathsf{e}_1, \ldots, \mathsf{e}_{2n}\}$.

v is said to be

- *spacelike* if $\mathbb{B}(\mathsf{v},\mathsf{v}) > 0$,
- *timelike* if $\mathbb{B}(\mathsf{v},\mathsf{v}) < 0$, and
- *lightlike* or *null* if $\mathbb{B}(\mathsf{v},\mathsf{v}) = 0$.

Now define the $2n$-linear $\mathbb{R}^{n+1,n}$ *cross product*

$$\boxtimes : (\mathbb{R}^{n+1,n})^{2n} \to \mathbb{R}^{n+1,n}$$

which can be symbolically represented as follows:

$$\boxtimes(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_{2n}) \mapsto \det\left(\begin{bmatrix} I_{n+1} & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} \mathsf{e}_0\, \mathsf{e}_1 \,\ldots\, \mathsf{e}_{2n+1} \\ \mathsf{v}_1 \\ \mathsf{v}_2 \\ \vdots \\ \mathsf{v}_{2n} \end{bmatrix} \right)$$

where the v_i's are viewed as row vectors written in terms of the standard basis of $\mathbb{R}^{n+1,n}$ (see [3] for explicit $\mathbb{R}^{2,1}$ cross product). It is easy to check that

$$\mathbb{B}(\boxtimes(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_{2n}), \mathsf{v}_i) = 0,$$

for all $1 \leq i \leq 2n$. That is, $\boxtimes(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_{2n})$ is $\mathbb{B}$-perpendicular to all of the v_i's.

An ordered basis $\{\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{2n}\}$ for $\mathbb{R}^{n+1,n}$ is *positively oriented* if

$$\mathbb{B}(\mathsf{v}_0, \boxtimes(\mathsf{v}_1, \ldots, \mathsf{v}_{2n})) > 0.$$

For any vector $\mathsf{v} \in \mathbb{R}^{n+1,n}$, its $\mathbb{B}$*-perpendicular plane* is denoted

$$\mathcal{P}(\mathsf{v}) = \{\mathsf{u} \in \mathbb{R}^{n+1,n} \mid \mathbb{B}(\mathsf{u},\mathsf{v}) = 0\}.$$

For $g \in \mathbf{SO}(n+1,n)$ define the following:

- $A^+(g)$ is the smallest subspace containing all eigenvectors corresponding to each eigenvalue whose absolute value is < 1;
- $A^0(g)$ is the smallest subspace containing all eigenvectors corresponding to each eigenvalue whose absolute value is $= 1$;
- $A^+(g)$ is the smallest subspace containing all eigenvectors corresponding to each eigenvalue whose absolute value is > 1.

We also define

$$D^{\pm}(g) = A^{\pm}(g) \oplus A^0(g).$$

g is called *purely hyperbolic* if $A^+(g)$, equivalently $A^-(g)$, is n-dimensional. If g is purely hyperbolic then $A^0(g)$ is 1-dimensional and it has $2n$ real eigenvalues, counting multiplicities, different than 1. Denote the eigenvalues of a purely hyperbolic g by $\lambda_i(g)$ for $-n \le i \le n$ so that

$$|\lambda_{-n}(g)| \le \ldots \le |\lambda_{-1}(g)| < \lambda_0(g) < |\lambda_1(g)| \le \ldots \le |\lambda_n(g)|.$$

Note that $\lambda_0(g) = 1$ and $\lambda_{-i}(g)\lambda_i(g) = 1$.

For g purely hyperbolic, choose eigenvectors $\mathsf{x}_i(g)$ so that:

- $\mathsf{x}_i(g)$ is an eigenvector with eigenvalue $\lambda_i(g)$;
- for $1 \le |i| \le n$,
$$\mathbb{B}\left(\mathsf{x}_i(g), \mathsf{x}_j(g)\right) = -\delta_{i,-j};$$
- for $i = 0$, $\mathbb{B}\left(\mathsf{x}_0(g), \mathsf{x}_0(g)\right) = 1$ and
$$\mathbb{B}\left(\mathsf{x}_0(g), \boxtimes(\mathsf{x}_{-n}(g), \ldots, \mathsf{x}_{-1}(g), \mathsf{x}_1(g), \ldots, \mathsf{x}_n(g))\right) > 0;$$
- and $\mathbb{B}\left(\mathsf{x}_0(g), \boxtimes(\mathsf{e}_{2n}, \ldots, \mathsf{e}_{n+1}, \mathsf{x}_1(g), \ldots, \mathsf{x}_n(g))\right) > 0$.

The choice of $\mathsf{x}_i(g)$ for $i \neq 0$ is not well defined for any particular i. However, the choice of $\mathsf{x}_0(g)$ is well defined. It is easy to see that $\mathsf{x}_0(g)$ varies continuously with g. By the properties of the cross product, if n is odd, then $\mathsf{x}_0(g) = -\mathsf{x}_0(g^{-1})$ and if n is even, then $\mathsf{x}_0(g) = \mathsf{x}_0(g^{-1})$.

$\{\mathsf{x}_0(g), \mathsf{x}_1(g), \ldots, \mathsf{x}_n(g)\}$ is a well defined orientation on $D^+(g)$. In particular, if $D^+(g_1) = D^+(g_2)$, then the defined orientations on the subspace are the same. The orientation on a spacelike line $\ell \subset D^+(g)$ induced by g is defined to be $\{\mathsf{w}\}$, where w is parallel to ℓ and $\mathbb{B}\left(\mathsf{w}, \mathsf{x}_0(g)\right) > 0$.

Two purely hyperbolic elements g_1 and g_2 are called *transversal* if

$$\mathbb{R}^{n+1,n} = A^+(g_1) \oplus D^+(g_2) = D^+(g_1) \oplus A^+(g_2),$$

and *ultra-transversal* if g_1, g_2, g_1^{-1} and g_2^{-1} are mutually transversal. Of course, any purely hyperbolic element and its inverse are transversal.

For g_1 and g_2 transversal, $\ell = D^+(g_1) \cap D^+(g_2)$ is a spacelike line. It can be shown that for n odd the orientation induced on ℓ by g_1 is opposite that induced by g_2. If n is even, then the orientations induced on ℓ by g_1 and g_2 are the same.

Now consider affine transformations whose linear part lies in $\mathbf{SO}(n+1,n)$. $h \in \mathbf{SO}(n+1,n) \ltimes \mathbb{V}$ can be written $h(\mathsf{w}) = g(\mathsf{w}) + \mathsf{v}$, where $\mathbb{L}(h) = g \in \mathbf{SO}(n+1,n)$ and $\mathsf{v} \in \mathbb{V}$. g is the *linear part of* h and v is the *translational part of* h.

h is said to be *purely hyperbolic* if its linear part is purely hyperbolic. Purely hyperbolic h_1 and h_2 are said to be *transversal* (*ultra-transversal*) if their linear parts are transversal (resp. ultra-transversal). If h is purely hyperbolic, then there exists a unique h-invariant 1-dimensional subspace $C(h)$ parallel to $\mathsf{x}_0(g)$. Let $E^{\pm}(h)$ be the subspaces containing $C(h)$ and parallel to and of the same dimension as $D^{\pm}(g)$.

If h is purely hyperbolic and has no fixed points, then $M = \mathbb{R}^{n+1,n}/\langle h \rangle$ is a manifold with $\pi_1(M) = \mathbb{Z}$ whose unique closed geodesic is the image of $C(h)$ under the identification. The $\mathbb{B}$ *-length* of the unique closed geodesic is defined to be

$$\alpha(h) = \mathbb{B}\,(h(\mathsf{x}) - \mathsf{x}, \mathsf{x}_0(g)),$$

for $\mathsf{x} \in C(h)$. It is easy to see that the above expression is the same for any $\mathsf{x} \in \mathbb{R}^{n+1,n}$. In particular, if $h(\mathsf{x}) = g(\mathsf{x}) + \mathsf{v}$, then

$$\alpha(h) = \mathbb{B}\,(\mathsf{v}, \mathsf{x}_0(g)).$$

The *sign* of h is the sign of $\alpha(h)$. It is interesting to note that if n is odd, then the signs of h and h^{-1} are the same, but if n is even, then the signs of h and h^{-1} are opposite.

3. Nonproper actions

For purely hyperbolic $h \in \Gamma \subset \mathbf{SO}(n+1,n) \ltimes \mathbb{R}^{n+1,n}$, $\alpha(h)$ is a simple and powerful tool which can be used to determine if Γ does not act properly on $\mathbb{R}^{n+1,n}$.

For instance, assume $\alpha(h) = 0$. If $h(\mathsf{w}) = g(\mathsf{w}) + \mathsf{v}$, then $\mathsf{v} \in \mathcal{P}(\mathsf{x}_0(g))$. Since $\mathsf{x}_i(g)$, $i \neq 0$, are the eigenvectors for g which form a basis for $\mathcal{P}(\mathsf{x}_0(g))$, there exist real numbers m_i such that

$$\mathsf{v} = \sum_{i \neq 0} m_i \mathsf{x}_i(g).$$

It can be shown by direct computation that

$$\sum_{i \neq 0} \frac{m_i}{1 - \lambda_i(g)}\, \mathsf{x}_i(g)$$

is fixed by h. Thus, $\langle h \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$.

Using the signs of affine transformations, Margulis proved the following in [7].

Theorem 3 (Margulis). *For n even, if Γ is a discrete group of $GL(2n+1) \ltimes \mathbb{V}$ such that the Zariski closure of $\mathbb{L}(\Gamma)$ is conjugate to $\mathbf{SO}(n+1,n)$, then Γ does not act properly on $\mathbb{R}^{n+1,n}$.*

Because of Theorem 3, we now restrict our attention to the cases where n is odd.

The following lemma was stated and proved in [7] for all n. The statement and proof shall be given here for only n odd. (see also [5] and [6] for $n = 1$)

Lemma 4 (Margulis). *If $h_1, h_2 \in \mathbf{SO}(n+1,n) \ltimes \mathbb{V}$ are transversal and have opposite signs, then $\langle h_1, h_2 \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$.*

Proof. Let h_1 and h_2 be transversal. Assume that $\alpha(h_1) > 0 > \alpha(h_2)$ and $g_i = \mathbb{L}(h_i)$. Choose w parallel to the spacelike line $\ell = E^+(h_1) \cap E^+(h_2)$ so that $\mathbb{B}(\mathsf{w}, \mathsf{x}_0(g_1)) > 0$ (and $\mathbb{B}(\mathsf{w}, \mathsf{x}_0(g_2)) < 0$).

Choose $\mathsf{u}_i \in C(h_i)$ and the closed $(n+1)$-dimensional parallelpiped neighborhoods $U_i \subset E^+(h_i)$ with vertices

$$\mathsf{u}_i \pm \frac{1}{2}\alpha(h_i)\mathsf{x}_0(g_i) + \sum_{j=1}^{n} \pm \mathsf{x}_j(g_i).$$

$h_i(U_i)$ is also an $(n+1)$-dimensional parallelpiped in $E^+(h_i)$, and $h_i(U_i) \cap U_i$ is a face of U_i. The Euclidean distance between $C(h_i)$ and the faces of $h_i^n(U_i)$ parallel to $C(h_i)$ increases exponentially with n. h_i^n acts on $C(h_i)$ by the translation $n[\alpha(h_i)\mathsf{x}_0(g_i)]$, i.e. linearly with n. Thus, the Euclidean distance between $C(h_i)$ and the faces of $h_i^n(U_i)$ parallel to $C(h_i)$ varies exponentially with the translation along $C(h_i)$ so that there exists an integer N_i such that

$$R_i = \left(\cup_{k=N_i}^{\infty} h_i^k(U_i)\right) \cap \ell$$

is a ray.

Since $\alpha(h_1) > 0$ and $\mathbb{B}(\mathsf{w}, \mathsf{x}_0(g_1)) > 0$, R_1 is in the same direction as w. But $\alpha(h_2) < 0$ and $\mathbb{B}(\mathsf{w}, \mathsf{x}_0(g_2)) < 0$ so that R_2 is also in the direction of w. Thus, $R_1 \cap R_2$ is a ray.

$\exists$ an infinite number of ordered pairs of positive numbers $\{m_{1,i}, m_{2,i}\}$ for which

$$h_1^{m_{1,i}}(U_1) \cap h_2^{m_{2,i}}(U_2) \neq \emptyset.$$

Let K be a compact set containing $U_1 \cup U_2$ and

$$h_2^{-m_{2,i}} h_1^{m_{1,i}}(K) \cap K \neq \emptyset.$$

Since h_1 and h_2 are transversal there are no positive integers k_1 and k_2 such that $h_2^{-k_2}h_1^{k_1}$ is the identity. There are an infinite number of elements $h \in G$ such that $h(K) \cap K \neq \emptyset$ and $\langle h_1, h_2 \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$. $\square$

For $\Gamma \subset \mathbf{SO}(n+1,n) \ltimes \mathbb{V}$ such that $\mathbb{L}(\Gamma)$ is Zariski dense in $\mathbf{SO}(n+1,n)$, $\mathbb{L}(\Gamma)$ contains at least one pair of purely hyperbolic elements which are ultra-transversal. It will be shown that if Γ acts freely and properly discontinuously on $\mathbb{R}^{n+1,n}$, then it cannot contain any *pure translations*, i.e. the linear part is the identity of $\mathbf{SO}(n+1,n)$ and the translational part is nonzero.

First, examine the case in which a pure translation is not $\mathbb{B}$-perpendicular to the fixed eigenvector of some $\mathbb{L}(h) \in \mathbf{SO}(n+1,n)$.

Lemma 5. *For n odd, let $t(\mathsf{w}) = \mathsf{w} + \mathsf{t}$ and $h(\mathsf{w}) = g(\mathsf{w}) + \mathsf{v}$, where $g \in \mathbf{SO}(n+1,n)$ is a purely hyperbolic transformation. If $\mathbb{B}(\mathsf{t}, \mathsf{x}_0(g)) \neq 0$, then $\langle t, h \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$.*

Proof. We can assume that $\alpha(h) > 0$.

$$(t^n h)(\mathsf{w}) = t^n(g(\mathsf{w}) + \mathsf{v}) = g(\mathsf{w}) + (\mathsf{v} + n\mathsf{t})$$

for any integer n, and

$$\alpha(t^n h) = \mathbb{B}(\mathsf{v} + n\mathsf{t}, \mathsf{x}_0(g)) = \alpha(h) + n\mathbb{B}(\mathsf{t}, \mathsf{x}_0(g)).$$

If $\mathbb{B}(\mathsf{t}, \mathsf{x}_0(g)) < 0$, then there is some positive integer N such that $\alpha(t^N h) < 0$. If $\mathbb{B}(\mathsf{t}, \mathsf{x}_0(g)) > 0$, then there is some negative integer N such that $\alpha(t^N h) < 0$. In either case, h and $t^N h$ have different signs but the same linear part g. But the signs of h^{-1} and h are the same. h^{-1} and $t^N h$ are transversal so $\langle h, t \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$ by Lemma 4. $\square$

Second is the case in which the pure translation is parallel to an expanding or contraction eigendirection of some $\mathbb{L}(h) \in \mathbf{SO}(n+1,n)$.

Lemma 6. *Let $t(\mathsf{w}) = \mathsf{w} + \mathsf{t}$ and $h(\mathsf{w}) = g(\mathsf{w}) + \mathsf{v}$, where $g \in \mathbf{SO}(n+1,n)$ is a purely hyperbolic transformation. If t is parallel to $\mathsf{x}_i(g)$ for any $i \neq 0$, then $\langle t, h \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$.*

Proof. Write t as $k\mathsf{x}_i(g)$.

$$\begin{aligned}
(hth^{-1})(\mathsf{w}) &= (ht)\left(g^{-1}(\mathsf{w}) - g^{-1}(\mathsf{v})\right) \\
&= h\left(g^{-1}(\mathsf{w}) - g^{-1}(\mathsf{v}) + k\mathsf{x}_i(g)\right) \\
&= g\left(g^{-1}(\mathsf{w}) - g^{-1}(\mathsf{v}) + k\mathsf{x}_i(g)\right) + \mathsf{v} \\
&= \mathsf{w} + k\lambda_i(g)\mathsf{x}_i(g)
\end{aligned}$$

is a pure translation. For any integer n, $h^n t h^{-n}$ is also a pure translation which is parallel to $\mathsf{x}_i(g)$. If $\mathsf{x}_i(g)$ is an expanding eigenvector, i.e. $|\lambda_i(g)| > 1$, then $h^n t h^{-n} \to 0$ as $n \to -\infty$. If $\mathsf{x}_i(g)$ is a contracting eigenvector, i.e. $|\lambda_i(g)| > 1$, then $h^n t h^{-n} \to 0$ as $n \to \infty$. In either case, the group generated by t and h does not act properly on $\mathbb{R}^{n+1,n}$. $\square$

Finally, consider the case in which the pure translation is $\mathbb{B}$-perpendicular to a fixed eigenvector of $\mathbb{L}(h_1)$ and $\mathbb{L}(h_2)$, which are ultra-transversal.

Lemma 7. *Let $t(\mathsf{w}) = \mathsf{w} + \mathsf{t}$ and $h_i(\mathsf{w}) = g_i(\mathsf{w}) + \mathsf{v}_i$ for $i \in \{1, 2\}$. If h_1 and h_2 are ultra-transversal and $\mathsf{t} \in \mathcal{P}(\mathsf{x}_0(g_1)) \cap \mathcal{P}(\mathsf{x}_0(g_2))$, then $\langle t, h_1, h_2\rangle$ does not act properly on $\mathbb{R}^{n+1,n}$.*

Proof. We can assume that t is not parallel to any eigenvector for either g_1 or g_2, otherwise $\langle t, h_1, h_2\rangle$ does not act properly by Lemma 6.

Note that

$$\left(h_1^n t h_1^{-n}\right)(\mathsf{w}) = \mathsf{w} + g_1^n(\mathsf{t})$$

is a pure translation with translational part $\mathsf{t}_n = g_1^n(\mathsf{t})$. As $n \to \infty$ the direction of t_n approaches some expanding eigendirection of g_1.

Since h_1 and h_2 are ultra-transversal, $\mathcal{P}(\mathsf{x}_0(g_1)) \cap \mathcal{P}(\mathsf{x}_0(g_2))$ does not intersect the null cone except at the origin. There is some n for which t_n is not contained in $\mathcal{P}(\mathsf{x}_0(g_2))$. By Lemma 5, $\langle t, h_1, h_2\rangle$ does not act properly on $\mathbb{R}^{n+1,n}$. $\square$

If the linear holonomy of M is Zariski dense in $\mathbf{SO}(n+1, n)$, then there exist ultra-transversal h_1 and h_2 in the holonomy of M. Lemma 5, Lemma 6, and Lemma 7 imply that if the linear holonomy of M is Zariski dense in $\mathbf{SO}(n{+}1, n)$, then the holonomy cannot contain a pure translation.

Suppose that the linear holonomy of M, Γ, has torsion. That is, there exists an $h \in \Gamma$ such that $g = \mathbb{L}(h)$ and $g^n = I$. h^n is a pure translation, which contradicts the previous discussion. Thus, the linear holonomy of M must be torsion free and Theorem 2 is proven.

4. Lorentz space-times

For $n = 1$, i.e. Lorentz space-times, we have the following [8]:

Theorem 8 (Mess). *A complete Lorentz space-time cannot have a fundamental group isomorphic to the fundamental group of a closed surface of genus ≥ 2.*

Suppose a torsion free $G \subset \mathbf{SO}(2, 1)$ is discrete and has 2 noncommuting elements. Then G is free of rank ≥ 2 or is a surface group.

Suppose $\Gamma \subset \mathbb{R}^{2,1} \ltimes \mathbb{V}$ acts freely and properly discontinuously on $\mathbb{R}^{2,1}$. Γ must be discrete by [4] and cannot contain a surface group by Theorem 8.

Thus, the linear holonomy of a complete Lorentz space-time is either cyclic or free of rank ≥ 2. Theorem 1 is proven.

G. Margulis has pointed out to the author that Theorem 1 is also an immediate consequence of the following theorem [4] :

Theorem 9 (Fried-Goldman). *If M is a compact complete flat 3-manifold, then $\pi_1(M)$ is virtually polycyclic.*

One can show that if the holonomy of M contains a translational element which is not parallel to any eigenvector of an element of the linear holonomy, then M must be compact and $\pi_1(M)$ must be virtually polycyclic.

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