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CANONICAL BASES FOR THE BRAUER CENTRALIZER ALGEBRA

S. FISHEL AND I. GROJNOWSKI

Introduction

In this paper we construct canonical bases for the Birman-Wenzl algebra BW_n, the q-analogue of the Brauer centralizer algebra, and so define left, right and two-sided cells. We describe these objects combinatorially (generalizing the Robinson-Schensted algorithm for the symmetric group) and show that each left cell carries an irreducible representation of BW_n. In particular, we obtain canonical bases for each representation, defined over $\mathbf{Z}$.

The same technique generalizes to an arbitrary tangle algebra and R-matrix [R]; in particular to centralizers of the quantum group action on $V^{\otimes r}$, for V a finite dimensional representation of a quantum group.

BW_n occurs for particular values of the parameters (q, r, x) as the centralizers of the action of $U_q\mathfrak{sp}_{2k}$ or $U_q\mathfrak{o}_k$ on the n-th tensor power of its standard representation V. One may presumably transfer the bases of the BW_n modules to give a basis of representations occurring in $V^{\otimes n}$ (as in [GL]), and it is natural to conjecture that the basis so obtained coincides with that of [L,§27].

Of the Weyl groups, only in the symmetric group are the cell representations irreducible. In this respect BW_n is similar to S_n. One would expect this because of the relation with quantum groups, which also behave like Hecke algebras of type A [L]. Moreover, our main new insight into the structure of BW_n is precisely of this form—we show that every representation is induced from a representation of a symmetric group in a precise way (see §6.5).

This paper is essentially self-contained, except for an appeal to the solution of the corresponding problem for S_n in [KL,1.4]. In particular, we make no further mention of quantum groups and use no previous work on the structure of BW_n (e.g. [BW,HR,W]) except for its description as a

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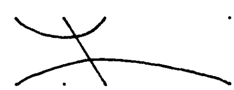
tangle algebra [Ka]. In a sequel to this paper we intend to derive further properties of BW_n from these techniques.

Finally, we remark that our main interest in this problem was to obtain information about a geometric description of the quantum group $U_q\mathfrak{g}$ analogous to the Beilinson-Lusztig-MacPherson construction for GL_n.

1. Brauer diagrams

Let F be a finite set and R a ring. We write RF for the free R-module with basis F; so an element of RF is a map from F to R, usually denoted $\sum_{f \in F} n_f f$. If $n \in \mathbf{N}$, write $2n!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$; and if S is a set, write $|S|$ for its cardinality.

1.1 A "Brauer diagram on n letters" is a partition of the set $\{1, \ldots, 2n\}$ into two element subsets. Write $B = B_n$ for the set of Brauer diagrams, so $|B| = 2n!!$. If $d \in B$, we represent d by a diagram in the plane



where there are n dots numbered $1, \ldots, n$ in the top row; n dots numbered $2n, \ldots, n+1$ in the bottom row, and the vertex i is joined to the vertex j if $\{i,j\} \in d$. We can draw this picture so two edges intersect at most once, there are no self-intersections, at most two edges intersect at any point, the only critical points of the functions representing the edges are the max (resp. min) of horizontal edges, etc. Call such a diagram *nice.*

1.2 If $d \in B$, write $\ell(d)$ for the number of pairs $\{i,j\}, \{k,l\}$ in d such that $i < k < j < l$. In our nice diagram representing d, this is just the number of crossings of edges.

1.3 Also for $d \in B$, write $h(d)$ for the number of pairs $\{i,j\}$ in d with $i \leq n$ and $j \leq n$. This is just the number of horizontal edges in the top row of the diagram of d; clearly this is also the number of horizontal edges in the bottom row.

1.4 Write $S_n = \{d \in B_n \mid h(d) = 0\}$. This is canonically isomorphic to the elements of the symmetric group on n letters.

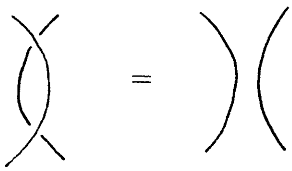
1.5 Let $\mathcal{A}$ be the ring

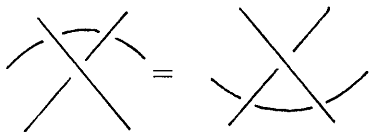
$$\mathcal{A} = \mathbf{Z}[r, r^{-1}, q, q^{-1}, x]/((1-x)(q-q^{-1}) + (r - r^{-1}))$$

and $\mathcal{A}' = \mathbf{Z}[x]$.

2. Tangles

2.1 A "tangle on n letters" is an equivalence class of certain pictures in the plane with $2n$ marked vertices $1, \dots, 2n$ [Ka]. Denote $\mathcal{T}_n$ for the set of n-tangles. A picture t in the plane with lines between the vertices $1, \dots, 2n$ (arranged as in a Brauer diagram), with over and undercrossings indicated and with some number of closed loops, represents a tangle t. If two such pictures differ only in the neighborhood of a crossing, where they are respectively of the form

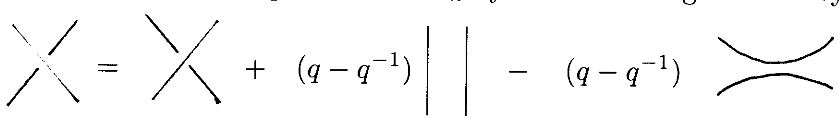
RII: 

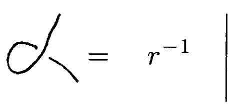
RIII: 

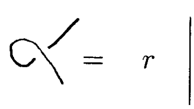
(or any diagram obtained by rotating these), then they represent the same element of $\mathcal{T}_n$; and the set of such pictures mod the equivalence relation generated by these two "Reidemeister moves" is $\mathcal{T}_n$. It is well known that $\mathcal{T}_n$ has an alternate description in terms of regular isotopy classes of links [Ka,R].

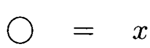
If $t_1, t_2 \in \mathcal{T}_n$, then we define $t_1 t_2$ to be (the equivalence class of) the tangle obtained by concatenating t_1 and t_2 (place t_1 above t_2 and join the dots). With this product, $\mathcal{T}_n$ is a monoid.

2.2 Write $\widetilde{BW}_n$ for the quotient of $\mathcal{A}\mathcal{T}_n$ by the relations generated by

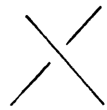
Q1: 

Q2: 

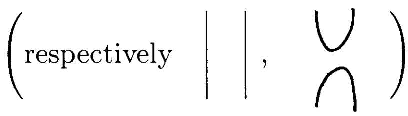
Q3: 

Q4: 

Here, by Q1 we mean that if t is a tangle with some crossing which looks like

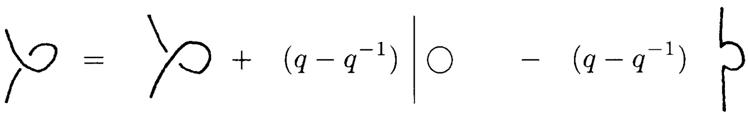


and t' (resp. t'', t''') represents the same tangle with this crossing modified to



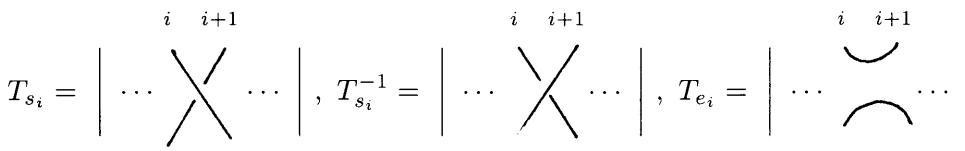
then $t = t' + (q - q^{-1})t'' - (q - q^{-1})t''$ in $\widetilde{BW_n}$. (These relations really do descend to $\mathcal{T}_n$.) Similarly for Q2–Q4.

For example,



whence $(r - r^{-1}) = (q - q^{-1})(x - 1)$ by Q2, Q3, and Q4.

2.2 Define elements T_{s_i}, $T_{s_i}^{-1}$, and T_{e_i}in $\mathcal{T}_n$ by



Define $BW = BW_n$ to be the submonoid of $\widetilde{BW_n}$ generated by T_{s_i}, $T_{s_i}^{-1}$, $T_{e_i}$for $1 \leq i < n$. This is an $\mathcal{A}$-algebra, the "Birman-Wenzl" algebra, and may be defined explicitly in terms of these generators and some relations. (See [BW].) Let us call a tangle $t \in \mathcal{T}_n$ "reachable" if its image in $\widetilde{BW_n}$ actually lies in BW_n. For example, if t is such that no two lines cross more than once, then t is clearly reachable.

2.3 If $t \in \mathcal{T}_n$, we define its Brauer diagram $\phi(t) \in B_n$ by $\{i, j\} \in \phi(t)$ if vertex i is joined to vertex j in t. In terms of pictures, we throw away cycles and ignore whether crossings are over or under.

Let $\mathcal{A} \to \mathcal{A}'$ be the ring homomorphism defined by $q \mapsto 1, r \mapsto 1, x \mapsto x$. Then it is immediate from Q1 that, restricting to tangles with no cycles or self intersection, ϕ descends to a map, also denoted ϕ,

$$\phi : BW_n \otimes_{\mathcal{A}} \mathcal{A}' \to \mathcal{A}' B_n$$

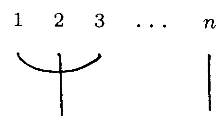
This is in fact an isomorphism of $\mathcal{A}'$-modules [Ka], and the resulting algebra structure on $\mathcal{A}'B_n$ is called the "Brauer centralizer algebra." (In particular, BW_n is a free $\mathcal{A}$-module of rank $2n!!$.)

2.4 We define a section $T : B_n \to \mathcal{T}_n$ of ϕ as follows. If $d \in B_n$, T_d is the picture obtained from a nice diagram for d by requiring $\{i, j\}$ to pass over $\{k, l\}$ if $i < k < j < l$. If is clear that $\phi(T_d) = d$, and that T_d is reachable. We also denote by T the map $B_n \to BW_n$; so that the elements T_d, $d \in B_n$, form an $\mathcal{A}$-basis of BW_n.

2.5 Let F^k be the $\mathcal{A}$-submodule of BW_n generated by the T_d with $h(d) \geq k$. Then $BW_n = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^{\lfloor n/2 \rfloor + 1} = 0$ is a decreasing filtration of BW_n by two-sided ideals. Write $gr_F^k = F^k / F^{k+1}$ for the kth piece of the associated graded algebra. For example $gr_F^0 = H_n$, the Hecke algebra of the symmetric group S_n, or rather, the usual Hecke algebra tensored with $\mathcal{A}$ over $\mathbf{Z}[q, q^{-1}]$ and $gr_F^{\lfloor n/2 \rfloor}$ is also especially simple. In the next section we will see how gr_F^k for general k is a combination of these two extremes.

3. Dangles

3.1 A "flat (n, k) dangle" is a subset of $\{1, \ldots, n\}$ of size $2k$, which is partitioned into k 2-element subsets. Write $D^k = D_n^k$ for the set of flat (n, k)-dangles, so $|D_n^k| = \binom{n}{2k} k!!$. If $d \in D^k$, we can represent d by a diagram in the plane



such that i is joined to j if $\{i, j\} \in d$ and there is a vertical line from i if $i \notin d$. We can insist that two edges intersect at most once, and no vertical edges intersect, etc. If $d \in D_n^k$, write $\ell(d)$ for the number of crossings of edges in the diagram of d, i.e. for the number $\ell_1 + \ell_2$ where ℓ_1 is the number of pairs $(i, j), (k, l)$ in d with $i < k < j < l$ and ℓ_2 is the number of pairs $k, (i, j)$ with $k \notin d$, $(i, j) \in d$, and $i < k < j$.

3.2 An "(n, k)-dangle" is an equivalence class of certain pictures in the plane, with n marked vertices $1, \ldots, n$. A picture p in the plane with k lines between the vertices $\{1, \ldots, n\}$, each line incident to precisely two vertices, with $n - 2k$ vertical lines, and with some number of cycles such that over and under crossings are indicated, represents a dangle if the vertical lines do not cross. The set of such pictures, modulo the equivalence relation generated by the Reidemeister moves RII and RIII is isomorphic to the set of (n, k)-dangles. We denote this $\mathcal{D}_n^k$. One may also describe this set in terms of isotopy classes of framed links.

Write $\widetilde{M}_n^k$ for the quotient of $\mathcal{A}\mathcal{D}_n^k$ by the relations generated by Q1–Q4;

and M_n^k for the $\mathcal{A}$-submodule of $\widetilde{M_n^k}$ spanned by the image of pictures in $\mathcal{D}_n^k$ such that two edges cross at most once and there are no cycles.

If $d \in \mathcal{D}_n^k$, write $\phi(d) \in D_n^k$ for the flat diagram obtained by forgetting over and under crossings, and cycles; i.e. $\{i,j\} \in \phi(d)$ if i is joined to j in d. If $d \in D_n^k$, write T_d for the dangle obtained from a nice diagram by requiring $\{i,j\}$ to pass over $\{k,l\}$ if $i < k < j < l$, and requiring horizontal lines to pass over vertical lines. It is clear that $\phi(T_d) = d$, and that the image of T_d in $\widetilde{M_n^k}$ lies in M_n^k. Again, it is immediate that ϕ descends to $\phi : M_n^k \otimes_{\mathcal{A}} \mathcal{A}' \to \mathcal{A}' D_n^k$, and that this is an isomorphism of $\mathcal{A}'$-algebras. In particular, $\{T_d \mid d \in D_n^k\}$ is a basis of M_n^k.

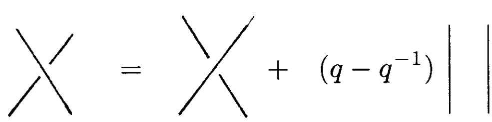
Define ${}^\circ\mathcal{D}_n^k$ to be $\mathcal{D}_n^k$, but draw the pictures dangling upward rather than down, and label the vertices $2n, \ldots, n+1$. (So these represent the bottom part of tangles.) Define the section $T : D_n^k \to {}^\circ\mathcal{D}_n^k$ by requiring horizontal lines to pass under vertical lines, and as always $\{i,j\}$ to pass over $\{k,l\}$ if $i < k < j < l$. Also define ${}^\circ M_n^k$.

3.3 We define maps $D_n^k \times S_{n-2k} \times {}^\circ D_n^k \to B_n$, $\mathcal{D}_n^k \times \mathcal{B}_{n-2k} \times {}^\circ\mathcal{D}_n^k \to \mathcal{T}_n$ (where $\mathcal{B}_{n-2k}$ is the braid group on $n-2k$ letters; i.e. those tangles with no cycles, no self intersection or critical points, and no horizontal edges) by concatenation, e.g.

It is clear that these maps descend to

$$\Phi : M_n^k \otimes_{\mathcal{A}} H_{n-2k} \otimes_{\mathcal{A}} {}^\circ M_n^k \to gr_F^k$$

as the relations Q1–Q4 in M_n^k are the same as those in BW_n, and the relation Q1 in gr_F^k becomes



if these two strands are vertical (i.e.  $\in F_{k+1}$), the defining relation in H_{n-2k}. Further $D^k \times S_{n-2k} \times {}^\circ D^k$ bijects to $\{d \in B_n \mid h(d) = k\}$. Write $d \mapsto (\tau(d), \pi(d), \mathfrak{b}(d))$ for its inverse. Then Φ is an isomorphism and

$$\Phi(T_{\tau(d)} \otimes T_{\pi(d)} \otimes T_{\mathfrak{b}(d)}) = T_d \tag{3.3.1}$$

Note that the section $T : S_{n-2k} \to H_{n-2k}$ defined in 2.4 agrees with $q^{-\ell(w)}\dot{T}_w$, where $\dot{T}_w$ is the usual basis of H_{n-2k}. Further, if $(d_1, \pi, d_2) \mapsto d \in B_n$, observe

$$\ell(d_1) + \ell(\pi) + \ell(d_2) = \ell(d) \tag{3.3.2}$$

with the length functions defined in 3.3, 1.2.

4. Verdier duality

4.1 If t is a picture representing a tangle or dangle, write $\bar{t}$ for the picture obtained from t by interchanging every over and under crossing. It is clear that $^-$ respects Reidemeister moves, and so this operation on pictures descends to tangles.

Also write $^- : \mathcal{A} \to \mathcal{A}$ for the $\mathbf{Z}$-linear ring homomorphism defined by

$$r \mapsto r^{-1}, \quad q \mapsto q^{-1}, \quad x \mapsto x.$$

This is an involution.

It is clear from Q1–Q4 that the $\mathcal{A}$-antilinear involution $^- : \mathcal{A}\mathcal{T}_n \to \mathcal{A}\mathcal{T}_n$, $\sum n_t t \mapsto \sum \bar{n}_t \bar{t}$ descends to an involution $^- : BW_n \to BW_n$; similarly we have $^- : M_n^k \to M_n^k$, $^- : H_n \to H_n$. Further, $\overline{t_1 t_2} = \bar{t}_1 \bar{t}_2$ whenever we can concatenate tangles or dangles t_1 and t_2; i.e. $^-$ is an algebra homomorphism whenever this makes sense. In particular,

$$\Phi(\bar{h}) = \overline{\Phi(h)}$$

where $\Phi : M_n^k \otimes H_{n-2k} \otimes {}^\circ M_n^k \tilde{\to} gr_F^k$, we define $\overline{a \otimes b \otimes c} = \bar{a} \otimes \bar{b} \otimes \bar{c}$, and $^- : gr_F^k \to gr_F^k$ is induced from that of BW_n, as it follows from Q1–Q4 that $\overline{F_k} \subseteq F_k$.

4.2 Observe that if d is a Brauer diagram or flat dangle,

$$\overline{T_d} = T_d + \sum_{d' : \ell(d') < \ell(d)} r_{d'd} T_{d'}$$

for certain $r_{d'd} \in \mathbf{Z}[(q - q^{-1})]$. This follows from Q1 by a straightforward induction.

5. Canonical Bases

5.1 We recall the following lemma of [KL].

Lemma. *Let M be a free $\mathbf{Z}[q,q^{-1}]$-module, with a given basis $(e_i)_{i\in I}$, I some index set. Suppose also given a semilinear involution $^-: M \to M$ such that $\overline{qm} = q^{-1}\bar{m}$, $\overline{m+m'} = \bar{m}+\bar{m}'$, and a partial order $\leq$ on I such that $\{j \mid j \leq i\}$ is finite and*

$$\bar{e}_i = \sum_{j\leq i} r_{ji}e_j, \qquad r_{ji} \in \mathbf{Z}[q,q^{-1}] \text{ and } r_{ii} = 1.$$

Then there is a unique basis $(b_i)_{i\in I}$ of M such that i) $\bar{b}_i = b_i$, and

$$\text{ii) } b_i = \sum_{j\leq i} P_{ji}e_j, \qquad \text{with } P_{ii} = 1, \text{ and } P_{ji} \in q^{-1}\mathbf{Z}[q^{-1}] \text{ if } j < i.$$

This basis is called the "canonical" (or Kazhdan-Lusztig) basis of M.

5.2 We apply the lemma to BW_n, (respectively M_n^k, H_n, gr_F^k) and to the involution $^-$, the standard basis T_d, and the partial order $d' \leq d$ if $d = d'$ or $\ell(d') < \ell(d)$. We may do this by 4.2. We denote the new basis by C_d (resp. C_d, C'_d, C''_d). Note that the basis C'_d of H_n we've just defined is precisely the basis of [KL], and that if $d \in B_n$, $h(d) = k$, then $C''_d = C_d + F_{k+1} \in gr_F^k$. (In particular, if $k = 0$, $C''_d = C'_d = C_d + F_1 \in H_n$, in an orgy of silly notation.)

Also observe that the polynomial $P_{d'd}$ are in $\mathbf{Z}[q^{-1}]$, that is they do not depend on r and x. For example, $C_{e_i} = T_{e_i}$, $C_1 = 1$, $C_{s_i} = T_{s_i} + q^{-1} - q^{-1}T_{e_i}$, $1 \leq i < n$.

Lemma 5.3. *Let $d \in B_n$, $h(d) = k$. Then in gr_F^k we have*

$$\Phi(C_{\tau(d)} \otimes C'_{\pi(d)} \otimes C_{\mathfrak{b}(d)}) = C''_d$$

Proof. As $\Phi(\bar{h}) = \overline{\Phi(h)}$, it is clear that the left hand side is fixed by $^-$. On the other hand, by 3.3.1 and 3.3.2 it is also clear that the left hand side is of the form $\sum_{d_1,\pi,d_2} \gamma_{d_1\pi d_2}\Phi(T_{d_1} \otimes T_\pi \otimes T_{d_2})$, where $\ell(d_1) + \ell(\pi) + \ell(d_2) \leq \ell(d)$ and that either $(d_1,\pi,d_2) = (\tau(d),\pi(d),\mathfrak{b}(d))$ and $\gamma_{d_1\pi d_2} = 1$ or $\ell(d_1) + \ell(\pi) + \ell(d_2) < \ell(d)$ and $\gamma_{d_1\pi d_2} \in q^{-1}\mathbf{Z}[q^{-1}]$. As the basis C''_d is the unique element of gr_F^k with these two properties, the lemma follows.

Note that $\phi(C_d)$, $d \in B_n$, gives a basis in the Brauer algebra $\mathcal{A}'B_n$ which is independent of the section $T : B_n \to BW_n$. We remark that Lusztig has informed us that in [L,27.3.10] a basis of BW_n for particular values of (q,r,x) is defined. We will compare this to our basis in a future article.

6. Cells

6.1 Let h_{xyz} be the structure constants for multiplication in BW_n with respect to the canonical bases; i.e.

$$C_x C_y = \sum_{z \in B_n} h_{xyz} C_z \qquad \text{for } x, y \in B_n.$$

Let $\leq_L$ (resp. $\leq_R$) be the preorder on B_n generated by the relations $z \leq_L y$ (resp. $z \leq_R x$) if there exists an $x \in B_n$ (resp. $y \in B_n$) such that $h_{xyz} \neq 0$. Let $\leq_{LR}$ be the preorder generated by the relation $x \leq_{LR} y$ if $x \leq_L y$ or $x \leq_R y$. Write $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$; similarly for $\sim_R$, $\sim_{LR}$. The equivalence classes for $\sim_L$, $\sim_R$, $\sim_{LR}$ are called respectively left, right or two-sided cells. Observe that if $x \sim_L y$, then $h(x) = h(y)$.

If Γ is a left cell in B_n, then if we set

$$F^\Gamma = \mathcal{A}\{C_x \mid x \leq_L \Gamma\},$$

F^Γ is a left ideal in BW_n. Write $F^{<\Gamma}$ for the sum of the $F^{\Gamma'}$ such that $\Gamma' \leq_L \Gamma$, $\Gamma \neq \Gamma'$; and write $gr^\Gamma = F^\Gamma / F^{<\Gamma}$. This is a left BW_n module. Similarly, for Γ a right or two-sided cell, the analogously defined F^Γ are right (resp. two-sided ideals), and gr^Γ is a right module (resp. $BW_n \times BW_n^\circ$ module).

Observe that these filtrations F^Γ refine the filtration F^k of 2.6. They are analogues of the definitions for Hecke algebras in [KL].

6.2 Our main result is an explicit description of the equivalence classes $\sim_L$, and hence an explicit construction of bases in the irreducible modules for BW_n with structure constants in $\mathcal{A}$. The proof will be given in 6.4.

Theorem 6.3. *We have $d \sim_L d'$ if and only if $h(d) = h(d')$, $\mathfrak{b}(d) = \mathfrak{b}(d')$, and $\pi(d) \sim_L \pi(d')$ in $S_{n-2h(d)}$. Further, if Γ and Γ' are two left cells in the same two-sided cell, then gr^Γ is isomorphic to $gr^{\Gamma'}$ as a BW_n-module with basis. Finally, let F be a field, $\alpha : \mathcal{A} \to F$ a homomorphism of rings, and suppose $BW_n \otimes_{\mathcal{A}} F$ is semisimple. Then each representation $gr^\Gamma \otimes_{\mathcal{A}} F$ is irreducible.*

Note that in light of the description of $\sim_L$ in the symmetric group in [KL], we may describe $\sim_L$ as follows. There is a bijection from the set $\{d \in B_n \mid h(d) = k\}$ to the set of tuples (d_1, d_2, P, Q) where $d_i \in D_n^k$, and P, Q are pairs of standard tableaux of the same shape and size $n-2k$. Under this bijection, $d_1 = \tau(d), d_2 = \mathfrak{b}(d)$ and (P, Q) is the pair of tableaux associated by the Robinson-Schensted algorithm to the permutation $\pi(d) \in S_{n-2k}$. Then (d_1, d_2, P, Q), (d_1', d_2', P', Q') are in the same left cell if $d_2 = d_2'$ and

$Q = Q'$; in the same right cell if $d_1 = d'_1$ and $P = P'$, and in the same two-sided cell if P and P' are of the same shape.

There is an alternate Robinson-Schensted algorithm, due to Sundaram [S], which bijects B_n onto pairs (p, q) of paths of length n in the Young lattice which end in partitions of the same shape. In this language, (p, q) and (p', q') are in the same left cell if $q = q'$, and are in the same two-sided cell if p, p' end at the same partition.

6.4 The action of BW_n on gr_F^k defines via Φ an action of BW_n on $M_n^k \otimes H_{n-2k} \otimes {}^\circ M_n^k$, again by composition of tangles. This action has the following property: If $h \in BW_n$, $a \otimes b \otimes c \in M_n^k \otimes H_{n-2k} \otimes {}^\circ M_n^k$, then

$$h \cdot (a \otimes b \otimes c) = \sum_{d \in D_n^k,\ \pi \in S_{n-2k}} \gamma_{d\pi}^{ha} C_d \otimes C'_\pi b \otimes c \tag{6.4.1}$$

where the coefficients $\gamma_{d\pi}^{ha}$ depend only on h, a (and d,π), and the multiplication $C'_\pi b$ is the usual one in H_{n-2k}. It follows that if $d' \leq_L d$ and if $h(d') = h(d)$, then $\pi(d') \leq_L \pi(d)$ and $\mathfrak{b}(d') = \mathfrak{b}(d)$.

Now suppose $d \in D_n^k$, $a \otimes b \otimes c \in M_n^k \otimes H_{n-2k} \otimes {}^\circ M_n^k$, $f \otimes g \in H_{n-2k} \otimes {}^\circ M_n^k$. Then (6.4.1) can be refined to

$$(T_d \otimes f \otimes g) \cdot (a \otimes b \otimes c) = T_d \otimes h(a, f, g) \cdot b \otimes c$$

where $h(a, f, g) \in H_{n-2k}$ depends only on (a, f, g). Hence

$$(C_d \otimes f \otimes g) \cdot (a \otimes b \otimes c) = C_d \otimes h(a, f, g) \cdot b \otimes c.$$

Now let $a = T_{d'}$ for some $d' \in D_n^k$. Then we can pick a $g \in {}^\circ M_n^k$ such that $ag = \lambda.1 \in H_{n-2k}$, where $\lambda \in \mathcal{A}$ and 1 is the tangle consisting of $n - 2k$ vertical lines. So $h(a, f, g) = f\lambda$ for this g. As the elements $T_{d'}$, $d' \in D_n^k$ form a basis of M_n^k, it follows that for arbitrary $a \in M_n^k$ there exists $g \in {}^\circ M_n^k$ and $\lambda' \in \mathcal{A}$, $\lambda' \neq 0$ with $h(a, f, g) = f\lambda'$.

It is immediate that if $(d_1, \pi_1, d''), (d_2, \pi_2, d'') \in D_n^k \times S_{n-2k} \times D_n^k$ satisfy $\pi_1 \leq_L \pi_2$ then $(d_1, \pi_1, d'') \leq_L (d_2, \pi_2, d'')$ and so, in particular, the left cells are as claimed.

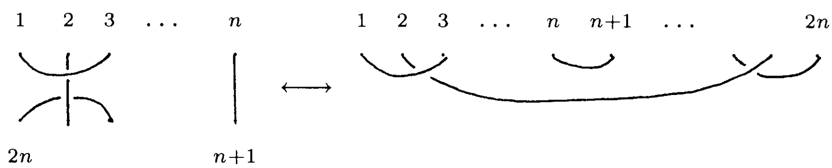
Now let Γ be a left cell in B_n. It is clear from the above that the representation gr^Γ does not depend on $\mathfrak{b}(\Gamma)$ and only depends on the representation in H_{n-2k} carried by the left cell $\pi(\Gamma)$. As this representation is the same for any $\pi(\Gamma)$ in a fixed two-sided cell in S_{n-2k} [KL,1.4], the second statement of the theorem follows.

Finally, as there are as many left cells Γ in a two-sided cell as elements in Γ, we have produced a decomposition of the regular representation of the form $\sum(\dim \rho)\rho$. It follows that each summand gr^Γ is irreducible.

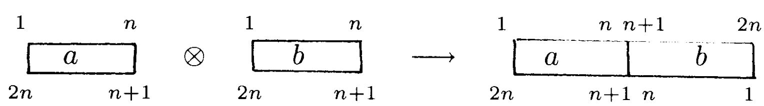
6.5 In the course of the proof of 6.3 we have shown that if V is an irreducible representation of H_{n-2k}, then we can give $M_n^k \otimes V$ the structure of an irreducible representation of BW_n, and that all irreducible representations of BW_n occur in this way for a unique k, V.

7. Extending the regular representation

7.1 Given a tangle $t \in \mathcal{T}_n$ one may represent it as a picture in which the dots $1, \ldots, 2n$ are placed in a line (by rotating the bottom line to the top).



As the relations Q1–Q4 are defined for tangles—they are rotation invariant, for example—concatenation defines an action of BW_{2n} on BW_n. By 6.4, this action makes BW_n an irreducible BW_{2n}-module; this action "extends" the regular representation as we have an algebra homomorphism $\psi : BW_n \otimes BW_n^\circ \hookrightarrow BW_{2n}$



by taking $a \otimes b$ to the tangle obtained by rotating b 180° and placing it next to a.

Note that, on the level of Brauer diagrams, the same procedure also gives an action of S_{2n} on B_n. Then S_{2n} acts transitively on B_n, the stabilizer of the identity being the hyperoctahedral Weyl group of rank n. So $|B_n| = (2n)!/2^n n! = 2n!!$.

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Southern Connecticut State University, New Haven, CT 06515
E-mail address: fishel@scsud.ctstateu.edu

Yale University, New Haven, CT 06520
E-mail address: grojnowski-ian@math.yale.edu