VOLUME GROWTH, ENTROPY AND THE GEODESIC STRETCH

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Dedicated to Prof. W. Klingenberg on the occasion of his 70th birthday.

1. Introduction

Let (M, g) be a compact Riemannian manifold with universal covering $\tilde{M}$. The simplest asymptotic invariant which can be associated to (M, g) is the exponential growth rate $h(g)$ of volume on the universal covering, called the volume entropy. If $B_r^g(p)$ denotes the geodesic ball of radius r about $p \in \tilde{M}$ and $vol_g(B_r^g(p))$ describes its volume with respect to the Riemannian metric g lifted to $\tilde{M}$ then volume entropy is given by

$$h(g) = \lim_{r \to \infty} \frac{\log vol_g(B_r^g(p))}{r}.$$

Due to the compactness of M, this limit exists and is independent of $p \in \tilde{M}$ [17]. Further asymptotic invariants are obtained by the dynamical properties of the geodesic flow induced by the metric g. For each flow invariant measure μ, $h_\mu(g)$ denotes the measure theoretic entropy and $h_{top}(g)$ denotes the topological entropy of the geodesic flow. In this note we compare the volume growth and the measure theoretic entropy for a pair of metrics on the same manifold. Some of the results are similar in spirit to the entropy comparison theorems of A. Katok [12]. Crucial for the proofs will be an asymptotic comparison of the distance functions for a pair of Riemannian metrics called the geodesic stretch. Given a pair g_0, g of Riemannian metrics on M, the geodesic stretch $I_\mu(g_0, g)$ measures the "stretching" of the metric g relative to a reference metric g_0 and a measure μ invariant under the geodesic flow induced by g_0 (see Section 2). In the general setting, this notion has been introduced in a paper by C. Croke and A. Fathi [5], where it is called intersection, since it can be viewed as a generalization of Thurston's intersection theory of curves and measured foliation on surfaces. Using those quantities, we will prove in Section 3 the following theorem.

Theorem 1.1. *If (M, g_0) is a compact space with no conjugate points and with ergodic geodesic flow, then for every metric g on M*

$$h(g) \geq \frac{1}{I_{\mu_L}(g_0, g)} h_{\mu_L}(g_0) \,,$$

where μ_L denotes the Liouville measure induced by the metric g_0.

Remark.
a) A. Manning [17] proved that the volume entropy $h(g)$ is less or equal to the topological entropy $h_{top}(g)$. Equality holds in general only if the metric has no conjugate points [9, 17].
b) The assumption of no conjugate points is necessary for the metric g_0. For instance, on the two dimensional sphere, there exist ergodic metrics with positive measure theoretic entropy [4, 7] but, of course, the volume entropy is zero for every metric. However, there might be a chance to remove the assumption of ergodicity for g_0.

Let g_0 now be a metric of strictly negative curvature. If we replace the Liouville measure by the Bowen-Margulis measure μ_{BM} of g_0 (this is the unique measure of maximal entropy, i.e. $h_{\mu_{BM}}(g_0) = h_{top}(g_0) = h(g_0)$), we will prove Theorem 1.2 in the third Section.

Theorem 1.2. *If (M, g_0) is a compact space of negative curvature, then for every metric g on M*

$$h(g) \geq \frac{1}{I_{\mu_{BM}}(g_0, g)} h(g_0) \,.$$

If g is also of negative curvature, equality holds if and only if the geodesic flows Φ_{g_0} of g_0 and Φ_g of g are up to a scaling time preserving conjugate, i.e. there exists a homeomorphism $F : (SM)_{g_0} \to (SM)_g$ such that

$$F \circ \Phi_{g_0}^{ct} = \Phi_g^t \circ F,$$

where $c = \frac{h(g)}{h(g_0)}$ and $(SM)_g$ denotes the unit tangent bundle with respect to g.

Remark.
a) The inequality in the case where both metrics are of negative curvature has been also obtained using different methods by M. Burger [3]. He shows that equality holds if and only if the marked length spectra are proportional. It is known [10] that this also implies the existence of a conjugating homeomorphism. However, so far only a sketch of his proof is available.
b) Recently Besson, Courtois, and Gallot [2] proved that a locally symmetric metric of negative curvature is the unique global minimum of h on the space of metrics with normalized volume. More precisely, they obtained:

If (M, g_0) is a compact locally symmetric space of negative curvature and $\dim M \geq 3$, then for all other metrics g on M

$$h(g) \geq h(g_0) \left(\frac{\mathrm{vol}_{g_0}(M)}{\mathrm{vol}_g(M)} \right)^{\frac{1}{n}} ,$$

where the inequality is strict unless g and g_0 are isometric up to a scaling. c) It has been proved by J.P. Otal [20] that two metrics of negative curvature on a surface with time preserving conjugacy of their geodesic flows are isometric.
Combining these results, we obtain:

Corollary 1.3. *Let M be a compact manifold and denote by $R_{ent}(M)$ the space of Riemannian metrics of negative curvature and with normalized entropy one. Then for all g_1 and $g_2 \in R_{ent}(M)$*

$$I_{\mu_{BM}}(g_1, g_2) \geq 1 ,$$

where the inequality is strict unless there exists a time preserving conjugacy between the geodesic flows of g_1 and g_2. This conjugacy is induced by an isometry if either $\dim M = 2$ or one of the metrics g_i is locally symmetric and the volume of both metrics coincides.

Remark.   It seems to be likely that the condition on the volumes is vacuous. For instance, if the conjugacy is C^1 the volume of the two metrics automatically coincides [6]. We conjecture that also for $\dim M \geq 3$ the geodesic stretch distinguishes between iso-entropy metrics of negative curvature up to isometry. By Corollary 1.3, this is equivalent to the statement that a time preserving conjugacy is induced by an isometry.

In [13], we proved that the topological entropy restricted to the metrics of negative curvature is a C^∞ functional. In [14], we derived an explicit formula for the first derivative. Moreover, if $R_{\mathrm{vol}}(M)$ denotes the space of Riemannian metrics of negative curvature with normalized volume one, we obtained: $h : R_{\mathrm{vol}}(M) \to \mathbf{R}$ has a critical point at $g_0 \in R_{\mathrm{vol}}(M)$ if and only if the Bowen-Margulis measure μ_{BM} and the Liouville measure μ_L coincide on symmetric two-forms. This leads to Corollary 1.4 proved in Section 4.

Corollary 1.4. *Let $g_0 \in R_{\mathrm{vol}}(M)$ be a critical point of the volume entropy. Then for any other metric g on M*

$$h(g)^2 \geq \frac{1}{\frac{1}{n} \int\limits_M tr_{g_0} g \, d\, \mathrm{vol}_{g_0}} h(g_0)^2 ,$$

where $tr_{g_0}g$ denotes the trace of g relative to g_0 and $vol_{g_0}(M)$ is the volume of M with respect to the metric g_0. Furthermore, equality holds if and only if $g_0 = g$ up to a constant.

Let g be a negatively curved metric on M and denote by $\mathrm{Conf}(g)$ the space of Riemannian metrics conformally equivalent to g. Then, as we have shown in [14], $h : R_{\mathrm{vol}}(M) \cap \mathrm{Conf}(g) \to \mathbf{R}$ has a critical point $g_0 \in R_{\mathrm{vol}}(M) \cap \mathrm{Conf}(g)$ if and only if the Bowen-Margulis and the Liouville measure of g_0 coincide on functions which are constant on the fibers of SM_{g_0}.

The following statement will be proved in Section 4.

Corollary 1.5. *Let $g_0 \in R_{\mathrm{vol}}(M) \cap \mathrm{Conf}(g)$ be a critical point of h. Then for all other metrics $\tilde{g} \in \mathrm{Conf}(g)$ with $vol_{\tilde{g}}(M) = 1$ (not necessarily of negative curvature)*

$$h(\tilde{g}) \geq h(g_0) ,$$

and equality holds if and only if $\tilde{g} = g_0$. In particular, a critical point is an absolute minimum in the conformal class of metrics with nomalized volume and, therefore, the critical point is uniquely determined if it exists.

Remark.
a) If $\mathrm{Conf}(g)$ contains a locally symmetric metric then this metric is the critical point since in this case the Liouville- and the Bowen-Margulis measure coincide. In particular, for surfaces of negative curvature, as has been shown in [14], the only critical points are metrics of constant negative curvature.
b) If $\mathrm{Conf}(g)$ does not contain a locally symmetric metric, it would be interesting to compute the minimal conformal entropy given by

$$\min h_{\mathrm{Conf}(g)} = \inf\{h(\tilde{g}) \mid \tilde{g} \in \mathrm{Conf}(g),\ vol_{\tilde{g}}(M) = 1\}.$$

Does there exist a metric $g_0 \in \mathrm{Conf}(g)$ with $vol_{g_0}(M) = 1$ which admits the minimal conformal entropy?

If $\lambda^0(g)$ denotes the bottom of the spectrum of the Laplacian on $\tilde{M}$ with respect to g and

$$\max \lambda^0_{\mathrm{Conf}(g)} = \sup\{\lambda_0(\tilde{g}) \mid \tilde{g} \in \mathrm{Conf}(g),\ vol_{\tilde{g}}(M) = 1\},$$

as shown in [1]

$$\max \lambda^0_{\mathrm{Conf}(g)} \leq \frac{1}{4} \min h^2_{\mathrm{Conf}(g)} .$$

c) Corollary 1.5 has been independently obtained by G. F. Robert [22] using convexity properties of the volume entropy.

Before proving the theorems, we would like to begin by recalling some basic facts about the geodesic stretch (see also [5]).

2. Geodesic stretch

Let $M = \tilde{M}/\Gamma$ be a compact manifold, where $\tilde{M}$ denotes the universal covering and Γ the group of covering transformations. Fix two Riemannian metrics g_0 and g on M. For each vector v in the unit tangent bundle $(SM)_{g_0}$ of g_0 and each $t \in \mathbf{R}$, consider the geodesic $c_v(s)$ with respect to the Riemannian metric g_0 and initial condition $\dot{c}_v(0) = v$. Denote its lift to the universal covering $\tilde{M}$ by $\tilde{c}_v(s)$. Let $a(v,t) = d_g(\tilde{c}_v(0), \tilde{c}_v(t))$ be the distance of the endpoints of the segment $\tilde{c}_v(s)$, $0 \le s \le t$ with respect to the metric g lifted onto $\tilde{M}$. It is an easy consequence of the triangle inequality that the map $a(v,t)$ is a subadditive cocycle for the geodesic flow Φ_{g_0} induced by the metric g_0, i.e.

Lemma 2.1. $a(v, t_1 + t_2) \le a(v, t_1) + a(\Phi_{g_0}^{t_1} v, t_2)$ for all $v \in S\tilde{M}$ and $t_1, t_2 \in \mathbf{R}$.

The following corollary is a consequence of the subadditive ergodic theorem [24].

Corollary 2.2. Let μ be a Φ_{g_0} invariant probability measure. Then

$$I_\mu(g_0, g, v) := \lim_{t \to \infty} \frac{a(v,t)}{t}$$

exists for μ almost all $v \in (SM)_{g_0}$ and defines a μ-integrable function on $(SM)_{g_0}$ invariant under the geodesic flow Φ_{g_0}.

Moreover, we have:

$$\int\limits_{(SM)_{g_0}} I_\mu(g_0, g, v)d\mu = \lim_{t \to \infty} \frac{1}{t} \int\limits_{(SM)_{g_0}} a(v,t)d\mu = \inf_{t > 0} \frac{1}{t} \int\limits_{(SM)_{g_0}} a(v,t)d\mu \ .$$

Definition 2.3.

$$I_\mu(g_0, g) := \int\limits_{(SM)_{g_0}} I_\mu(g_0, g, v)d\mu$$

is called the geodesic stretch of the metric g relative to g_0 and the measure μ. Furthermore,

$$E_\mu(g_0, g) := \int\limits_{(SM)_{g_0}} g(v,v)d\mu$$

is called the energy with respect to μ, g_0, g.

Lemma 2.4. *With the notations above, we obtain the following inequality:*

$$\left(I_\mu(g_0, g)\right)^2 \le E_\mu(g_0, g).$$

Equality implies the existence of a constant $e > 0$ such that

$$g(v, v) = e \cdot g_0(v, v)$$

for almost all vectors v in the tangent bundle TM of M. If the measure is positive on open sets, equality holds if and only if g and g_0 are identical up to a constant and are metrics without conjugate points.

Proof. From Corollary 2.2, the Φ_{g_0} invariance of μ and the fact that

$$a(v, t) \le \int_0^t ||\Phi_{g_0}^s v||_g ds \quad \text{for all} \quad t > 0,$$

we obtain:

$$
\begin{aligned}
I_\mu(g_0, g) \quad & \le \quad \frac{1}{t} \int_{(SM)_{g_0}} a(v, t) d\mu \le \frac{1}{t} \int_{(SM)_{g_0}} \int_0^t ||\Phi_{g_0}^s v||_g \, ds \, d\mu \\
& = \quad \int_{(SM)_{g_0}} ||v||_g d\mu \le \Big(\int_{(SM)_{g_0}} g(v, v) d\mu \Big)^{\frac{1}{2}}.
\end{aligned}
$$

This yields the inequality. Since the last estimate above is the Cauchy-Schwarz inequality, equality implies $g(v, v) = e$ for a positive constant e and almost all $v \in (SM)_{g_0}$. If the measure μ is positive on open sets, equality implies $g = e \cdot g_0$. Furthermore, we have:

$$\sqrt{e} = \frac{1}{t} \int_0^t ||\Phi_{g_0}^s v||_g ds = \frac{a(v, t)}{t} = \frac{d_g(\tilde{c}_v(0), \tilde{c}_v(t))}{t} = \sqrt{e} \frac{d_{g_0}(\tilde{c}_v(0), \tilde{c}_v(t))}{t} \ .$$

Therefore, each g_0-geodesic $\tilde{c}_v$ on the universal covering is minimizing. Hence, there are no conjugate points. $\square$

3. Entropy comparison

Since any two Riemannian metrics g, g_0 on a compact manifold M are equivalent, there exists a constant $a > 1$ such that for all measurable sets A in the universal covering $\tilde{M}$, the estimate

$$\frac{1}{a}\,\mathrm{vol}_{g_0}(A) \leq \mathrm{vol}_g(A) \leq a\;\mathrm{vol}_{g_0}(A)$$

holds. Therefore, we can take the volume element of any metric to measure the exponential growth rate of balls on the universal covering. More precisely, we have:

Lemma 3.1. *Let g, g_0 be a pair of Riemannian metrics on a compact manifold M. Then*

$$h(g) = \lim_{r \to \infty} \frac{\log \mathrm{vol}_{g_0}(B_r^g(p))}{r}\;,$$

where $B_r^g(p)$ is the geodesic ball of radius r about $p \in \tilde{M}$ with respect to the metric g.

The measure theoretic entropy $h_{\mu_L}(g)$ with respect to the Liouville measure can be obtained as an integral over the sum of the Lyapunov exponents. If g is a metric without conjugate points, there is a geometric description. For each vector $v \in SM$, let $J_v(t)$ be the Jacobi tensor (i.e. $J_v(t)$ is a linear endomorphism of the normal space of $\dot{c}_v(t)$ solving the Jacobi equation) along the geodesic $c_v(t)$ with initial conditions $J_v(0) = 0$, $J'_v(0) = Id$. If $\det J_v(t)$ denotes the determinant of this endomorphism, the limit

$$\chi(v) = \lim_{t \to \infty} \frac{1}{t} \log \det J_v(t)$$

exists for almost all $v \in SM$ and defines an integrable function such that $h_{\mu_L}(g) = \int_{SM} \chi(v) d\mu_L$. See [9] for details.

Now we are able to prove Theorem 1.1 stated in the introduction.

Proof of Theorem 1.1. Since the geodesic flow of g_0 is ergodic with respect to g_0, we have:

$$\lim_{t \to \infty} \frac{a(v, t)}{t} = I_{\mu_L}(g_0, g) \quad \text{and} \quad \lim_{t \to \infty} \frac{\log \det J_v(t)}{t} = h_{\mu_L}(g_0)$$

for almost all $v \in (SM)_{g_0}$, where J_v denotes the Jacobi tensor induced by the metric g_0 and with initial conditions as above.

For $\epsilon > 0$ and $T > 0$ consider the sets

$$A^{T,\epsilon} = \{v \in (SM)_{g_0} \mid \left| \frac{a(v,t)}{t} - I_{\mu_L}(g_0, g) \right| < \epsilon,\ t \geq T\} ,$$

and

$$B^{T,\epsilon} = \{v \in (SM)_{g_0} \mid \left| \frac{\log \det J_v(t)}{t} - h_{\mu_L}(g_0) \right| < \epsilon,\ t \geq T\}.$$

For all δ, $0 < \delta < 1$ and $\epsilon > 0$ we can find a $T(\epsilon, \delta)$ such that

$$\mu_L(A^{T,\epsilon}) \geq 1 - \delta \quad \text{and} \quad \mu_L(B^{T,\epsilon}) \geq 1 - \delta.$$

Since μ_L is a probability measure, we have that $\mu_L(A^{T,\epsilon} \cap B^{T,\epsilon}) \geq 1 - 2\delta$.

Using the canonical decomposition of the Liouville measure, we obtain for a fixed $\delta < \frac{1}{2}$:

$$\forall \epsilon > 0, \exists p \in M \text{ such that } \theta_p(A_p^{T,\epsilon} \cap B_p^{T,\epsilon}) > d_1 > 0$$

for a constant d_1, where θ_p is the canonical measure of the fiber S_pM and $A_p^{T,\epsilon} = A^{T,\epsilon} \cap S_pM$, $B_p^{T,\epsilon} = B^{T,\epsilon} \cap S_pM$. Given $v \in A_p^{T,\epsilon}$

$$d_g(\tilde{c}_v(0), \tilde{c}_v(t)) = a(v,t) \leq (\epsilon + I_{\mu_L}(g_0, g))t ,$$

and, therefore, we obtain for $\tilde{p} = \tilde{c}_v(0) \in \tilde{M}$

$$B_{g_0}^{T,\epsilon}(\tilde{p}, t) \quad := \quad \{exp_{\tilde{p}}(vs) \mid v \in A_p^{T,\epsilon} \cap B_p^{T,\epsilon}, T \leq s \leq t\}$$

$$\subseteq \quad B_g(\tilde{p}, (\epsilon + I_{\mu_L}(g_0, g))t)$$

for all $t \geq T$.

$v \in B_p^{T,\epsilon}$ implies:

$$\log \det J_v(t) \geq (h_{\mu_L} - \epsilon)t \quad \text{for all} \quad t \geq T.$$

Integrate this relation; there exists a constant d_2 such that

$$d_2 \cdot e^{(h_{\mu_L} - \epsilon)t} \leq \int_T^t \int_{A_p^{T,\epsilon} \cap B_p^{T,\epsilon}} \det J_v(s) d\theta_p(v) ds$$

$$= \text{vol}_{g_0}(B_{g_0}^{T,\epsilon}(\tilde{p}, t)$$

$$\leq \text{vol}_{g_0} B_g(\tilde{p}, (\epsilon + I_{\mu_L}(g_0, g))t) ,$$

and, therefore, using Lemma 3.1

$$
\begin{aligned}
h_{\mu_L} - \epsilon \quad &\leq \quad \lim_{t \to \infty} \frac{\mathrm{vol}_{g_0} B_g(\tilde{p}, (\epsilon + I_{\mu_L}(g_0, g))t)}{(\epsilon + I_{\mu_L}(g_0, g))t} \cdot \frac{(\epsilon + I_{\mu_L}(g_0, g))t}{t} \\
&= \quad h(g)(I_{\mu_L}(g_0, g) + \epsilon) \ .
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, this proves the theorem. $\quad\square$

In order to prove the next theorems, we need to recall some basic facts about the asymptotic geometry of a simply connected space $\tilde{M}$ of negative curvature.

Given a metric g of negative curvature, the ideal boundary $\tilde{M}(\infty)$ is defined to be the set of equivalence classes of asymptotic geodesic rays [8]. If one chooses a point $p \in \tilde{M}$, the map $G_p(g) : S_p\tilde{M} \to \tilde{M}(\infty)$ which associates to each $v \in S_p M$ the geodesic ray $c_v(t)$ with initial condition $\dot{c}_v(0) = v$ is a bijection. The topology on $\tilde{M}(\infty)$ induced by $G_p(g)$ is called the sphere topology. It is well known that the sphere topology is independent of p. It is a consequence of a Lemma by M. Morse that the ideal boundary depends only on the quasi-isometric structure of $\tilde{M}$, i.e. it can be described in terms of quasi-rays.

Definition 3.2. Let g be a Riemannian metric on $\tilde{M}$ with corresponding distance function d_g. A quasi-ray is a quasi-isometric map $\varphi : \mathbf{R}^+ \to \tilde{M}$, i.e. there exists a constant $b > 1$ such that

$$
\frac{1}{b}|t - s| \leq d_g(\varphi(t), \varphi(s)) \leq b|t - s| \ .
$$

A quasi-isometric map $\varphi : \mathbf{R} \to \tilde{M}$ is called a quasi-geodesic.

Theorem 3.3. (*Morse Lemma*) *Let g be a metric on $\tilde{M}$ such that the sectional curvature is bounded from above by a negative constant. For each quasi-geodesic (quasi-ray) φ, there exists a geodesic (ray) c such that the Hausdorff distance of their images is bounded by a constant $r > 0$ depending only on b and the upper bound of the sectional curvature. In the case of quasi-geodesics, the geodesic c is unique up to parametrization. If we consider quasi-rays φ, the ray c is uniquely determined if we require $\varphi(0) = c(0)$.*

Proof. See [19] and [15]. $\quad\square$

Call two quasi-rays equivalent if their Hausdorff distance is bounded. From the Morse Lemma, it follows that we can identify the set of equivalence classes of asymptotic geodesic rays and the equivalence classes of

quasi-rays. In particular, this implies that the ideal boundary of two equiv-
alent Riemannian metrics g_1 and g_2 of negative curvature on the same
manifold coincide. The sphere topology does not change, since for each
$p \in \tilde{M}$, the map $G_p(g_1) \circ G_p^{-1}(g_2) : S_p\tilde{M} \to S_p\tilde{M}$ is a homeomorphism.

Denote by $B(\tilde{M}(\infty))$ the set of finite Borel measures on $\tilde{M}(\infty)$. If Γ
is a discrete group acting on $\tilde{M}$ with compact quotient $M = \tilde{M}/\Gamma$, we
can associate to each metric g of negative curvature a family of measures
$\{\mu_p^g\}_{p \in \tilde{M}}$ on $\tilde{M}(\infty)$ constructed in the following way.

Consider the Poincaré series

$$P_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd_g(x, \gamma y)} .$$

This series converges for $s > h(g)$ and diverges for $s \leq h(g)$. For a fixed
p, a fixed $x \in \tilde{M}$ and $s > h(g)$, define a finite measure on $\tilde{M} \cup \tilde{M}(\infty)$ by

$$\mu_{p,x}^g(s) = \frac{\sum\limits_{\gamma \in \Gamma} e^{-sd_g(p, \gamma x)} \delta_{\gamma x}}{P_s(x, x)} .$$

$\mu_{p,x}^g(s)$ measures the distribution of the orbit of x under Γ as viewed from
$p \in \tilde{M}$. We obtain a measure at infinity if we consider a weak limit

$$\mu_p^g = \lim_{s_n \to h} \mu_p^g(s_n) ,$$

where $s_n > h(g)$. Since $P_s(x, y)$ is diverging for $s = h(g)$, it follows that
μ_p^g is concentrated on the accumulation points of the orbit Γx. These
measures have the following two characterizing properties:
a) μ_p^g and μ_q^g are in the same measure class and their Radon-Nikodym
derivative is for almost all $\xi \in \tilde{M}(\infty)$ given by

$$\frac{d\mu_p^g}{d\mu_q^g}(\xi) = e^{-h(g)b_q(p, \xi)} ,$$

where $b_q(p, \xi)$ is the Busemann function with respect to the metric g,
normalized in such a way that $b_q(q, \xi) = 0$ and $b_q(p, \xi)$ tends to $-\infty$ as p
tends to ξ. $b_q(p, \xi)$ can be interpreted as the signed distance between the
horospheres through p and q with common center in ξ.
b) The map $\mu^g : \tilde{M} \to B(\tilde{M}(\infty))$ is Γ-equivariant, i.e.

$$\mu_p^g \circ \gamma = \mu_{\gamma^{-1}(p)}^g$$

for all $\gamma \in \Gamma$, if $\mu_p^g \circ \gamma(A) = \mu_p^g(\gamma A)$.

One should think of μ^g as measuring a given set at infinity from different
observation points. It turns out that the map μ^g with property a) and b) is

uniquely determined up to a constant. This follows from the fact that the action of Γ is ergodic with respect to the measure class defined by any such family μ^g. The ergodicity of the action of Γ on $\tilde{M}(\infty)$ can be proved along the lines of [23]. This measure class (Patterson-Sullivan measure) has been first considered by Patterson [21] in the case of Fuchsian groups and by Sullivan [23] in the study of discrete group actions on the hyperbolic space. However, the construction can be extended to general compact manifolds of negative curvature [11]. For a different representation of μ^g see [16, 25]. For us the following local property of μ^g will be crucial [23]. Define $pr_p^g :$ $\tilde{M} \to \tilde{M}(\infty)$ to be the projection of $\tilde{M}$ onto $\tilde{M}(\infty)$ by geodesic rays with respect to g emanating from p (the image of a set $A \subset \tilde{M}$ can be interpreted as the shadow of a set at infinity produced by a "light" source at p). For a fixed constant $R > 0$, there exists a constant $c > 1$ such that

$$\frac{1}{c} e^{-h(g)d_g(p,x)} \leq \mu_p^g(pr_p^g B(x,R)) \leq c e^{-h(g)d_g(p,x)} \; ,$$

where $B(x,R)$ is a ball of radius R about x for a fixed metric on M equivalent to g. Now we will prove the first part of Theorem 1.2.

Theorem 3.4. *Let (M, g_0) be a compact manifold of negative sectional curvature. Then for every metric g on M*

$$h(g) \geq \frac{1}{I_{\mu_{BM}}(g_0, g)} h(g_0) \; .$$

Proof. Since the geodesic flow of g_0 is ergodic with respect to the Bowen-Margulis measure μ_{BM}, we have

$$\lim_{t \to \infty} \frac{a(v,t)}{t} = I_{\mu_{BM}}(g_0, g)$$

for μ_{BM} almost all $v \in (SM)_{g_0}$. We remark that the triangle inequality and the fact that the two metrics are equivalent imply the existence of the limit for all vectors on a fixed weak stable manifold if it exists for one vector.

The measure μ_{BM} can be realized as

$$d\mu_{BM} = e^{h(g_0)\beta_p(\xi,\eta)} d\mu_p^{g_0}(\xi) d\mu_p^{g_0}(\eta) dt \; ,$$

where $(\xi, \eta) \in (\tilde{M}(\infty) \times \tilde{M}(\infty)) \setminus diagonal$, $\mu_p^{g_0}$ is the normalized Patterson-Sullivan measure with respect to g_0 and dt denotes the integration along the flow lines. Furthermore, $\beta_p(\xi, \eta)$ is twice the Gromov product and is given by

$$\beta_p(\xi, \eta) = \lim_{t \to \infty} (2t - d(c_{p,\xi}(t), c_{p,\eta}(t))) \; ,$$

where $c_{p,\xi}$ is the geodesic with $c_{p,\xi}(0) = p$ and $c_{p,\xi}(\infty) = \xi$. If $v_p(\xi) = G_p^{-1}(g_0)(\xi) = \dot{c}_{p,\xi}(0) \in S_pM$ is the initial vector of the geodesic $c_{p,\xi}$, the above remark and the structure of μ_{BM} imply that

$$\lim_{t \to \infty} \frac{a(v_p(\xi), t)}{t} = I_{\mu_{BM}}(g_o, g)$$

for $\mu_p^{g_0}$ almost all $\xi \in \tilde{M}(\infty)$. For all $\epsilon > 0$ and $T > 0$, we define the set

$$A_p^{T,\epsilon} = \{\xi \in \tilde{M}(\infty) \mid \left| \frac{a(v_p(\xi), t)}{t} - I_{\mu_{BM}}(g_0, g) \right| \leq \epsilon,\ t \geq T\}\ .$$

From the above statement follows that for a fixed $d \in (0,1)$ and all $\epsilon > 0$, there exists $T > 0$ such that $\mu_p^{g_0}(A_p^{T,\epsilon}) \geq d$. For $t \geq T$ consider the subset $\{c_{p,\xi}(t) \mid \xi \in A_p^{T,\epsilon}\}$ of the geodesic sphere $S_t(p)$ of radius t about p.

Let $\{B(x_i, R) \mid i \in I\}$ be a covering of this subset by balls of fixed radius $R > 0$ such that $x_i \in S_t(p)$ and $B(x_i, R/4)$ are pairwise disjoint. Then, by the local behavior of $\mu_p^{g_0}$, there exists a constant $c > 1$ independent of t such that

$$\frac{1}{c} e^{-h(g_0)t} \leq \mu_p(pr_p B(x_i, R)) \leq c e^{-h(g_0)t}\ .$$

Obviously, $A_p^{T,\epsilon}$ is contained in $\bigcup_{i \in I} pr_p B(x_i, R)$ and, therefore,

$$* \qquad d \leq \mu_p\left(\bigcup_{i \in I} pr_p B(x_i, R)\right) \leq \sum_{i \in I} \mu_p(pr_p B(x_i, R)) \leq c \cdot card(I) e^{-h(g_0)t}.$$

From the definition of the set $A_p^{T,\epsilon}$, the triangle inequality, and since the metrics g and g_0 are equivalent, all the balls $B(x_i, R/4)$ are contained in the geodesic ball $B^g(p, c_1 + t(I_{\mu_{BM}}(g_0, g) + \epsilon))$ with respect to the metric g for a t-independent constant c_1. Now we can estimate the volume of the large B^g-ball by the volume of the disjoint union $\bigcup_{i \in I} B(x_i, R/4)$ and

obtain, using $*$

$$d_2 \cdot e^{h(g_0)t} \quad \leq \sum_{i \in I} vol_g(B(x_i, R/4))$$

$$\leq vol_g\, B^g(p, t(I_{\mu_{BM}}(g_0, g) + \epsilon) + c_1)\ ,$$

where d_2 is another positive and t-independent constant. This concludes the proof of Theorem 3.4, by the same argument as in Theorem 1.1. $\square$

To prove the second part of Theorem 1.2, we need some preparations.

Lemma 3.5. *Let (X, μ) be a probability space, $T : X \to X$ a μ-preserving ergodic transformation, and $a : X \times \mathbf{N} \to \mathbf{R}$ a measurable subadditive cocycle. Assume that there exists a constant $b > 0$ such that*

$$a(x, n + l) \leq a(x, n) + a(T^n x, l) \leq b + a(x, n + l) .$$

Then there is a constant $a \in \mathbf{R}$ such that for μ-almost all $x \in X$ we can find a sequence $n_j \to \infty$ such that

$$|a(x, n_j) - n_j a| \leq c = 2b .$$

Proof. We have the impression that Lemma 3.5 is known [3] but since we did not find a reference for the proof, we include a proof in the appendix. $\square$

Let (M, g_0) be a compact manifold of negative curvature and g any other metric. Consider two points $p, q \in \tilde{M}$ and the geodesic segment $c_{g_0} : I \to \tilde{M}$ with respect to g_0 connecting p and q. Then, by the Lemma of M. Morse, there exists a constant $r > 0$ not depending on the choice of p and q and a g-geodesic segment $c_g : J \to \tilde{M}$ such that the Hausdorff distances of their images is bounded by r. Therefore, the subadditive cocycle $a(v, t) = d_g(\pi(v), \pi\Phi_{g_0}^t v)$ has the property $a(v, t_1) + a(\Phi_{g_0}^{t_1} v, t_2) \leq b + a(v, t_1 + t_2)$ stated in the lemma with $b = 2r$. This yields the following corollary.

Corollary 3.6. *If $p \in \tilde{M}$ and $\mu_p^{g_0}$ denotes the Patterson-Sullivan measure with respect to p and the metric g_0, there exists a constant L such that for $\mu_p^{g_0}$ almost all $\xi \in \tilde{M}$ there is a sequence $t_n \to \infty$ such that*

$$|d_g(p, \pi\Phi_{g_0}^{t_n} v_p(\xi)) - I_{\mu_{BM}}(g_0, g)t_n| \leq L .$$

Proof. The proof is a consequence of Lemma 3.5 and arguments similar to those given in the proof of Theorem 3.4. $\square$

Proposition 3.7. *Let (M, g_0) be a compact manifold of negative curvature and g another negatively curved metric. Then*

$$h(g) = \frac{1}{I_{\mu_{BM}}(g_0, g)} h(g_0)$$

implies that the Patterson-Sullivan measures of g_0 and g are equivalent.

Proof. Choose a point $p \in \tilde{M}$. We will find for $\mu_p^{g_0}$ almost all ξ a sequence $y_n = \pi\Phi_{g_0}^{t_n} v_p(\xi)$ such that

$$|d_g(p, y_n) - I_{\mu_{BM}}(g_0, g)t_n| \leq L .$$

For a fixed constant $R > 0$, we obtain from the local property of the Patterson-Sullivan measure μ_p^g

$$\frac{1}{c_1} e^{-h(g)d_g(p,y_n)} \leq \mu_p^g(pr_p^g(B(y_n, R))) \leq c_1 e^{-h(g)d_g(p,y_n)}$$

for a constant $c_1 > 1$.

Since $\frac{h(g_0)}{h(g)} = I_{\mu_{BM}}(g, g_0)$, there is a further constant $c_2 > 1$ such that

$$\frac{1}{c_2} e^{-h(g_0)d_{g_0}(p,y_n)} \leq \mu_p^g(pr_p^g(B(y_n, R))) \leq c_2 e^{-h(g_0)d_{g_0}(p,y_n)} .$$

Choose the constant R to be bigger than the maximal Hausdorff distance r of corresponding geodesic rays with respect to g_0 and g. Then

$$pr_p^{g_0}(B(y_n, R+r)) \supset pr_p^g(B(y_n, R)) \supset pr_p^{g_0}(B(y_n, R-r)) .$$

Therefore,

$$c_3 e^{-h(g_0)d_{g_0}(p,y_n)} \geq \mu_p^{g_0}(pr_p^{g_0}(B(y_n, R+r))) \geq \mu_p^{g_0} pr_p^g(B(y_n, R))$$

and

$$\mu_p^{g_0}(pr_p^g(B(y_n, R))) \geq \mu_p^{g_0}(pr_p^{g_0}(B(y_n, R-r))) \geq c_4 e^{-h(g_0)d_{g_0}(p,y_n)}$$

for suitable constants c_3 and c_4.

Hence,

$$\frac{1}{c_5} \leq \frac{\mu_p^g(pr_p^g(B(y_n, R)))}{\mu_p^{g_0}(pr_p^g(B(y_n, R)))} \leq c_5 .$$

Since $pr_p^g(B(y_n, R)) \to \xi$ for $n \to \infty$, the measures μ_p^g and $\mu_p^{g_0}$ are equivalent. $\square$

The following proposition can be derived from a slightly modified version of arguments given in [10]. For the sake of completeness and the benefit of the reader, we would like to include a sketch of the proof.

Proposition 3.8. *If the Patterson-Sullivan measures of a pair of Riemannian metrics g_0 and g are equivalent, there is up to scaling, a time preserving conjugacy of the geodesic flows with respect to g_0 and g.*

Proof. Let v be a vector in the unit tangent bundle of g_0 and $\eta = \Phi_{g_0}^{-\infty}v$, $\xi = \Phi_{g_0}^{+\infty}v$ be the endpoints of the corresponding g_0-geodesic. Then there exists a unique vector w in the unit tangent bundle of g such that the corresponding g-geodesic is joining η and ξ, i.e. $\Phi_g^{-\infty}w = \eta$, $\Phi_g^{+\infty}w = \xi$ and that the measures $\mu_{\pi v}^{g_0}$ and $\mu_{\pi w}^g$ have the same density at $\xi \in \tilde{M}(\infty)$, i.e. $\frac{d\mu_{\pi v}^{g_0}}{d\mu_{\pi w}^g}(\xi) = 1$.

The map $F : (S\tilde{M})_{g_0} \to (S\tilde{M})_g$ defined by $F(v) = w$ is up to a scaling a time preserving conjugacy as the following argument shows. Let v_1, v_2 be vectors of the tangent bundle of g_0 such that $v_2 = \Phi_{g_0}^{t_0} v_1$, $\xi = \Phi^\infty v_1$. We have

$$e^{h(g_0)t_0} = \frac{d\mu_{\pi v_2}^{g_0}}{d\mu_{\pi v_1}^g}(\xi) = \frac{d\mu_{\pi F v_2}^{g_0}}{d\mu_{\pi F v_1}^g}(\xi) = e^{h(g)t}$$

if $Fv_2 = \Phi_g^t Fv_1$. Therefore, $t_0 = \frac{h(g)}{h(g_0)}t$ and we get

$$F(\Phi_{g_0}^{\frac{h(g)}{h(g_0)}t} v_1) = \Phi_g^t Fv_1 \ .$$

The Γ-equivariance of F is an easy consequence of the Γ-equivariance of the measures. $\square$

The second part of Theorem 1.2 is now a consequence of the Propositions 3.7 and 3.8.

4. Further applications

The following results are a combination of applications of Theorem 1.1 and the first part of Theorem 1.2, and are consequences of the fact that the geodesic stretch can be estimated from above by the energy (Lemma 2.4 and [5]).

Theorem 4.1. *Let (M, g_0) be a compact n-dimensional manifold with no conjugate points and ergodic geodesic flow. Then for all other metrics g on M*

$$h(g)^2 \geq \frac{\mathrm{vol}_{g_0}(M)}{\frac{1}{n}\int_M tr_{g_0} g \, d\,\mathrm{vol}_{g_0}} h_{\mu_L}(g_0)^2 \ .$$

If g is conformally equivalent to g_0

$$h(g) \geq \left(\frac{\mathrm{vol}_{g_0}(M)}{\mathrm{vol}_g(M)} \right)^{\frac{1}{n}} h_{\mu_L}(g_0) \ .$$

Moreover, both inequalities are strict, unless g and g_0 are identical up to a constant and $h_{\mu_L}(g_0) = h(g_0)$.

Proof. From Lemma 2.4, it follows that

$$I_{\mu_L}(g_0, g)^2 \leq E_{\mu_L}(g_o, g) = \int_{(SM)_{g_0}} g(v, v) d\mu_L = \frac{1}{\mathrm{vol}_{g_0} M} \int_M \frac{tr_{g_0} g}{n} d\,\mathrm{vol}_{g_0} \ .$$

Together with Theorem 1.1, this implies:

$$h(g)^2 \geq \frac{1}{I_{\mu_L}(g_0,g)^2} h_{\mu_L}(g_0)^2 \geq \frac{\mathrm{vol}_{g_0} M}{\frac{1}{n} \int\limits_M tr_{g_0} g \, d\,\mathrm{vol}_{g_0}} h_{\mu_L}(g_0)^2 \;.$$

By Lemma 2.4, equality holds if and only if $g = eg_0$ for a positive constant e. But then we have $h(g)^2 = \frac{h(g_0)^2}{e}$ and, therefore, it follows that

$$\frac{h(g_0)^2}{e} = \frac{\mathrm{vol}_{g_0}(M)}{e \; \mathrm{vol}_{g_0}(M)} h_{\mu_L}(g_0)^2 = \frac{h_{\mu_L}(g_o)^2}{e} \;.$$

Let g be conformally equivalent to g_0, i.e. $g = \varphi g_0$ for a function $\varphi : M \to \mathbf{R}$. Then it follows, using Jensen's inequality,

$$\frac{1}{n} \frac{\int\limits_M tr_{g_0} g \, d\,\mathrm{vol}_{g_0}}{\mathrm{vol}_{g_0} M} = \frac{\int\limits_M \varphi \, d\,\mathrm{vol}_{g_0}}{\mathrm{vol}_{g_0} M} \leq \left(\frac{\int\limits_M \varphi^{\frac{n}{2}} d\,\mathrm{vol}_{g_0}}{\mathrm{vol}_{g_0} M} \right)^{\frac{2}{n}} = \left(\frac{\mathrm{vol}_g M}{\mathrm{vol}_{g_0} M} \right)^{\frac{2}{n}} \;.$$

□

Remark. Theorem 4.1 generalizes a result by A. Katok in two ways [12, Theorem A]. We are only assuming that our reference metric g_0 has no conjugate points. Instead of comparing the measure entropy of g_0 with the topological entropy of g we compare it with the smaller volume entropy.

Theorem 4.2. *Let (M, g_0) be a compact manifold of negative curvature. Then for any other metric g on M*

$$h(g)^2 \geq \frac{1}{\int\limits_{(SM)_{g_0}} g(v,v) \, d\mu_{BM}(g_0)} h(g_0)^2 \;,$$

where equality holds if and only if g and g_0 are identical up to a constant.

Proof. The proof follows immediately from the first part of Theorem 1.2, Lemma 2.4, and the fact that the Bowen-Margulis measure is positive on open sets [18]. □

Corollary 4.3. *Let (M, g_0) be a compact manifold of negative curvature. If g is a metric with $Ric_g \geq Ric_{g_0}$, we obtain:*

$$\int\limits_{(SM)_{g_0}} g(v,v) \, d\mu_{BM}(g_0) \geq 1 \;.$$

Furthermore, the second inequality is strict unless $g_0 = g$.

Proof. This follows from the Theorem 4.2 and Bishop's inequality. □

Let $R_{\text{vol}}(M)$ be the space of Riemannian metrics with negative sectional curvature and volume one. In [14], we proved, that g_0 is a critical point of $h : R_{\text{vol}}(M) \to M$ if and only if

$$\int\limits_{(SM)_{g_0}} \xi(v,v) \, d\mu_{BM}(g_0) = \int\limits_{(SM)_{g_0}} \xi(v,v) \, d\mu_L(g_0)$$

for all symmetric two-forms ξ on M.

Corollary 4.4. *Let g_0 be a critical point of $h : R_{\text{vol}}(M) \to \mathbf{R}$. Then for all metrics g on M, it follows that*

$$h(g)^2 \geq \frac{1}{\frac{1}{n}\int\limits_{M} tr_{g_0} g \, d\,\text{vol}_{g_0}} h(g_0)^2 \ ,$$

where equality holds if and only if the metrics are identical up to a constant.

Proof. The proof is an immediate consequence of Theorem 4.2. $\square$

For each metric $g \in R_{\text{vol}}(M)$, let $\text{Conf}(g)$ be the space of Riemannian metrics conformally equivalent to g. Then $h : R_{\text{vol}}(M) \cap \text{Conf}(g) \to \mathbf{R}$ has a critical point $g_0 \in R_{\text{vol}}(M) \cap \text{Conf}(g)$ if and only if

$$\int\limits_{(SM)_{g_0}} f \circ \pi(v) \, d\mu_{BM}(g_0) = \int\limits_{(SM)_{g_0}} f \circ \pi(v) \, d\mu_L(g_0)$$

for all continuous functions $f : M \to \mathbf{R}$, where $\pi : SM_{g_0} \to M$ is the canonical projection [14].

This leads to the proof of Corollary 1.5 stated in the Introduction.

Corollary 4.5. *Let g_0 be a critical point of $h : R_{\text{vol}}(M) \cap \text{Conf}(g) \to \mathbf{R}$. Then for all metrics $\tilde{g} \in \text{Conf}(g)$ with $vol_{\tilde{g}}(M) = 1$, it follows that*

$$h(\tilde{g}) \geq h(g_0) \ ,$$

where equality holds if and only if $\tilde{g} = g_0$. In particular, a critical point is an absolute minimum in the conformal class of metrics with nomalized volume and, therefore, the critical point is uniquely determined if it exists.

Proof. Let $\tilde{g}$ be a metric conformally equivalent to g_0, i.e., there exists a positive function $\varphi : M \to \mathbf{R}$ such that $\tilde{g} = \varphi g_0$. If g_0 is a critical point of $h : R_{\text{vol}}(M) \cap \text{Conf}(g) \to \mathbf{R}$, we have

$$\int\limits_{(SM)_{g_0}} \tilde{g}(v,v) d\mu_{BM}(g_0) = \int\limits_{(SM)_{g_0}} \varphi \cdot \pi(v) d\mu_L$$

$$= \int\limits_{M} \varphi d\,\text{vol}_{g_0} \leq \left(\int\limits_{M} \varphi^{n/2} d\,\text{vol}_{g_0} \right)^{2/n} = 1$$

and, therefore, the corollary is a consequence of Theorem 4.2. $\quad\square$

Remark. This shows that in each conformal class there can be only one metric for which the Bowen-Margulis measure and Liouville measure projected onto M coincide (see also [22]).

5. Appendix

Proof of Lemma 3.5. Since $a(x, n)$ is a subadditive cocycle,

$$\lim_{n \to \infty} \frac{a(x, n)}{n} = a \text{ for almost all } x \in X.$$

Without loss of generality, we can assume $a = 0$ (otherwise consider $a(x, n) - na$). For $j \in \mathbf{N}$, let

$$A = A_j = \{x \in X \mid a(x, n) \geq c \text{ for all } n \geq j\}.$$

It follows that $\mu(A) = 0$. If not, define $n_A(x) = \inf\{n \geq j \mid T^n(x) \in A\}$ and consider the map $T_A : A \to A$ given by $T_A(x) = T^{n_A(x)}(x)$. If

$$N_k(x) = n_A(x) + n_A(Tx) + \ldots + n_A(T_A^{k-1}x),$$

one obtains that $\operatorname{card}\{1 \leq l \leq N_k(x) \mid T^l x \in A\} \leq kj$. Hence,

$$\liminf_{k \to \infty} \frac{kj}{N_k(x)} \leq \lim_{k \to \infty} \frac{1}{N_k(x)} \sum_{l=0}^{N_k(x)} \chi_A(T^l x) = \mu(A) .$$

On the other hand, we have, for all $x \in A$, $\quad a(x, N_k(x)) \geq kb$. This is easily obtained by induction since $a(x, n_A(x)) \geq c = 2b$ and

$$\begin{aligned} a(x, N_k(x)) =& a(x, N_{k-1}(x)) + a(T_A^{k-1}x, n_A(T_A^{k-1}x) \\ \geq& -b + (k-1)b + 2b = kb . \end{aligned}$$

But then $\lim_{k \to \infty} \frac{a(x, N_k(x))}{N_k(x)} \geq \liminf_{k \to \infty} \frac{kb}{N_k(x)} > 0$, which contradicts the assumption.

In a similar way, one proves that

$$A_j = \{x \in X \mid a(x, n) \leq -c \text{ for all } n \geq j\}$$

has measure 0. Let B be the set of all $x \in X$ which does not have the desired property, i.e.

$$B = \{x \in X \mid \exists n_0(x) \in \mathbf{N} \text{ such that } |a(x, n)| \geq c \; \forall n \geq n_0(x)\}.$$

If we define $C_l = \{x \in X \mid |a(x, n)| \geq c, \ n \geq l\}$, we can write $B = \cup C_l$. Since $\mu(C_l) = 0$, we have that $\mu(B) = 0$. This implies the assertion of the lemma. $\square$

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