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REMARKS ON RATIONAL POINTS OF VARIETIES WHOSE COTANGENT BUNDLES ARE GENERATED BY GLOBAL SECTIONS

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Abstract. In this short note, we will give several remarks on rational points of varieties whose cotangent bundles are generated by global sections. For example, we will show that if the sheaf of differentials $\Omega^1_{X/k}$ of a projective variety X over a number field k is ample and generated by global sections, then the set of k-rational points of X is finite.

0. Introduction

In [Fa1] and [Fa2], G. Faltings proved the following theorem.

Theorem A. *Let A be an abelian variety over a number field k and X a subvariety of A. Then there are a finite number of translated abelian subvarieties $B_1, \dots, B_n$ over k such that $B_i \subset X$ and the closure $\overline{X(k)}$ of $X(k)$ in X is contained in $\bigcup_i B_i$.*

In this short note, as applications of the above Faltings' theorem, we will give several remarks on rational points of varieties whose cotangent bundles are generated by global sections. The first remark is the following theorem which is a slight generalization of Theorem A.

Theorem B. *Let X be a projective variety over a number field k, A an abelian variety over k, and $\alpha : X \to A$ a morphism over k. If $\alpha^*(\Omega^1_{A/k}) \to \Omega^1_{X/k}$ is surjective, then every irreducible component of $\overline{X(k)}$ is geometrically irreducible and isomorphic to an abelian variety.*

As a corollary of Theorem B, we have the following.

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Corollary C. *Let X be a connected smooth projective variety over a number field k. If the cotangent bundle $\Omega^1_{X/k}$ is generated by global sections, then every irreducible component of $\overline{X(k)}$ is geometrically irreducible and isomorphic to an abelian variety.*

Thus, if we assume further that $\Omega^1_{X/k}$ is ample in Corollary C, then X contains no abelian variety. Hence, the set of k-rational points of X is finite. This is a partial answer to the following conjecture (due to S. Lang) under the additional assumption that "$\Omega^1_{X/k}$ is generated by global sections."

Conjecture D (cf. [La]**).** Let X be a smooth projective variety over a number field k. If the cotangent bundle $\Omega^1_{X/k}$ is ample, then $X(k)$ is finite.

Our partial answer holds even on a singular variety. Namely, we have

Theorem E. *Let X be a projective variety over a number field k. If the sheaf of differentials $\Omega^1_{X/k}$ of X over k is ample and generated by global sections, then the set of k-rational points of X is finite, where ampleness of $\Omega^1_{X/k}$ means that the tautological line bundle on* $\mathrm{Proj}(\bigoplus_{d\geq 0}\mathrm{Sym}^d(\Omega^1_{X/k}))$ *is ample.*

1. Non-denseness

In this section, we will give a powerful and simple non-denseness theorem.

First of all, we fix notation. Let X be a connected smooth projective variety over a field k of characteristic zero. Assume $X(k) \neq \emptyset$ and fix $x_0 \in X(k)$. Let $\mathrm{Alb}_{X/k}$ be the dual abelian variety of the Picard variety $\mathrm{Pic}^0_{X/k}$ of X. Let Q be the universal line bundle on $X \times \mathrm{Pic}^0_{X/k}$ with $Q|_{\{x_0\}\times \mathrm{Pic}^0_{X/k}} = \mathcal{O}_{\mathrm{Pic}^0_{X/k}}$. Q gives rise to an X-valued point of $\mathrm{Alb}_{X/k}$, that is, a morphism $\alpha : X \to \mathrm{Alb}_X$. Actually, α is given by $\alpha(x) = Q|_{\{x\}\times \mathrm{Pic}^0_{X/k}}$ for $x \in X$. In particular, $\alpha(x_0) = 0$. The abelian variety $\mathrm{Alb}_{X/k}$ is called the Albanese variety of X over k and α is called the Albanese map. Moreover, the dimension of $\mathrm{Alb}_{X/k}$, denoted by $q(X)$, is called the irregularity of X. Further, we denote by $d(X)$ the dimension of $\alpha(X)$, i.e. $d(X) = \dim \alpha(X)$. In general, $d(X) \leq q(X)$ and $d(X) \leq \dim X$. It is well known that $q(X) = \dim_k H^0(X, \Omega^1_{X/k}) = \dim_k H^1(X, \mathcal{O}_X)$.

Let Y be a geometrically irreducible projective variety over k and $\mu : X \to Y$ a desingularization of Y. It is easy to see that $q(X)$ and $d(X)$ do not depend on the choice of X. Thus, we can define $q(Y)$ and $d(Y)$ by $q(X)$ and $d(X)$ respectively.

As a consequence of Theorem A, we have the following non-denseness theorem.

Theorem 1.1. *Let X be a geometrically irreducible projective variety over a number field k. If $d(X) < q(X)$, then the closure of $X(k)$ in X is a proper closed subset.*

Proof. Considering a desingularization of X, we may assume that X is smooth over k. Moreover, we may assume $X(k) \neq \emptyset$. Let $\alpha : X \to \mathrm{Alb}_{X/k}$ be the Albanese map. Since $q(X) > d(X)$, α is not surjective. If $X(k)$ is dense in X, then so is $\alpha(X)(k)$ in $\alpha(X)$. Thus, by Theorem A, $\alpha(X)$ is an abelian subvariety of $\mathrm{Alb}_{X/k}$. This is a contradiction because $\alpha(X)(\bar{k})$ generates $\mathrm{Alb}_{X/k}(\bar{k})$. $\square$

2. Lemmas

In this section, we will prepare three lemmas for the proof of Theorem E.

Let X be a projective scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. We denote $\mathrm{Proj}(\bigoplus_{d\geq 0} \mathrm{Sym}^d(\mathcal{F}))$ by $\mathbb{P}(\mathcal{F})$. Let $\pi_F : \mathbb{P}(\mathcal{F}) \to X$ be the natural morphism and $\mathcal{O}_{\mathcal{F}}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{F})$. We say $\mathcal{F}$ is *ample* if $\mathcal{O}_{\mathcal{F}}(1)$ is ample.

Lemma 2.1. *Let X be a projective scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

(1) *Let $\mathcal{F} \to \mathcal{G}$ be a surjective homomorphism of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is ample, then so is $\mathcal{G}$.*
(2) *Let $f : Y \to X$ be a finite morphism of projective schemes. If $\mathcal{F}$ is ample, then so is $f^*(\mathcal{F})$.*

Proof. (1) Since $\mathcal{F} \to \mathcal{G}$ is surjective, $\mathbb{P}(\mathcal{G})$ is a subscheme of $\mathbb{P}(\mathcal{F})$ and $\mathcal{O}_{\mathcal{G}}(1) = \mathcal{O}_{\mathcal{F}}(1)|_{\mathbb{P}(\mathcal{G})}$. Thus, $\mathcal{O}_{\mathcal{G}}(1)$ is ample.

(2) Let $f' : \mathbb{P}(f^*(\mathcal{F})) \to \mathbb{P}(\mathcal{F})$ be the induced morphism. Since

$$\mathcal{O}_{f^*(\mathcal{F})}(1) = {f'}^*(\mathcal{O}_{\mathcal{F}}(1))$$

and f' is finite, $\mathcal{O}_{f^*(\mathcal{F})}(1)$ is ample. $\square$

Lemma 2.2. *Let X be a scheme over a field k and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

(1) *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a surjective homomorphism of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is generated by global sections, then so is $\mathcal{G}$.*
(2) *Let $f : Y \to X$ be a morphism of schemes over k. If $\mathcal{F}$ is generated by global sections, then so is $f^*(\mathcal{F})$.*
(3) *If $\mathcal{F}$ is generated by global sections, then so is $\mathcal{O}_{\mathcal{F}}(1)$.*

Proof. (1) Let us consider the following commutative diagram:

$$\begin{array}{ccc} H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_X & \longrightarrow & H^0(X,\mathcal{G}) \otimes_k \mathcal{O}_X \\ \big\downarrow & & \big\downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

Since $H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{G}$ are surjective, so is

$$H^0(X,\mathcal{G}) \otimes_k \mathcal{O}_X \to \mathcal{G}.$$

(2) Since $H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F}$ is surjective, we have a surjection $H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_Y \to f^*(\mathcal{F})$. Here we consider the following diagram:

$$\begin{array}{ccc} H^0(X,\mathcal{F}) \otimes_k \mathcal{O}_Y & \longrightarrow & f^*(\mathcal{F}) \\ \big\downarrow & & \big\| \\ H^0(Y,f^*(\mathcal{F})) \otimes_k \mathcal{O}_Y & \longrightarrow & f^*(\mathcal{F}) \end{array}$$

Thus, $H^0(Y,f^*(\mathcal{F})) \otimes_k \mathcal{O}_Y \longrightarrow f^*(\mathcal{F})$ is surjective.

(3) By (2), $\pi_{\mathcal{F}}^*(\mathcal{F})$ is generated by global sections. On the other hand, there is the natural surjective homomorphism $\pi_{\mathcal{F}}^*(\mathcal{F}) \to \mathcal{O}_{\mathcal{F}}(1)$. Thus, by (1), $\mathcal{O}_{\mathcal{F}}(1)$ is generated by global sections. □

Lemma 2.3. *Let Y be a geometrically irreducible projective variety over a field k of characteristic zero. If $\dim Y \geq 1$ and $\Omega^1_{Y/k}$ is ample and generated by global sections, then $q(Y) \geq 2\dim Y$.*

Proof. Let $\mu : Y' \to Y$ be a desingularization of Y such that

$$\mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor}$$

is locally free. We set

$$P = \mathbb{P}(\mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor}) \quad \text{and} \quad L = \mathcal{O}_{\mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor}}(1).$$

By Lemma 2.2, $\mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor}$ is generated by global sections. Thus, so is L by Lemma 2.2. Hence we have a morphism $\phi_{|L|} : P \to \mathbb{P}^N$ with $\phi_{|L|}^*(\mathcal{O}_{\mathbb{P}^N}(1)) = L$, where $N = \dim_k H^0(P,L) - 1$. Let ν be the composition of maps $P \hookrightarrow \mathbb{P}(\mu^*(\Omega^1_{Y/k})) \to \mathbb{P}(\Omega^1_{Y/k})$. Then, $L = \nu^*(\mathcal{O}_{\Omega^1_{Y/k}}(1))$. Since ν gives a birational morphism from P to the image $\nu(P)$ and $\mathcal{O}_{\Omega^1_{Y/k}}(1)$

is ample, L is big, i.e. $(L^{\dim P}) > 0$. It follows that $\phi_{|L|}$ is generically finite. Therefore,

$$\dim P = \dim \phi_{|L|}(P) \leq \dim \mathbb{P}^N = \dim_k H^0(P, L) - 1,$$

which implies

$$\begin{aligned} \dim_k H^0(Y', \mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor}) &= \dim_k H^0(P, L) \\ &\geq \dim P + 1 = 2 \dim Y. \end{aligned}$$

On the other hand, the natural homomorphism $\mu^*(\Omega^1_{Y/k}) \to \Omega^1_{Y'/k}$ induces an injection $\mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor} \to \Omega^1_{Y'/k}$. Hence,

$$\dim_k H^0(Y', \Omega^1_{Y'/k}) \geq \dim_k H^0(Y', \mu^*(\Omega^1_{Y/k})/\mu^*(\Omega^1_{Y/k})_{tor}).$$

Therefore, $\dim_k H^0(Y', \Omega^1_{Y'/k}) \geq 2 \dim Y$, which says $q(Y) \geq 2 \dim Y$. $\square$

3. Proofs of Theorem B, Corollary C and Theorem E

3.1. Proof of Theorem B. Let Y be an irreducible component of $\overline{X(k)}$. Since $Y(k)$ is dense in Y, Y is geometrically irreducible. For, let $Y_{\bar{k}} = \Gamma_1 \cup \cdots \cup \Gamma_r$ be the irreducible decomposition of $Y_{\bar{k}}$. Then, it is easy to see $Y(k) \subset \bigcap_i \Gamma_i$. Thus, if $r \geq 2$, we have a contradiction.

We set $B = \alpha(Y)$. Since $Y(k)$ is dense in Y, so is $B(k)$ in B. Thus, by Faltings' theorem (Theorem A), B is a translated abelian subvariety of A. Let us consider the following diagram:

$$\begin{tikzcd} \alpha^*(\Omega^1_{A/k})\big|_Y \arrow[r, "\beta"] \arrow[d, "\delta"'] & \Omega^1_{X/k}\big|_Y \arrow[d, "\epsilon"] \\ (\alpha|_Y)^*(\Omega^1_{B/k}) \arrow[r, "\gamma"'] & \Omega^1_{Y/k} \end{tikzcd}$$

Since β, δ and ϵ are surjective, so is γ. On the other hand, $\operatorname{rank} \Omega^1_{B/k} \leq \operatorname{rank} \Omega^1_{Y/k}$ and $\Omega^1_{B/k}$ is locally free. It follows that γ gives an isomorphism between $(\alpha|_Y)^*(\Omega^1_{B/k})$ and $\Omega^1_{Y/k}$. Thus, Y is smooth over k and $\alpha|_Y$ is etale. Therefore, by a theorem due to S. Lang (cf. [Mu, Chapter IV, 18]), Y is an abelian variety. $\square$

3.2. Proof of Corollary C. Let us consider the Albanese map

$$\alpha : X \to \mathrm{Alb}_{X/k}.$$

Since $H^0(X, \Omega^1_{X/k}) \otimes \mathcal{O}_X \to \Omega^1_{X/k}$ is surjective and

$$H^0(X, \Omega^1_{X/k}) \otimes \mathcal{O}_{\mathrm{Alb}_{X/k}} \simeq \Omega^1_{\mathrm{Alb}_{X/k}},$$

$\alpha^*(\Omega_{\mathrm{Alb}_{X/k}}) \to \Omega^1_{X/k}$ is surjective. Therefore, we can apply Theorem B to get our corollary. $\square$

3.3. Proof of Theorem E. Assume that $\overline{X(k)}$ has an irreducible component Y with $\dim Y \geq 1$. In the same way as in the proof of Theorem B, Y is geometrically irreducible. By (2) of Lemma 2.1 and (2) of Lemma 2.2, $\left.\Omega^1_{X/k}\right|_Y$ is ample and generated by global sections. Here there is the natural surjection $\left.\Omega^1_{X/k}\right|_Y \to \Omega^1_{Y/k}$. Thus, by virtue of (1) of Lemma 2.1 and (1) of Lemma 2.2, $\Omega^1_{Y/k}$ is ample and generated by global sections. Therefore, by Lemma 2.3, $q(Y) \geq 2\dim Y$. Hence, by Theorem 1.1, $Y(k)$ is not dense in Y. This is a contradiction. □

3.4. Remark. Over a function field, a positive answer to Conjecture D was obtained by Noguchi (cf. [No] and [Mo]). In this case, the set of rational points is however not finite in general. They are concentrated on a proper closed subset.

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