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FOUR-MANIFOLDS WITHOUT SYMPLECTIC STRUCTURES BUT WITH NONTRIVIAL SEIBERG-WITTEN INVARIANTS

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It was proved in [9] that every closed symplectic four-manifold has a nontrivial Seiberg-Witten invariant. Combining this result with the arguments of [5], we show here that the converse is false. In fact, there are closed oriented four-manifolds with nontrivial Seiberg-Witten (and Donaldson [5]) invariants which do not admit symplectic structures. For some examples the Seiberg-Witten invariants are the same as those of symplectic manifolds, whereas for others they do not satisfy the structure results for the invariants of symplectic manifolds proved in [10].

Our examples are connected sums in which one summand has a negative definite intersection form and a nontrivial fundamental group. Conversely, there are symplectic four-manifolds whose fundamental group splits as a free product [2], but we show that in most cases this splitting is not realized by any decomposition of the manifold as a smooth connected sum. This provides large classes of counterexamples to the four-dimensional Kneser conjecture, which was disproved only recently [7] and only for a few specific groups.

We begin with an observation about the possible connected sum decompositions of closed symplectic four-manifolds. This was proved in [5] for the Kähler case.

Proposition 1. *Let X be a closed symplectic 4-manifold which decomposes as a smooth connected sum. Then one of the summands, call it N, has a negative definite intersection form and is algebraically simply connected, i.e. its fundamental group has no nontrivial finite quotients.*

Proof. As noted in [9], the nontriviality of the Seiberg-Witten invariants of X implies that in any smooth connected sum decomposition of X one of the summands has a negative definite intersection form.

Let $X = Y\#N$ be such a decomposition. If $\pi_1(N)$ has a finite quotient G of order $d \geq 1$, then G is also a quotient of $\pi_1(X) = \pi_1(Y) * \pi_1(N)$

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and X has a connected d-sheeted covering X' which decomposes as a smooth connected sum $X' = dY\#N'$, where N' is a connected d-sheeted covering of N. We have $b_2^+(Y) = b_2^+(X) > 0$ because X is symplectic. The symplectic structure of X lifts to a symplectic structure on X', so that applying the result of [9] to X' implies $d = 1$. □

Starting with an arbitrary symplectic four-manifold Y, we can blow up a point to get another symplectic manifold $X = Y\#\overline{\mathbb{C}P^2}$. Strengthening the conjecture made in [5], it is possible that this is the only kind of connected sum decomposition a symplectic four-manifold allows:

Conjecture 1. *If $X = Y\#N$ is a decomposition of a closed symplectic four-manifold as a smooth connected sum, then Y is symplectic and X is obtained from Y by blowing up some number of points.*

The last author [11] has recently made progress in this direction.

Next we show that the nontriviality of Seiberg-Witten invariants is preserved by forming the connected sum with (certain) negative definite four-manifolds. The analogous result for Donaldson invariants was proved in [5].

Proposition 2. *Let Y and N be a closed oriented 4-manifolds. If Y has a nontrivial Seiberg-Witten invariant and if $b_1(N) = b_2^+(N) = 0$, then $X = Y\#N$ also has a nontrivial Seiberg-Witten invariant.*

Proof. Let $n = b_2(N)$. It is a theorem of Donaldson [1] that there are classes $e_1, \dots, e_n \in H^2(N, \mathbb{Z})$ descending to a basis of $H^2(N, \mathbb{Z})/Tor$ with respect to which the cup product form is diagonal[1]. We may choose the e_i such that $c = \sum_{i=1}^n e_i$ is characteristic. Then there is a $Spin^c$-structure on N with first Chern class c. With respect to this $Spin^c$-structure, the Seiberg-Witten equations for a generic Riemannian metric on N have a unique (up to gauge transformations) solution $(A_c, 0)$ given by the zero-section of the positive spin bundle and a connection A_c whose curvature is the harmonic form representing the image of c in $H^2(N, \mathbb{R})$.

Recall that the Seiberg-Witten invariants are smooth invariants of closed oriented four-manifolds with $b_2^+ > 1$ endowed with a choice of $Spin^c$-structure [12], [8]. By assumption, Y has a $Spin^c$-structure for which the invariant is nontrivial. We can extend this $Spin^c$-structure to X using the $Spin^c$-structure with Chern class c on N. The solutions of the Seiberg-Witten equations with respect to this $Spin^c$-structure and a Riemannian metric formed by connecting generic metrics on $Y - D^4$ and on $N - D^4$ by a sufficiently long cylinder are given by gluing the solutions on Y to the unique solution $(A_c, 0)$ on N. After perturbing the Dirac equation if

[1] Donaldson's proof used Yang-Mills theory. One can give an alternative proof using moduli spaces of solutions to the Seiberg-Witten equations.

necessary, this gluing is unobstructed and the gluing parameter in $U(1)$ is absorbed by the stabilizer of $(A_c, 0)$ in the gauge group. We conclude that there are $Spin^c$-structures on X and on Y for which the Seiberg-Witten invariants take the same (nonzero) value, up to sign. □

Combining Propositions 1 and 2, we obtain:

Theorem 1. *Let Y be a manifold with a nontrivial Seiberg-Witten invariant, e.g. a symplectic manifold with $b_2^+(Y) > 1$, and let N be a manifold with $b_1(N) = b_2^+(N) = 0$ whose fundamental group has a nontrivial finite quotient. Then $X = Y\#N$ has a nontrivial Seiberg-Witten invariant but does not admit any symplectic structure.*

It is easy to give examples of manifolds N satisfying the assumptions of Theorem 1. If M is a rational homology 3-sphere, surgery on $M \times S^1$ killing the generator of $\pi_1(S^1)$ produces a rational homology 4-sphere N with $\pi_1(N) = \pi_1(M)$. As 3-manifold groups tend to have linear representations, they also tend to have nontrivial finite quotients[2].

Remark 1. If N is an integral homology 4-sphere there is a map $X = Y\#N \to Y$ inducing an identification of $Spin^c$-structures which preserves the Seiberg-Witten invariants. If $\pi_1(N)$ has a nontrivial finite quotient, e.g. if N is obtained by surgery on $P \times S^1$, where P is the Poincaré homology 3-sphere, then X is not symplectic, although it has the same Seiberg-Witten invariants as Y which we can take to be symplectic.

Remark 2. If Y is symplectic and N has the integral homology of $n\overline{\mathbb{C}P^2}$, then $X = Y\#N$ has the same Seiberg-Witten invariants as the n-fold blowup of Y.

Remark 3. If $|H_1(N, \mathbb{Z})| = r$, there are r distinct $Spin^c$-structures on N that can be used in the proof of Proposition 2. This shows that the $Spin^c$-structures on X fall into groups of r so that the Seiberg-Witten invariants are constant on each group. On the other hand, the first Chern classes of the structures in a given group are the same modulo torsion and so have the same pairing with any cohomology class. This cannot happen for the Seiberg-Witten invariants of symplectic manifolds, where there is always a unique $Spin^c$-structure with nontrivial Seiberg-Witten invariant and with maximal degree with respect to the symplectic form [10].

[2] A 3-manifold with nontrivial fundamental group and no finite coverings would be a counterexample to Thurston's geometrization conjecture.

Remark 4. If $|H_1(N,\mathbb{Z})| = r$ is even, then the proof of Proposition 2 and the previous remark show that there are distinct $Spin^c$-structures having isomorphic determinant bundles and the same nontrivial Seiberg-Witten invariants. This is different from, for example, the case of minimal surfaces of general type, for which Kronheimer and Mrowka have shown that there are two determinant bundles, $K^{\pm 1}$, with underlying $Spin^c$-structures with nontrivial Seiberg-Witten invariants, and for each bundle there is only one $Spin^c$-structure with this property, regardless of the possible presence of 2-torsion.

Remark 5. Theorem 1 should be contrasted with the following result of Gromov [3]: if X has the rational homology of $\mathbb{C}P^2$ and is not diffeomorphic to $\mathbb{C}P^2$, then either X is not symplectic, or it is symplectic and the generator of $H_2(X)$ can not represented by a pseudo-holomorphic rational curve. For example, this applies to $X = \mathbb{C}P^2 \# N$ if N is a rational homology 4-sphere. Compare also [11].

We will now show that for most finitely presentable groups Γ which decompose as free products $\Gamma_1 * \Gamma_2$ with both Γ_i nontrivial there are smooth four-manifolds X with $\pi_1(X) = \Gamma$ admitting no smooth connected sum decomposition $X = X_1 \# X_2$ with $\pi_1(X_i) = \Gamma_i$ for both $i = 1$ and 2, although such decompositions can always be found [4] after passing from X to $X \# k(S^2 \times S^2)$ for some k (which trivializes the Seiberg-Witten invariants, by Proposition 1). This phenomenon was first discovered in [7] for very special choices of Γ_1 and Γ_2. If (a weak version of) Conjecture 1 is true, our argument gives counterexamples for all choices of Γ_1 and Γ_2.

Before stating our result, recall the following:

Definition 1. ([5],[6]) For a finitely presentable group Γ let

$$p(\Gamma) = \inf_X \{\chi(X) - |\sigma(X)|\} \ ,$$

where the infimum is taken over all smooth closed oriented 4-manifolds X with $\pi_1(X) = \Gamma$, and where χ denotes the Euler characteristic and σ the signature.

Proposition 3. ([6], cf. also [5]) *For every finitely presentable group, the invariant $p(\Gamma)$ is an even integer satisfying*

$$p(\Gamma) \geq 2 - 2b_1(\Gamma) \ ,$$

with equality if and only if Γ is the fundamental group of a manifold with (negative) definite intersection form. Furthermore, $p(\Gamma) \leq 2 - 2b_1(\Gamma) + 2a$ if and only if Γ is the fundamental group of a smooth manifold with $b_2^+ \leq a$.

The following theorem gives our counterexamples to the Kneser conjecture:

Theorem 2. *Let* $\Gamma = \Gamma_1 * \Gamma_2$ *be a free product for which each factor* Γ_i *satisfies at least one of the following conditions:*
(1) Γ_i *has a nontrivial finite quotient; or*
(2) $p(\Gamma_i) > 2 - 2b_1(\Gamma_i)$.

Then there exist infinitely many closed oriented smooth (in fact, symplectic) 4*-manifolds* X*, representing distinct homotopy types, with* $\pi_1(X) = \Gamma$ *which have no smooth connected sum decompositions* $X = X_1 \# X_2$ *with* $\pi_1(X_i) = \Gamma_i$ *for both* $i = 1$ *and* 2*.*

Proof. For every finitely presentable Γ, Gompf [2] has constructed infinitely many symplectic 4-manifolds X with $\pi_1(X) = \Gamma$ and with distinct homotopy types.

If such an X splits smoothly as $X_1 \# X_2$ with $\pi_1(X_i) = \Gamma_i$ for both $i = 1$ and 2, then it follows from Proposition 1 that one summand, say X_1, has a negative definite intersection form and its fundamental group Γ_1 does not satisfy condition (1). By Proposition 3, it cannot satisfy condition (2) either, which contradicts our assumptions. □

There are many examples of groups satisfying condition (2). In [6], there is a list of sufficient conditions implying $p > 2 - 2b_1$ for many classes of groups. For example, if G is amenable or if it contains an infinite sequence of subgroups of finite index with bounded first Betti number (this is true if it is poly-(finite or cyclic)), then $p(G) \geq 0$. On the other hand, $2 - 2b_1(G)$ is negative if the rank of the Abelianization of G is at least 2. When the first Betti number is positive, one can always satisfy condition (1). However, this does not mean that (2) implies (1).

Remark 6. At least in the case when condition (1) holds for both $i = 1$ and 2, the conclusion of Theorem 2 can be strengthened, to say that the splitting of the fundamental group can not be realized by any decomposition of X along a 3-manifold with trivial Floer homology corresponding to the Seiberg-Witten invariants. For example, this rules out spherical space forms because they have positive scalar curvature, cf. [8], [12].

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