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A REMARK ON THE FOURIER-MUKAI TRANSFORM

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This note arose from the attempt to understand the claim of Mukai in [4] that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the derived category $\mathcal{D}^b(A)$ of coherent sheaves on an abelian variety A endowed with a principal symmetric polarisation modulo shifts. Namely, this action is determined in terms of the standard generators of $\mathrm{SL}_2(\mathbb{Z})$, one of which acts by tensoring with a line bundle defining the polarisation and another by the Fourier-Mukai transform. Then one can check easily that the defining relations are satisfied, which means that there is a homomorphism from SL_2 to the quotient of the group of autoequivalences of $\mathcal{D}^b(A)$ considered up to an isomorphism modulo the subgroup $\mathbb{Z}$ generated by shifts. With some additional work one can prove that there is an action of the central extension $\widetilde{\mathrm{SL}_2}$ of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$ on $\mathcal{D}^b(A)$ in the more strict sense due to J.-L. Verdier [7]. Namely, we say that a group G acts on a category $\mathcal{C}$ if $\mathcal{C}$ is a fibered category over $\mathrm{Cat}(G)$—the category with one object corresponding to G. Explicitly this means that there is a system of functors F_g, where $g \in G$, from $\mathcal{C}$ to itself and isomorphisms $\alpha(g,h) : F_g \circ F_h \to F_{gh}$, for each $g, h \in G$, which satisfy certain cocycle condition for triples of elements of G. Though in the case of $\widetilde{\mathrm{SL}_2}$ it is not a big problem to construct such a system, this definition leads to the search for a more natural setup in which the above action occurs. In this note a variant of such a setup is suggested using quadratic modules over $\mathbb{Z}$.

1. Biextensions and quadratic modules

The complete proofs of the results of this section will be given elsewhere.

Let Bil be the category whose objects are pairs (V, b) where V is a finitely generated free $\mathbb{Z}$-module, b is a bilinear form on V; a morphism $(V, b) \to (V', b')$ is a homomorphism $f : V \to V'$ such that $f^{-1}(b') = b$. There is an obvious (commutative) monoidal structure on this category given by $(V, b) \otimes (V', b') = (V \oplus V', b \oplus b')$, for which $(0, 0)$ is the neutral object. There is a natural forgetting (monoidal) functor $p : \mathrm{Bil} \to \mathrm{Mod}$ from Bil to the category Mod of finitely generated free $\mathbb{Z}$-modules, and the functor $t : \mathrm{Mod} \to \mathrm{Bil} : V \mapsto (V, 0)$.

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Let $\widetilde{\mathrm{Sch}}$ be the category of pairs (X, L) where X is a scheme over a base scheme S, L is a $\mathbb{G}_m$-torsor over X, where a morphism from (X, L) to (X', L') is a pair (ϕ, α) consisting of a morphism $\phi : X \to X'$ and an isomorphism $\alpha : L \to \phi^* L'$. This category is also monoidal: $(X, L) \otimes (X', L') = (X \times_S X', p_1^* L \otimes p_2^* L')$ where $\otimes$ denotes the sum of $\mathbb{G}_m$-torsors (with the obvious neutral object). Let $p : \widetilde{\mathrm{Sch}} \to \mathrm{Sch}$ be the forgetting functor to the category of schemes over S, $t : \mathrm{Sch} \to \widetilde{\mathrm{Sch}}$ be the functor sending a scheme X to a pair $(X, \mathcal{O}_X)$.

Note that the category of monoidal functors Mod $\to$ Sch is equivalent to the category of commutative group schemes over S. For a commutative group scheme G and a $\mathbb{Z}$-module V we denote by $V \otimes G$ the value of the corresponding functor on V. More precisely, $V \otimes G$ is defined as some universal object in the category of commutative group schemes in the same spirit as the usual tensor product of modules is defined ($V \otimes G$ is noncanonically isomorphic to G^r where $V \simeq \mathbb{Z}^r$). Let Fun be the category of pairs F, F_0 of compatible symmetric monoidal functors from Bil to $\widetilde{\mathrm{Sch}}$ and from Mod to Sch (more precisely, one should consider isomorphisms of monoidal functors $p \circ F \simeq F_0 \circ p$ and $F \circ t \simeq t \circ F_0$ as a part of the data). Such a pair is essentially defined by a commutative group scheme $G = F_0(\mathbb{Z})$ and a $\mathbb{G}_m$-torsor L over G^2 obtained from the standard bilinear form b_0 on $\mathbb{Z}^2$: $b_0((x, y), (x', y')) = xy'$. Indeed any bilinear form b on $V \simeq \mathbb{Z}^r$ has form $b(v, v') = < v, \beta(v') >$ for some homomorphism $\beta : V \to V^*$, so that b is induced by the homomorphism $(\mathrm{id}, \beta) : V \to V \oplus V^*$ and the standard form on $V \oplus V^* \simeq (\mathbb{Z}^2)^r$ which is isomorphic to $(b_0)^r$. Considering the morphisms

$$(\mathbb{Z}^3, b') \to (\mathbb{Z}^2, b_0) \oplus (\mathbb{Z}^2, b_0) : (x, y, z) \mapsto ((x, y), (x, z)),$$
$$(\mathbb{Z}^3, b') \to (\mathbb{Z}^2, b_0) : (x, y, z) \mapsto (x, y + z)$$

in Bil we deduce an isomorphism of $\mathbb{G}_m$-torsors $p_{12}^* L \otimes p_{13}^* L$ and $m_{23}^* L$ on G^3 where $m_{23} = (\mathrm{id} \times m) : G^3 \to G^2$, $m : G^2 \to G$ is the composition law, $p_{12}, p_{13} : G^3 \to G^2$ are the projections. Symbolically this isomorphism can be written as $L_{x,y} L_{x,y'} \simeq L_{x,y+y'}$. Analogously we have an isomorphism $L_{x,y} L_{x',y} \simeq L_{x+x',y}$. By the same method one obtains the commutativity of certain diagrams which express the fact that L defines a *biextension* of $G \times G$ by $\mathbb{G}_m$ (see [6],[2],[1]). Moreover, one can prove the following theorem.

Theorem 1.1. *The category* Fun *is equivalent to the category of biextensions L of $G \times G$ by $\mathbb{G}_m$ where G is some commutative group scheme over S; a morphism $(G, L) \to (G', L')$ in the latter category is given by a homomorphism $\phi : G \to G'$ and an isomorphism $L \simeq (\phi \times \phi)^* L'$.*

Remark. Analogously, one can characterize symmetric biextensions in the sense of [1] (which are roughly speaking the biextensions L of G^2 equipped with a symmetry isomorphism $s^*L \simeq L$) via monoidal functors from the category of even quadratic modules (that is the category of $\mathbb{Z}$-valued symmetric forms b on finitely generated free $\mathbb{Z}$-modules such that $b(v, v)$ is even).

Let Sym be the category of quadratic modules (that is the category of $\mathbb{Z}$-valued symmetric forms), Fun_G be the category of monoidal functors $\mathrm{Sym} \to \widetilde{\mathrm{Sch}}$ compatible with the functor $F_0 : \mathrm{Mod} \to \mathrm{Sch}$ associated with a commutative group scheme G over S. A *cube structure* on a $\mathbb{G}_m$-torsor L over G is a symmetric $\mathbb{G}_m$-biextension structure on the $\mathbb{G}_m$-torsor

$$\Lambda(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$$

where $m : G^2 \to G$ is the group law, $p_1, p_2 : G^2 \to G$ are the projections, satisfying some natural compatibility (see [1]). Now assume that L is a symmetric $\mathbb{G}_m$-torsor on G which means that an isomorphism $i^*L \simeq L$ is given where $i : G \to G$ is the inversion. Then a Σ-structure on L is a cube structure compatible with the symmetry $i^*L \simeq L$ (see [1]). With this terminology we have the following result.

Theorem 1.2. *The category* Fun_G *is equivalent to the category of* $\mathbb{G}_m$-*torsors on* G *endowed with* Σ-*structure.*

We omit the proof of this theorem. Note only that the $\mathbb{G}_m$-torsor L corresponding to a functor $F \in \mathrm{Fun}_G$ is defined by the formula $F(\mathbb{Z}, q_1) = (G, L)$ where q_1 is the simplest symmetric form on $\mathbb{Z}$: $q_1(e, e) = 1$ where e is the generator of $\mathbb{Z}$.

2. Correspondences

For any smooth projective variety X over a field k we denote by $\mathcal{D}^b(X)$ the (bounded) derived category of coherent sheaves on X. Let $\mathrm{Cor}(X)$ denotes the category $\mathcal{D}^b(X \times X)$ with the following monoidal structure:

$$\mathcal{G} * \mathcal{F} = p_{13*}(p_{12}^*\mathcal{F} \otimes p_{23}^*\mathcal{G})$$

where $p_{ij} : X^3 \to X^2$ are the projections; all functors are the derived ones. The neutral object of $\mathrm{Cor}(X)$ is $\Delta_*\mathcal{O}_X$ where $\Delta : X \to X^2$ is the diagonal embedding. There is a natural action of $\mathrm{Cor}(X)$ on the category $\mathcal{D}^b(X)$: each object $\mathcal{F} \in \mathrm{Cor}(X)$ gives rise to a functor

$$\Phi(\mathcal{F}) = p_{2*}(p_1^*(\cdot) \otimes \mathcal{F}) : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

where $p_1, p_2 : X^2 \to X$ are the natural projections. A morphism $\mathcal{F} \to \mathcal{F}'$ induces a morphism of functors $\Phi(\mathcal{F}') \to \Phi(\mathcal{F})$, and for a pair of morphisms the usual compatibility holds showing that we have a functor from $\mathrm{Cor}(X)$ to the category of functors from $\mathcal{D}^b(X)$ to itself. One can easily show that this functor is monoidal which justifies the phrase "$\mathrm{Cor}(X)$ acts on $\mathcal{D}^b(X)$" (see [4] for more general setup).

Let $\mathrm{Cor}(V)$ be the category whose objects are triples: a quadratic module (W,q), a homomorphism of $\mathbb{Z}$-modules $h : W \to V^2$ and an integer number n, such that $p_1 h$ is surjective where $p_1 : V^2 \to V$ is the projection on the first factor. A morphism $\mathcal{W} = (W,q,h,n) \to \mathcal{W}' = (W',q',h',n')$ in $\mathrm{Cor}(V)$ is a quadruple $((U,q_U),i,f,\mu)$ where $(U,q_U) \in \mathrm{Sym}$ is a quadratic module, $i : U \to W$ and $f : U \to W'$ are homomorphisms such that $i^{-1}(q) = f^{-1}(q') = q_U$ (that is i and f are morphisms in Sym), $h \circ i = h' \circ f$, i is an embedding such that $W/i(U)$ is a free $\mathbb{Z}$-module of rank $(n-n')$, $\mu : \mathbb{Z} \widetilde{\to} \det(W/i(U))$ is an isomorphism (where for a free $\mathbb{Z}$-module P of rank r we denote by $\det(P) = \bigwedge^r(P)$ the top degree wedge power of P). One can easily define the composition of such things. There is a monoidal structure on this category: $\mathcal{W}' * \mathcal{W} = (W * W', q * q', h * h', n + n')$ where $W * W' = W \times_V W'$ is the fibered product with respect to the morphisms $p_2 h : W \to V$ and $p_1 h' : W' \to V$, $q * q' = p_W^{-1}(q) + p_{W'}^{-1}(q')$ where p_W and $p_{W'}$ are the projections of $W * W'$ on W and W', the morphism $h * h' : W * W' \to V^2$ has components $p_1 h p_W : W * W' \to V$ and $p_2 h' p_{W'} : W * W' \to V$. The neutral object is $(V, 0, \Delta_V, 0)$ where Δ_V is the diagonal embedding $V \to V^2$.

From now on let A be a connected abelian variety of dimension g over a field k, $F \in \mathrm{Fun}_A$ be a monoidal functor from Sym to $\widetilde{\mathrm{Sch}}$ as above, corresponding to a line bundle L on A with a Σ-structure (which in this case is equivalent to the trivialization of the fiber of L at $0 \in A$ compatible with the symmetry $(-1)^*L \simeq L$).

Theorem 2.1. *Fix a trivialization $\mathcal{O}_A \simeq \omega_A = \bigwedge^g(\Omega_A)$. Then there is a natural monoidal contravariant functor $\mathrm{Cor}_F : \mathrm{Cor}(V) \to \mathrm{Cor}(V \otimes A)$ such that for an object $\mathcal{W} = (W,q,h,n) \in \mathrm{Cor}(V)$ we have*

$$\mathrm{Cor}_F(\mathcal{W}) = (Fh)_*(\mathcal{L}_W)[ng]$$

where $F(W,q) = (W \otimes A, \mathcal{L}_W) \in \widetilde{\mathrm{Sch}}$, $Fh : W \otimes A \to V \otimes A$ is the morphism induced by h. Thus, there is an action of $\mathrm{Cor}(V)$ on $\mathcal{D}^b(V \otimes A)$.

Proof. Let $(U, q_U, i, f, \mu) : \mathcal{W} \to \mathcal{W}'$ be a morphism in $\mathrm{Cor}(V)$. Let $W_0 = (W \oplus W')/U$ be the coproduct of W and W' over U in the category of $\mathbb{Z}$-modules (here the embedding $U \to W \oplus W'$ is given by $u \mapsto (-i(u), f(u))$), so that W_0 is a free $\mathbb{Z}$-module (as an extension of W/U by W') equipped with the natural maps $f_0 : W \to W_0$ and $i_0 : W' \to W_0$ such that $i_0 f = f_0 i$. Also there is a morphism $h_0 : W_0 \to V^2$ such that $h_0 i_0 = h'$, $h_0 f_0 = h$. As above for each quadratic module (W, q) we denote by $\mathcal{L}_W$ the line bundle on $W \otimes A$ given by the functor F. Note that $\mu : \mathbb{Z} \simeq \det(W/i(U))$ together with the fixed trivialization $\mathcal{O}_A \simeq \omega_A$ induces an isomorphism $\mathcal{O}_{U\otimes A} \simeq \omega_{U\otimes A/W\otimes A}$ where for an embedding of locally complete intersection $Y \subset X$ of codimension r we denote following [3] $\omega_{Y/X} = \bigwedge^r (J_Y/J_Y^2)^\vee$ where J_Y is the ideal sheaf of Y. Thus we have an isomorphism of functors $(Fi)^! \simeq (Fi)^*[-r]$ where $Fi : U \otimes A \to W \otimes A$ is the embedding induced by i, $r = (n - n')g$ is the codimension of $U \otimes A$ in $W \otimes A$ (see [3], Cor.7.3). The composition of this isomorphism with the trace morphism $(Fi)_*(Fi)^! \to \mathrm{id}$ induces a morphism $(Fi)_*(Fi)^*\mathcal{L}_W \to \mathcal{L}_W[r]$. Now we compose this morphism with the natural morphism

$$(Ff_0)^*(Fi_0)_*\mathcal{L}_{W'} \to (Fi)_*(Ff)^*\mathcal{L}_{W'} \simeq (Fi)_*\mathcal{L}_U \simeq (Fi)_*(Fi)^*\mathcal{L}_W$$

to obtain a morphism $(Ff_0)^*(Fi_0)_*\mathcal{L}_{W'} \to \mathcal{L}_W[r]$ which by adjunction induces a morphism $(Fi_0)_*\mathcal{L}_{W'} \to (Ff_0)_*\mathcal{L}_W[r]$ and therefore a morphism

$$\mathrm{Cor}_F(\mathcal{W}') = (Fh')_*(\mathcal{L}_{W'})[n'g] \simeq (Fh_0)_*(Fi_0)_*\mathcal{L}_{W'}[n'g] \to$$
$$(Fh_0)_*(Ff_0)_*\mathcal{L}_W[n'g + r] \simeq (Fh)_*(\mathcal{L}_W)[ng] = \mathrm{Cor}_F(\mathcal{W}).$$

If we apply the functor Cor_F to the product $\mathcal{W}' * \mathcal{W}$ then we get

$$\mathrm{Cor}_F(\mathcal{W}' * \mathcal{W}) = (F(h * h'))_*(p_W^*\mathcal{L} \otimes p_{W'}^*\mathcal{L}')[(n + n')g]$$

where we put $\mathcal{L} = \mathcal{L}_W$, $\mathcal{L}' = \mathcal{L}_{W'}$. On the other hand

$$\mathrm{Cor}_F(\mathcal{W}') * \mathrm{Cor}_F(\mathcal{W}) = p_{13*}(p_{12}^*((Fh)_*\mathcal{L}) \otimes p_{23}^*((Fh')_*\mathcal{L}'))[ng + n'g]$$

where $p_{ij} : (V \otimes A)^3 \to V \otimes A$ are the projections. Now by the flat base change ([3]) we have an isomorphism $p_{23}^*((Fh')_*\mathcal{L}') \simeq (\mathrm{id} \times (Fh'))_* p_2^*\mathcal{L}'$ where $\mathrm{id} \times (Fh') : (V \otimes A) \times (W' \otimes A) \to (V \otimes A)^3$ is induced by the morphism $Fh' : W' \otimes A \to (V \otimes A)^2$, $p_2 : (V \otimes A) \times (W' \otimes A) \to W' \otimes A$ is the projection. So

$$\begin{aligned} &p_{13*}(p_{12}^*((Fh)_*\mathcal{L}) \otimes p_{23}^*((Fh')_*\mathcal{L}')) \\ &\qquad \simeq p_{13*}(p_{12}^*((Fh)_*\mathcal{L}) \otimes (\mathrm{id} \times (Fh'))_*(p_2^*\mathcal{L}')) \\ &\qquad \simeq p_{13*}(\mathrm{id} \times (Fh'))_*((\mathrm{id} \times F(p_1 h'))^*((Fh)_*\mathcal{L}) \otimes p_2^*\mathcal{L}') \\ &\qquad \simeq (\mathrm{id} \times F(p_2 h'))_*((\mathrm{id} \times F(p_1 h'))^*((Fh)_*\mathcal{L}) \otimes p_2^*\mathcal{L}'). \end{aligned}$$

Here we used the projection formula and the equalities

$$p_{12}(\mathrm{id} \times F(h')) = (\mathrm{id} \times F(p_1 h')) : V \otimes A \times W' \otimes A \to (V \otimes A)^2,$$
$$p_{13}(\mathrm{id} \times F(h')) = (\mathrm{id} \times F(p_2 h')).$$

Now apply the base change formula to the morphism $\phi = Fh : W \otimes A \to (V \otimes A)^2$ and the flat (due to surjectivity of $p_1 h'$) base change $u = \mathrm{id} \times F(p_1 h') : (V \otimes A) \times (W' \times A) \to (V \otimes A)^2$. Then the corresponding fiber product is $(W * W') \otimes A$ with the structural morphisms $\phi' = (F(p_1 h p_W), F(p_{W'})) : (W * W') \otimes A \to (V \times W') \otimes A$ and $u' = F(p_W) : (W * W') \otimes A \to W \otimes A$. Thus we get an isomorphism

$$\begin{aligned}(\mathrm{id} \times F(p_2 h'))_*(u^*(\phi_* \mathcal{L}) \otimes p_2^* \mathcal{L}') &\simeq (\mathrm{id} \times F(p_2 h'))_*(\phi'_*(u')^* \mathcal{L} \otimes p_2^* \mathcal{L}') \\ &\simeq (\mathrm{id} \times F(p_2 h'))_* \phi'_*((u')^* \mathcal{L} \otimes F(p_{W'})^* \mathcal{L}') \\ &\simeq F(h * h')_*(F(p_W)^* \mathcal{L} \otimes F(p_{W'})^* \mathcal{L}').\end{aligned}$$

Here we used the projection formula and the equalities

$$p_2 \phi' = F(p_{W'}),\ (\mathrm{id} \times F(p_2 h'))\phi' = F(h * h') : (W * W') \otimes A \to (V \otimes A)^2.$$

Finally we obtain an isomorphism

$$\mathrm{Cor}_F(\mathcal{W}') * \mathrm{Cor}_F(\mathcal{W}) \simeq (F(h * h'))_*(p_W^* \mathcal{L} \otimes p_{W'}^* \mathcal{L}')[(n + n')g]$$

which shows that Cor_F respects monoidal structures (we omit the proof of all the remaining compatibilities). □

Let $F(\mathbb{Z}^2, q_0) = (A^2, \mathcal{L}_0)$, where q_0 is the standard quadratic form on $\mathbb{Z}^2$ (the symmetrization of b_0), then $\mathcal{L}_0$ gives a homomorphism $\lambda : A \to A'$ where A' is the dual abelian variety to A. Assume that λ is an isomorphism and L is ample (in other words L gives a principal polarisation of A). With this assumption we are going to show that certain morphisms in $\mathrm{Cor}(V)$ become isomorphisms in $\mathrm{Cor}(V \otimes A)$.

Proposition 2.2. *Let (W, q) be a quadratic module and $U \subset W$ be an isotropic submodule (that is $q|_U = 0$) such that the map $W/U \to U^* = \mathrm{Hom}(U, \mathbb{Z})$ induced by q is an isomorphism. Let $p : W \otimes A \to W/U \otimes A$ be the natural projection, $i : S = \mathrm{Spec}(k) \to W/U \otimes A$ be the embedding corresponding to the neutral element of $W/U \otimes A$. Then under the assumption above we have $p_* \mathcal{L}_W \simeq i_* \mathcal{O}_S[-mg]$ where m is the rank of W/U.*

Proof. One can easily check that there exists an isomorphism $\alpha : W \to U \oplus U^*$ of the quadratic modules (where the symmetric form on $U \oplus U^*$ is the standard one) such that $\alpha(U) = (U, 0) \subset U \oplus U^*$. So it is sufficient to prove the statement in the case of the standard quadratic module $(W, q) = (\mathbb{Z}^2, q_0)$ which amounts to the well-known computation of $p_{2*}(\mathcal{L}_0)$ where $p_2 : A^2 \to A$ is the projection (see [5]). □

Corollary 2.3. *Let (W,q) be a quadratic module, $U \subset W$ be an isotropic submodule, such that W/U has no torsion and the map $W \to U^*$ induced by q is surjective. Let $p : W \otimes A \to (W/U) \otimes A$ be the natural projection, $i : W' \otimes A \to (W/U) \otimes A$ be the natural embedding where $W' = U^{\perp}/U$, $U^{\perp} = \{w \in W | \ q(w,U) = 0\}$. Then $p_*\mathcal{L}_W \simeq i_*\mathcal{L}_{W'}[-mg]$ where m is the rank of U, the line bundle $\mathcal{L}_{W'}$ comes from the quadratic form q' on W' induced by q.*

Proof. Choose a decomposition of $\mathbb{Z}$-modules $W = U \oplus P$, then $W/U \simeq P$ and by the projection formula we may assume that $q|_P = 0$ (since the contribution of $q|_P$ to $\mathcal{L}_W$ and $\mathcal{L}_{W'}$ is a pull-back from $(W/U)\otimes A$). Then $U^{\perp} = U \oplus K$ where $K \subset P$ is the kernel of the (surjective) map $P \to U^*$ induced by q. Now all the picture is obtained by the base change $P \otimes A \to (P/U) \otimes A$ from the one considered in the previous proposition. $\square$

Proposition 2.4. *Let (W,q) be a quadratic module, $(W',q') = (W,q) \oplus (\mathbb{Z}^n, q_n)$ be a direct sum of quadratic modules where $q'|_{\mathbb{Z}^n} = q_n$ is the standard symmetric form on $\mathbb{Z}^n$, that is $(\mathbb{Z}^n, q_n) = (\mathbb{Z}, q_1)^n$. Let $p : W' \to W$ be the projection. Then $(Fp)_*\mathcal{L}_{W'} \simeq \mathcal{L}_W$.*

Proof. Obviously we may assume that $n = 1$. Then the assertion follows from the isomorphism $H^0(A,L) \simeq k$ together with the vanishing $H^{>0}(A,L) = 0$. $\square$

Let $\mathcal{S}_1$ be the class of morphisms $\mathcal{W} \to \mathcal{W}'$ in $\mathrm{Cor}(V)$ which are given by the quadruples $((U^{\perp}, q|_{U^{\perp}}), i, f, \mu)$ where $U \subset W$ is an isotropic submodule such that the map $W \to U^*$ induced by q is surjective, $i : U^{\perp} \to W$ is the natural embedding, a map $f : U^{\perp} \to W'$ vanishes on U and induces an isomorphism $U^{\perp}/U \simeq W'$. Let $\mathcal{S}_2$ be the class of morphisms $\mathcal{W} \to \mathcal{W}'$ in $\mathrm{Cor}(V)$ such that $i = \mathrm{id}$, $f : (W,q) \to (W',q')$ is an embedding and there exists a quadratic submodule $(P, q_P) \subset (W',q')$ such that $h'(P) = 0$, $(P,q_P) \simeq (\mathbb{Z}^n, q_n)$ and $(W',q') = (W,q) \oplus (P,q_P)$. Then one can see easily that $\mathcal{S}_1$ and $\mathcal{S}_2$ are compatible with the monoidal structure on $\mathrm{Cor}(V)$ and the functor Cor_F sends morphisms from $\mathcal{S}_1$ and $\mathcal{S}_2$ to isomorphisms in $\mathrm{Cor}(V \otimes A)$. Therefore if we consider the multiplicative class of morphisms $\mathcal{S}$ generated by $\mathcal{S}_1$ and $\mathcal{S}_2$ then the localized category $\widetilde{\mathrm{Cor}}(V) = \mathrm{Cor}(V)[\mathcal{S}^{-1}]$ is monoidal and we have a monoidal functor $\widetilde{\mathrm{Cor}}_F : \widetilde{\mathrm{Cor}}(V) \to \mathrm{Cor}(V \otimes A)$.

3. Comparison with Mukai's picture

Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ can be defined as the group with two generators S and T and the following defining relations:

$$S^2 = T^3,\ S^4 = T^6 = 1.$$

Namely, these generators correspond to the following matrices:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now the central extension $\widetilde{\mathrm{SL}}_2$ of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$ can be defined as the group with generators S and T and with the defining relation $S^2 = T^3$. The central element $C = S^2$ generates the subgroup isomorphic to $\mathbb{Z}$ in $\widetilde{\mathrm{SL}}_2$ and the quotient by C^2 is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. So we can realize $\widetilde{\mathrm{SL}}_2$ as the group of sequences $(\varepsilon_1, \dots, \varepsilon_k)[n]$ where ε_i is either 0 or 1, n is an integer, and the sequence $(\varepsilon_1, \dots, \varepsilon_k)$ satisfies the following property: there can be at most two 1's and at most one 0 in a row. For example, (1101) is allowed while (001) is forbidden. The group law is described as follows:

$$(\varepsilon_1, \dots, \varepsilon_k)[n] \cdot (\varepsilon'_1, \dots, \varepsilon'_l)[m] = (\varepsilon_1, \dots, \varepsilon_k, \varepsilon'_1, \dots, \varepsilon'_l)[n+m]$$

if the combined sequence is allowed. If $\varepsilon_k = \varepsilon'_1 = 0$ then one should throw out this pair and replace $n+m$ by $n+m+1$. Analogously if the combined sequence contains (111) as a subsequence one should throw it out and replace $n+m$ by $n+m+1$. The neutral element is given by the empty sequence $()[0]$. The relation with the previous description is the following: one should replace $\varepsilon_i = 0$ by S, $\varepsilon_i = 1$ by T, $[n]$ by C^n and take the corresponding product. Let $\widetilde{\mathcal{SL}}_2$ be the category whose objects are elements of the group $\widetilde{\mathrm{SL}}_2$ with only identity morphisms and the monoidal structure given by the opposite group law, that is $g_1 * g_2 = g_2 g_1$.

Theorem 3.1. *There exists a monoidal functor $\Psi : \widetilde{\mathcal{SL}}_2 \to \widetilde{\mathrm{Cor}}(\mathbb{Z})$. Thus the group $\widetilde{\mathrm{SL}}_2$ acts on the category $\mathcal{D}^b(A)$ of an abelian variety A endowed with a principal symmetric polarisation.*

Proof. Let $\varepsilon[n] = (\varepsilon_1, \dots, \varepsilon_k)[n]$ be an element of $\widetilde{\mathrm{SL}}_2$. Define an object $\Psi(\varepsilon[n]) \in \mathrm{Cor}(\mathbb{Z})$ as follows:

$$\Psi(\varepsilon[n]) = (\mathbb{Z}^{k+1}, q_\varepsilon, p^\epsilon_{0k}, -n)$$

where the symmetric form on $\mathbb{Z}^{k+1}$ is given by

$$q_\varepsilon(e_0,e_0)=0,\ q_\varepsilon(e_i,e_i)=\varepsilon_i\ (i=1,\dots,k),$$
$$q_\varepsilon(e_{i-1},e_i)=1\ (i=1,\dots,k),$$
$$q_\varepsilon(e_i,e_j)=0\ (|i-j|>1)$$

where $e_0,e_1,\dots,e_k$ is the basis of $\mathbb{Z}^{k+1}$. The map $p^\epsilon_{0k}:\mathbb{Z}^{k+1}\to\mathbb{Z}^2$ where $\epsilon=(-1)^{\sum_i\varepsilon_i}$ is defined by the formulas

$$p^\epsilon_{0k}(e_0)=(1,0),\ p^\epsilon_{0k}(e_k)=(0,\epsilon),\ p^\epsilon_{0k}(e_i)=0\ (i=1,\dots,k-1)$$

if $k>0$ and by $p^\epsilon_{00}(e_0)=(1,\epsilon)$ if $k=0$ (note that $p^1_{0k}=p_{0k}$ is the usual projection).

To show the monoidal structure of the functor Ψ we consider two examples (expressing the relations $S^2=C$ and $T^3=C$). First, as $\Psi((0)[0])=(\mathbb{Z}^2,q_0,p_{01},0)$ one computes easily that

$$\mathcal{W}_0=\Psi((0)[0])*\Psi((0)[0])\simeq(\mathbb{Z}^3,q_{(00)},p_{02},0)$$

where the form $q_{(00)}$ is given by the formulas above (though the sequence (00) is forbidden). Now $e_1\in\mathbb{Z}^3$ is an isotropic vector such that $p_{02}(e_1)=0$ so we may apply the reduction of the corollary 2.3. Namely, we have a morphism of the class $\mathcal{S}_1$ from $\mathcal{W}_0$ to

$$\mathcal{W}_0'=((e_1)^\perp/(e_1),q',h',-1)\simeq(\mathbb{Z},0,p^{-1}_{00},-1)=\Psi(()[1])$$

where the latter isomorphism corresponds to an element $e_0-e_2\in(e_1)^\perp$. Second, $\Psi((1)[0])=(\mathbb{Z}^2,q_{(1)},p^{-1}_{01},0)$ so one can easily compute its cube with respect to the monoidal structure $*$ on $\mathrm{Cor}(\mathbb{Z})$:

$$\mathcal{W}_1=\Psi((1)[0])^{*3}\simeq(\mathbb{Z}^4,q_{(111)},p^{-1}_{03},0).$$

Now we can apply 2.3 to the isotropic vector $e_1-e_2\in\mathbb{Z}^4$ to get a morphism of the class $\mathcal{S}_1$:

$$\mathcal{W}_1\to\mathcal{W}_1'=((e_1-e_2)^\perp/(e_1-e_2),q',h',-1)\simeq(\mathbb{Z}^2,q',h',-1)$$

where the isomorphism $\mathbb{Z}^2\simeq(e_1-e_2)^\perp/(e_1-e_2)$ is given by the vectors $f_0=e_0+e_3-e_1$, $f_1=e_1\in\mathbb{Z}^2$, so that $q'(f_0,f_i)=0$ $(i=0,1)$, $q'(f_1,f_1)=1$, $h'(f_0)=(1,-1)$, $h'(f_1)=0$. Thus we can apply the second reduction (see 2.4) to the decomposition $(\mathbb{Z}^2,q')\simeq(\mathbb{Z},0)\oplus(\mathbb{Z},q_1)$. Namely, the embedding of $\mathbb{Z}$ in $\mathbb{Z}^2$ given by f_0 induces a morphism $\Psi(()[1])\to\mathcal{W}_1'$ of the class $\mathcal{S}_2$.

One can check that the isomorphisms in $\widetilde{\mathrm{Cor}}(\mathbb{Z})$ constructed above extend to the monoidal structure on the functor Ψ. $\square$

Remark. Notice that the twisting functor $\otimes L$ is obtained in our picture as $\mathrm{Cor}_F(\Psi(01)[-1])$. Also for any symmetric form q on $\mathbb{Z}^{k+1}$ with the properties $|q(e_i, e_{i+1})| = 1$ for any i, $q(e_i, e_j) = 0$ for $|i-j| > 1$, the corresponding object of $\widetilde{\mathrm{Cor}}(\mathbb{Z})$ (where we consider $\mathbb{Z}^{k+1}$ as a correspondence via the projection p_{0k} as above) belongs to the monoidal subcategory generated by the essential image of Ψ and by the object $(\mathbb{Z}, 0, p_{00}^{-1}, 0)$.

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