HOMOGENEOUS REAL HYPERSURFACES

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Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. The hypersurface M is called *weighted homogeneous* if it is locally equivalent, via a biholomorphic map which preserves the origin, to a hypersurface given by an equation of the form

$$(0.1) \qquad\qquad\qquad P(z, \overline{z}) = 0$$

where P is a polynomial which is homogeneous with respect to a non-isotropic group of dilations (see Section 1). If M is weighted homogeneous and m is the weight of the polynomial P in (0.1), we say M is *weighted homogeneous of weight m*. In this paper, we study the question of how one can tell whether M is weighted homogeneous. Our main result, Theorem 4.1, gives a sufficient condition for M to be weighted homogeneous.

For the case of hypersurfaces in $\mathbf{C}^2$ which are rigid in the sense of [BRT] (see Section 4), we proved Theorem 4.1 in [S2,S3] (see also [S1]). In [S4] we proved the theorem for holomorphically nondegenerate rigid hypersurfaces in the case that the group of dilations is $\delta_t(z) = (tz_1, \ldots, tz_{n-1}, t^m z_n)$. Unlike our earlier results, Theorem 4.1 does not require that M be rigid and there is no nondegeneracy hypothesis or finite type hypothesis on M. However, if M is weighted homogeneous, then it is rigid (see Proposition 4.3).

After some preliminaries in Section 1, we prove some lemmas concerning *allowable* maps in Section 2, maps which are the identity map modulo terms of higher weight. In Section 3 we show that certain holomorphic vector fields, approximate holomorphic infinitesimal dilations, can be linearized. We use this result in Section 4 to prove the sufficient condition for weighted homogeneity.

1. Preliminaries

Throughout this paper we will assume that M is a real analytic real hypersurface through the origin in $\mathbf{C}^n$. We fix positive integers $m_1, \ldots, m_n$. The non-isotropic group of dilations determined by $(m_1, \ldots, m_n)$ is the group $\{\delta_t : t > 0\}$ where

$$(1.1) \qquad \delta_t(z) = (t^{m_1} z_1, \ldots, t^{m_n} z_n).$$

For a multiindex $A = (\alpha_1, \ldots, \alpha_n)$, we let $||A|| = \sum m_i \alpha_i$ and $|A| = \sum \alpha_i$.

Definition 1.2. A function h is *homogeneous of weight* j if $h \circ \delta_t = t^j h$. A vector field Y is *homogeneous of weight* j if

$$Y(f \circ \delta_t) = t^{-j}(Yf) \circ \delta_t.$$

We say a function $h = O(j)$ if every term in its Taylor expansion about 0 has weight $\geq j$. Similarly, if Y is a vector field, we write $Y = O(j)$ if, when we expand the coefficients of Y in a Taylor series about 0, each term is a vector field of weight $\geq j$. If Y is a vector field with $Y = O(j)$ and h is a function with $h = O(k)$, it follows that $Yh = O(j + k)$.

We say a function is $O(|z|^j)$ if it vanishes to order j at the origin. If Y is a vector field, we write $Y = O(|z|^j)$ if every coefficient of Y is $O(|z|^j)$.

Definition 1.3. A biholomorphic map ϕ is *allowable* if

$$\phi(z) = (\phi_1(z), \ldots, \phi_n(z)), \qquad \phi_j(z) = z_j + O(m_j + 1).$$

Let

$$Z_0 = \sum_{j=1}^n m_j z_j \frac{\partial}{\partial z_j}$$

and let

$$Y_0 = 2 \operatorname{Re} Z_0.$$

We call Y_0 an *infinitesimal dilation* because the one-parameter group generated by Y_0 is the group of non-isotropic dilations $\{\delta_{e^t} : t \in \mathbf{R}\}$. We call Z_0 a *holomorphic infinitesimal dilation*. If f is a polynomial homogeneous of weight k then $Y_0 f = kf$.

A smooth vector field X on M is called an *infinitesimal CR automorphism* of M if the local one-parameter group it generates is a local group of CR automorphisms of M. Let $\operatorname{aut}(M)$ denote the space of infinitesimal CR automorphisms of M which are defined in a neighborhood of the

origin. If Z is a holomorphic vector field, i.e., a vector field of type $(1,0)$ with holomorphic coefficients, then the local group generated by $Y = \operatorname{Re} Z$ is a local group of holomorphic maps [KN, remarks preceding Proposition IX.2.10]. Hence, if Y is a vector field defined on a neighborhood of the origin in M and $Y = (\operatorname{Re} Z)|_M$ for some holomorphic vector field Z, then $Y \in \operatorname{aut}(M)$. We denote the subspace of such infinitesimal CR automorphisms by $\operatorname{hol}(M)$.

Definition 1.4. A vector field $Y \in \operatorname{aut}(M)$ is an *approximate infinitesimal dilation* if $Y = Y_0 + O(1)$. A holomorphic vector field Z is an *approximate holomorphic infinitesimal dilation* if $Z = Z_0 + O(1)$.

If M is given by an equation of the form (0.1) where P is a polynomial homogeneous with respect to the group $\{\delta_t\}$, then $Y_0 \in \operatorname{hol}(M)$. We will see in Proposition 2.2 that if M is equivalent to (0.1) via an allowable map, then M has an approximate infinitesimal dilation in $\operatorname{hol}(M)$.

2. Allowable maps

In this section, we give some results concerning allowable biholomorphic maps. The first is that the inverse of an allowable map is allowable and that an allowable map takes an infinitesimal approximate dilation to an infinitesimal approximate dilation.

Lemma 2.1. *Suppose ϕ is an allowable biholomorphic map. Then $\psi = \phi^{-1}$ is allowable. If Z is an approximate holomorphic dilation then $d\phi Z$ also is.*

We omit the proof, which is straightforward. As a corollary, we have the following proposition.

Proposition 2.2. *Suppose M is a real analytic real hypersurface through the origin in $\mathbf{C}^n$ and that M is equivalent to (0.1) via the restriction to M of an allowable biholomorphic map ϕ. Then there is an approximate infinitesimal dilation $Y \in \operatorname{hol}(M)$.*

Proof. Let $Y = d\phi^{-1}Y_0$. Then $Y \in \operatorname{hol}(M)$. Because ϕ is allowable, by Lemma 2.1, $Y = Y_0 + O(1)$. $\square$

Next we show that if Z is an approximate holomorphic dilation then the linear part of Z is diagonalizable.

Lemma 2.3. *Suppose*

$$(2.4) \qquad Z = \sum_i m_i z_i \frac{\partial}{\partial z_i} + O(1).$$

Then there is an allowable linear transformation $z' = Sz$ such that

$$(2.5) \qquad dS\, Z = \sum_i m_i z'_i \frac{\partial}{\partial z'_i} + O(1) \cap O(|z'|^2).$$

Proof. Write

$$Z = \sum_{i,j} a_{ij} z_j \frac{\partial}{\partial z_i} + O(|z|^2).$$

Let A be the linear transformation with matrix (a_{ij}). After reordering the coordinates if necessary, we may assume that $m_i \leq m_{i+1}$. Because Z satisfies (2.4),

$$(2.6) \qquad a_{ij} = m_i \delta_{ij} \qquad \text{if } m_i \geq m_j.$$

(So, in particular, A is upper triangular.) We will find a linear transformation $T = (t_{ij})$ such that

$$(2.7) \qquad t_{ij} = \delta_{ij} \qquad \text{if } m_i \geq m_j$$

which diagonalizes A,

$$(2.8) \qquad T^{-1} A T z = D z$$

where D is the diagonal matrix with $d_{ii} = m_i$. Then by (2.7), T is allowable. Hence, by Lemma 2.1, $S = T^{-1}$ is allowable. If $z' = Sz$ then, by (2.8) and Lemma 2.1, $dS\, Z$ satisfies (2.5).

By (2.6), (2.8) is equivalent to

$$(2.9_k) \qquad (m_k - m_i) t_{ki} = - \sum_{\{j:\, m_k < m_j\}} a_{kj} t_{ji}$$

holding for $k = 1, \ldots, n$. Because $m_n \geq m_j$ for all j, we can solve (2.9_n) by taking

$$t_{ni} = \begin{cases} 1, & i = n \\ 0, & i < n \end{cases}.$$

Suppose we have found t_{ji}, $j = k+1, \ldots, n$, $i = 1, \ldots, n$, satisfying (2.7) and (2.9_j). The right hand side of (2.9_k) is known. Hence, t_{ki} satisfies

$$(2.10) \qquad t_{ki} = \frac{-1}{m_k - m_i} \sum_{\{j:\, m_k < m_j\}} a_{kj} t_{ji}, \qquad m_k \neq m_i.$$

If $m_j > m_k \geq m_i$, then $j > k$ and, by (2.7), $t_{ji} = 0$, so

$$(2.11) \qquad \sum_{\{j:m_k<m_j\}} a_{kj}t_{ji} = 0, \qquad m_k \geq m_i.$$

Hence, we can take

$$(2.12) \qquad t_{ki} = \delta_{ki}, \qquad m_k = m_i.$$

By (2.10), (2.11) and (2.12), we have t_{ji}, $j = k, \ldots, n$, $i = 1, \ldots, n$, satisfying (2.7) and (2.9_j). Thus, by downward induction on k, we can find T satisfying (2.7) and (2.9_k), and hence (2.8). $\square$

3. Linearization

In this section, we prove the following linearization result.

Theorem 3.1. *If Z is an approximate holomorphic infinitesimal dilation, there is an allowable biholomorphic map ϕ such that $Z_0 = d\phi\, Z$ is a holomorphic infinitesimal dilation.*

By the Poincaré-Dulac Theorem [Po, Deuxième Partie, Théorème III], [Du] (see also [A, Theorem 24E]), there is a biholomorphic map taking Z to a polynomial vector field in which all monomials of degree greater than 1 are resonant. It does not follow from this that Z can be linearized, nor that the map may be taken to be allowable. However, Theorem 3.1 does follow from Poincaré's method of proof. One constructs the map as a formal power series and then uses a domination argument to prove convergence. Once one has a formal power series solution, one could invoke a more general result of Pliss [Pl] to obtain convergence. However, Pliss's proof uses successive approximations and is much more complicated.

Proof. By Lemma 2.3, we may assume that

$$Z = \sum (m_i z_i + h_i(z))\frac{\partial}{\partial z_i}, \qquad h_i \in O(m_i + 1) \cap O(|z|^2).$$

Let $z' = \phi(z) = (\phi_1(z), \ldots, \phi_n(z))$ be an allowable biholomorphic map,

$$\phi_j(z) = z_j + f_j(z), \qquad f_j \in O(m_j + 1).$$

Then

$$(3.2) \qquad d\phi\, Z = \sum_{i,k} \left(m_i z_i + h_i(z)\right)\left(\delta_{ik} + \frac{\partial f_k}{\partial z_i}(z)\right)\frac{\partial}{\partial z_k'}.$$

Hence, if

(3.3) $$d\phi Z = \sum m_k z_k' \frac{\partial}{\partial z_k'},$$

equating coefficients of $\frac{\partial}{\partial z_k'}$ in (3.2) and (3.3) gives

(3.4) $$\sum_i \big(m_i z_i + h_i(z)\big) \frac{\partial f_k}{\partial z_i}(z) = m_k f_k(z) - h_k(z).$$

Because $h_i \in O(m_i + 1) \cap O(|z|^2)$, we look for a solution f_k of (3.4) with $f_k \in O(m_k + 1) \cap O(|z|^2)$. We first find a formal power series solution. Write

(3.5) $$h_j = - \sum_{\substack{|\alpha| \geq 2 \\ ||\alpha|| > m_j}} b_{j\alpha} z^\alpha, \quad f_k = \sum_\alpha a_{k\alpha} z^\alpha.$$

Set $a_{k\alpha} = 0$ if $|\alpha| = ||\alpha|| \leq 1$. Fix $N \geq 2$ and suppose we have found $\{a_{k\gamma} : ||\gamma|| < N\}$ with $a_{k\gamma} = 0$ if $|\gamma| \leq 1$ or $||\gamma|| \leq m_k$. Let $||\alpha|| = N$. Substitute (3.5) into (3.4) and equate the coefficients of z^α. This gives

(3.6) $$(||\alpha|| - m_k) a_{k\alpha} = P_\alpha'(b_{j\beta}, a_{k\gamma} : ||\beta|| \leq N, ||\gamma|| \leq N - 1)$$

where P_α' is a polynomial with nonnegative coefficients and $P_\alpha'(0) = 0$. The terms on the right side only involve $a_{k\gamma}$ with $||\gamma|| \leq N - 1$ and hence are known. If $N \leq m_k$, the right side of (3.6) is 0 so we may take $a_{k\alpha} = 0$. Thus, by induction, we see that

(3.7) $$a_{k\alpha} = P_{k\alpha}(b_{j\beta} : ||\beta|| \leq ||\alpha||)$$

where $P_{k\alpha}$ is a polynomial with nonnegative coefficients, and we can find a formal power series solution f_k to (3.4) such that $a_{k\alpha} = 0$ if $|\alpha| \leq 1$ or $||\alpha|| \leq m_k$.

We can find $K' > 0$ and $r > 0$ such that for all j and α,

$$|b_{j\alpha}| \leq \frac{K'}{r^{|\alpha|}}.$$

Let $K = \frac{K'}{r^2}$ and set

$$\mu(z) = \frac{Kr \left(\sum_{j=1}^n z_j\right)^2}{r - \sum_{j=1}^n z_j} = Kr^2 \sum_{|\alpha| \geq 2} \frac{|\alpha|!}{\alpha! r^{|\alpha|}} z^\alpha.$$

Then the series for μ dominates the series for h_j in $\{|z| < r\}$. Let A denote the set of multiindices α such that $2 \leq |\alpha| < 2m_k$, $||\alpha|| > m_k$, and let

$$c = \min\left(\left\{\frac{||\alpha|| - m_k}{|\alpha| - 1} : \alpha \in A\right\}, \frac{1}{2}\right).$$

Then $c > 0$ because A is finite. Suppose f solves the equation

(3.8) $$\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j} = \frac{\mu}{c}\left(\sum_{j=1}^{n} \frac{\partial f}{\partial z_j} + 1\right) + f$$

with $f(0) = 0$, $df\,|_0 = 0$. Then the coefficient of z^α in the power series for f is given by the right side of (3.7) with $b_{j\beta}$ replaced by $b_\beta = Kr^2 \frac{|\beta|!}{c\beta! r^{|\beta|}}$ and $||\gamma|| - m_k$ replaced by $|\gamma| - 1$. Hence, it is of the form

$$\widetilde{P}_\alpha(b_\beta : |\beta| \leq |\alpha|)$$

where each coefficient of $\widetilde{P}_\alpha$ is larger than the corresponding coefficient of $P_{k\alpha}$ because $c(|\alpha| - 1) < ||\alpha|| - m_k$ if $|\alpha| > 1$, $||\alpha|| > m_k$. Hence, the series for f majorizes the series for f_k.

To solve (3.8), we write $f(z) = \sum_{j=1}^{n} z_j\, v(\sum_{k=1}^{n} z_k)$ and solve the resulting regular ordinary differential equation for v with $v(0) = 0$. This gives

$$f(z) = \frac{1}{n} \sum_{j=1}^{n} z_j\left(\left(1 - (c + Knr)\sum_{k=1}^{n} \frac{z_k}{cr}\right)^{-Knr/(c+Knr)} - 1\right)$$

which solves (3.8) in a neighborhood of 0 with $f(0) = 0$, $df\,|_0 = 0$ and has a convergent power series expansion in the neighborhood. Thus, because the series for f dominates the one for f_k, the series for f_k also converges. $\square$

4. A criterion for homogeneity

In this section, we prove our main result, Theorem 4.1, which gives a criterion for M to be homogeneous. We also show, Proposition 4.3, that a homogeneous hypersurface is rigid.

Theorem 4.1. *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$ and suppose there is an approximate infinitesimal dilation $Y \in \mathrm{hol}(M)$. Then M is weighted homogeneous.*

Proof. Because $Y \in \mathrm{hol}(M)$, $Y = 2\,\mathrm{Re}\,Z$ where Z is an approximate holomorphic infinitesimal dilation. By Theorem 3.1 there is an allowable biholomorphic change of coordinates $\widehat{z} = \phi(z)$ such that

$$d\phi\, Z = \sum m_j \widehat{z}_j \frac{\partial}{\partial \widehat{z}_j}.$$

After reordering the coordinates and replacing $\widehat{z}_n$ by $i\widehat{z}_n$ if necessary, we may assume that M is given by an equation of the form

(4.2) $\operatorname{Im}\widehat{z}_n - F(\widehat{z}', \overline{\widehat{z}'}, \operatorname{Re}\widehat{z}_n) = 0, \qquad \widehat{z}' = (\widehat{z}_1, \ldots \widehat{z}_{n-1}).$

Write $F = \sum F_k$ where F_k is a polynomial homogeneous of weight k. Because Y is tangent to M, applying $d\phi\, Y = Y_0$ to (4.2) gives

$$Y_0\big(\operatorname{Im}\widehat{z}_n - F(\widehat{z}', \overline{\widehat{z}'}, \operatorname{Re}\widehat{z}_n)\big)\big|_M = -\sum_k (k - m_n) F_k(\widehat{z}', \overline{\widehat{z}'}, \operatorname{Re}\widehat{z}_n) = 0.$$

Hence,
$$F_k(\widehat{z}', \overline{\widehat{z}'}, \operatorname{Re}\widehat{z}_n) = 0, \qquad k \neq m_n$$

and F is homogeneous of weight m_n. Thus, (4.2) is a polynomial equation homogeneous of weight m_n and M is weighted homogeneous.   □

Following the terminology of Baouendi, Rothschild and Treves [BRT], we call M *rigid* if there are coordinates $(z_1, \ldots, z_{n-1}, w)$ such that M is given by an equation of the form $\operatorname{Im} w = F(z, \overline{z})$ or, equivalently, if there is a transversal infinitesimal CR automorphism on M. Rigid hypersurfaces are called *regular* by Tanaka [T] and *T-regular* by D'Angelo [DA].

Proposition 4.3. *If M is weighted homogeneous then M is rigid.*

Proof. We can assume that, after a CR diffeomorphism if necessary, M is given by a homogeneous polynomial equation of weight m. As in the proof of Theorem 4.1, we can assume that this equation is of the form

$$\operatorname{Im} z_n = F(z', \overline{z}', \operatorname{Re} z_n), \qquad z' = (z_1, \ldots, z_{n-1}).$$

Then
$$\operatorname{Im} z_n - a\operatorname{Re} z_n = p(z', \overline{z}')$$

for some polynomial p homogeneous of weight m and some $a \in \mathbf{R}$. Let

$$\widehat{z}_j = z_j, \; j = 1.., n-1, \; \widehat{z}_n = \frac{1+a^2}{1+ia} z_n.$$

In the new coordinates, the equation of M is

$$\operatorname{Im}\widehat{z}_n = p(\widehat{z}', \overline{\widehat{z}'}).$$

Hence M is rigid.   □

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