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SMOOTH COCYCLES RIGIDITY FOR LATTICE ACTIONS

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Abstract. We apply the duality method to prove a rigidity theorem for smooth cocycles of actions of higher rank cocompact lattices on a class of homogeneous spaces. In particular, in the split case, we explicitly find all the smooth cocycles of actions of lattices on the imaginary boundaries of symmetric spaces.

1. Introduction

Cocycles of a group action are a fundamental tool for understanding the action. They lie at the heart of many questions about rigidity, existence of invariant structures, and other properties of the action (see [3], [1], [8], [7], [9].)

We developed a new duality method for calculating cocycles of group actions. The main idea is to try to lift the action to a well understood action on a bigger space. This lifting induces a map F on cocycle spaces. Then, naturally, the image and the kernel of this map contain most of the information about the cocycles of the original action. Assuming we have a good understanding of the lifted action, we may hope to easily obtain significant information about its cocycles and thus, the image of F. Then the task becomes to control the kernel of F.

It turns out that the kernel subspaces enjoy some very useful duality properties which make it possible, in some cases, to substitute a question about cocycles of an action by a question about cocycles of another action, which dynamically may be completely different from the original one, and might turn out to be easier to study.

Successful implementation of this program leads to different results on cohomological, differential and infinitesimal rigidity of actions of lattices which will be presented in [4] and [5].

In this paper, we apply our methods to prove a rigidity theorem (Theorem 6.1) for the C^∞ real cocycles of the actions of cocompact lattices in a semi-simple Lie group G of higher rank, without compact factors (with some additional conditions), on $N\backslash G$ for a class of subgroups N in G.

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In the case when N is a minimal parabolic subgroup of G and thus $N\backslash G$ is the maximal boundary of the symmetric space corresponding to G, we obtain a complete description of all C^∞ real cocycles of the Γ action on it (Theorem 7.1).

To prove this result we use our duality to transform the question about cocycles of the Γ action on $N\backslash G$ (which does not seem to have any "nice" properties) into a question about cocycles of the N action on G/Γ (which is "nice" in many respects: it is partially hyperbolic, it preserves the natural measure μ on G/Γ, etc.)

Also, along the way, we prove some facts about trivializations of the cocycles for certain classes of actions (Theorem 4.1 and Theorem 4.2) and some Katok-Spatzier type constant cocycles theorem (Theorem 5.1).

Some results for Hölder cocycles are mentioned in Section 8.

2. General duality theorem

Recall that if a group G acts on a space X then the cocycle of this action with coefficients in a group H is a map $\alpha : G \times X \to H$ such that

$$\alpha(g_1 g_2, m) = \alpha(g_1, g_2(m))\alpha(g_2, m).$$

Two cocycles α and β will be called cohomologous if there exists a function $P : X \to H$ such that

$$\beta(g, m) = P(gm)^{-1}\alpha(g, m)P(m).$$

By $\mathrm{H}(G, X)$ we will denote a set of equivalence classes of cocycles.

Imposing additional regularity conditions on the functions $\alpha(g, \cdot)$ and P (C^∞, measurable, continuous, etc.), in the above definition, we define the spaces of cocycles of the corresponding regularity.

Most of the time we will work with the cocycles with coefficients in $\mathbb{R}$, but it is easy to see that all our results will automatically be true for cocycles with coefficients in $\mathbb{R}^l, l \in \mathbb{N}$ as well. Also, notice that for real cocycles, we will use additive notation.

Suppose that G acts on X_1 and X_2 in such a way that the action on X_2 is a factor of the action on X_1. Let us try to study $\mathrm{H}(G, X_2)$ by lifting its elements to X_1. By $\mathrm{H}^{tr}(G, X_2)$ we will denote a set of those elements of $\mathrm{H}(G, X_2)$ that lift to a trivial element in $\mathrm{H}(G, X_1)$.

Also, assume that there is a certain structure on X_1 and X_2 preserved by the factor map. (It can be a structure of a topological space, measurable space, smooth manifold, etc.) Suppose that we consider some class of cocycles with respect to this structure (for example, continuous cocycles, measurable, smooth, etc.). If the reference to which class of cocycles we are working with is absent, it means that we work with any of the above described classes, which are applicable to the factor maps involved. In that case, cohomologous means cohomologous in the corresponding class.

It turns out that the spaces of H^{tr}-cocycles enjoy the following duality property:

Theorem 2.1. *Let G_1 and G_2 be two groups acting on a space M. Consider cocycles with values in an arbitrary group H.*

If those actions commute, then:

$$H^{tr}(G_1, M/G_2) \cong H^{tr}(G_2, G_1\backslash M).$$

To be more precise, there is a canonically defined map

$$K(G_1, G_2) : H^{tr}(G_1, M/G_2) \to H^{tr}(G_2, G_1\backslash M)$$

such that $K(G_2, G_1) \circ K(G_1, G_2) = Id$ on $H^{tr}(G_1, M/G_2)$.

Where M/G_2 and $G_1\backslash M$ are the factor spaces on which the other group acts in the obvious way, and H^{tr} spaces are defined with respect to the lift of actions to the whole M.

Rather then proving Theorem 2.1, we will prove its particular case, Theorem 3.1, which will be sufficient for all our applications. The proof of Theorem 3.1 is very similar to the proof of Theorem 2.1 but has the advantage of being much more transparent and much less technically involved.

The full proof of Theorem 2.1 will be presented in [4] where, unlike in this article, we will need to use it in its general form.

3. Duality theorem for subgroup actions

Let G be a topological group (Lie group), P and Q its subgroups. Let P act on G/Q from the left, and Q act on $P\backslash G$ from the right. Lift both actions to the whole G.

Then, we prove the following:

Theorem 3.1. *For real cocycles:*

(1) *There is a naturally defined, linear imbedding*

$$K(P, Q) : H^{tr}(P, G/Q) \to H(Q, P\backslash G).$$

(2) *$Im(K(P, Q)) = H^{tr}(Q, P\backslash G)$.*

(3) *$K(P, Q) \circ K(Q, P) = Id$ on $H^{tr}(P, G/Q)$, i.e., $K(P, Q)$ defines a natural duality between $H^{tr}(P, G/Q)$ and $H^{tr}(Q, P\backslash G)$.*

Corollary 3.1. *If*

$$H^{tr}(P,G/Q)=H(P,G/Q) \quad \textit{and} \quad H^{tr}(Q,P\backslash G)=H(Q,P\backslash G),$$

then there is a natural duality between $H(P,G/Q)$ *and* $H(Q,P\backslash G)$.

Remark. Later, we will see quite often that not only are all lifts of the cocycles to the whole G trivial but also that all the cocycles of the action of a subgroup on the whole group are trivial (Theorems 4.2 and the remark at the end of Section 4).

Proof. Let $\alpha : P\times G/Q\to\mathbb{R}$ be a cocycle, β: $P\times G\to\mathbb{R}$ be its lifting to G. Assume that β is trivial. We will say that a function $f:G\to\mathbb{R}$ represents β if $\beta(p,g)=f(g)-f(pg)$.

Let C_P (resp. C_Q) be the set of functions on $P\backslash G$ (resp. G/Q), which is naturally identified with the set of all left P (resp. right Q) invariant functions on G.

Lemma 3.1. *If* f_1 *and* f_2 *represent* β *then* f_1-f_2 *belongs to* C_P.

Proof. If

$$\beta(p,g)=f_1(g)-f_1(pg)=f_2(g)-f_2(pg)$$

then

$$f_1(pg)-f_2(pg)=f_1(g)-f_2(g).$$

So, f_1-f_2 is P-invariant. □

Let f be some function representing β. Then since β is a lift of α, we have

$$\beta(p,g)=\beta(p,gq^{-1})=f(gq^{-1})-f(pgq^{-1}).$$

So, $g\to f(gq^{-1})$ also represents β. So,

$$f(gq^{-1})=f(g)+\phi(q,g),$$

where $\phi(q,g)$ belongs to C_P.

Lemma 3.2. $\phi(q,g)$ *defines a cocycle of the action of* Q *on* $P\backslash G$.

Proof. We have, $f(g(q_1q_2)^{-1})=f(g)+\phi(q_1q_2,g)$. On the other hand,

$$f(gq_2^{-1}q_1^{-1})=f(gq_2^{-1})+\phi(q_1,gq_2^{-1})=f(g)+\phi(q_2,g)+\phi(q_1,gq_2^{-1}).$$

So, $\phi(q_1q_2,g)=\phi(q_1,gq_2^{-1})+\phi(q_2,g)$. □

We have constructed ϕ using a particular f representing β. Now we will show that, in fact, the equivalence class of ϕ is independent from the choice of f.

Lemma 3.3. *If f_1 and f_2 represent β, they generate equivalent ϕ_1 and ϕ_2.*

Proof. If $f_2 = f_1 + \eta$, and η is in C_P, then

$$f_2(gq^{-1}) = f_1(gq^{-1}) + \eta(gq^{-1}) = f_1(g) + \phi_1(q,g) + \eta(gq^{-1})$$
$$= f_2(g) + \phi_1(q,g) + \eta(gq^{-1}) - \eta(g).$$

So, $\phi_2(q,g) = \phi_1(q,g) - \eta(gq^{-1}) + \eta(g)$. □

Thus, we have defined a map $\mathrm{K}(P,Q)$ from $\mathrm{H}^{tr}(P, G/Q)$ to $\mathrm{H}(Q, P\backslash G)$.

From the definition and the formula $f(gq^{-1}) = f(g) + \phi(q,g)$, we see that ϕ is in $\mathrm{H}^{tr}(Q, P\backslash G)$ and is represented by the function $-f$. Applying our construction we have $\beta(p,g) = f(g) - f(pg) = \mathrm{K}(Q,P)(\phi)$, which immediately implies that $\mathrm{K}(P,Q) \circ \mathrm{K}(Q,P) = \mathrm{Id}$ on $\mathrm{H}^{tr}(P, G/Q)$, which proves Theorem 3.1. □

4. Trivializations of cocycles on the whole group

In this section we prove some results about trivializations of cocycles of the action of a subgroup on a group. We will consider only the left action, although all the results are, of course, true for the right action as well. For brevity, we only deal with the C^∞ cocycles, although all the results have natural analogs in other cases. The statements and proofs are completely analogous to the C^∞ case, except for some minor technical differences. So, we will not go into it.

Theorem 4.1. *Let Γ be a discrete group, acting freely and totally discontinuously on a Riemanian manifold M. Assume that*

(1) *It acts by isometries.*
(2) *There exists $\delta > 0$ such that $\forall m \in M$, $\forall \gamma \in \Gamma$, and $\forall \epsilon \leq \delta$, $\gamma B(m,\epsilon)$ are disjoint, where $B(m,\epsilon)$ is an open ball of radius ϵ around m.*

Then $H(\Gamma, M) = 0$.

First we prove two technical lemmas.

Call a set $U = \bigcup_{\gamma\in\Gamma} \gamma B$, where B is an open ball in M such that all γB are pairwise disjoint, a Γ-ball.

Call a radius of B a radius of U.

Call a cover of M by Γ-balls a Γ-cover.

Lemma 4.1. *Under the assumptions of the Theorem 4.1, for every $0 < \epsilon \leq \delta$ there exists a locally finite Γ-cover U_n, $n \in \mathbb{N}$, such that the radii of all U_n are equal to ϵ.*

Proof. Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset K_n \subset \cdots M$ be such that K_n are all compact and $\bigcup_{n=1}^{\infty} K_n = M$.

Then we will construct the Γ-cover U_n in the following way.

Step 1 : take any point $m_1 \in K_1$ and put $U_1 = \bigcup_{\gamma\in\Gamma} \gamma B(m_1, \epsilon)$.

Step k : Let $A = \bigcup_{n=1}^{k-1} U_n$. Suppose $A \bigcap K_i = K_i, i < j$ and $A \bigcap K_j \neq K_j$ for some $j > 0$. Then take any point $m_k \in K_j \backslash (A \bigcap K_j)$ and put $U_k = \bigcup_{\gamma\in\Gamma} \gamma B(m_k, \epsilon)$.

Now, notice that the set $S = \{\gamma m_n, n \in \mathbb{N}, \gamma \in \Gamma\}$ is ϵ-separated set in M. Therefore, $S \bigcap K$ is finite for any compact set $K \subset M$.

Thus, K_i is covered by a finite number of U_n, and so $K_i \subset \bigcup_{n=1}^{\infty} U_n$. Therefore, U_n covers M.

Let us prove that U_n is locally finite. Suppose it is not. Let $x \in M$ be covered by infinitely many U_n. Then x is less than ϵ distant from infinitely many points in S; in other words, $S \bigcap \overline{B(x, \epsilon)}$ is an infinite set, which is impossible. The lemma is proved by contradiction. □

"Blow-up" the cover U_n a little. Namely, fix ϵ_1 such that $\epsilon < \epsilon_1 < \delta$ and consider the new cover $U'_n = \bigcup_{\gamma\in\Gamma} \gamma B(m_n, \epsilon_1)$. Then, from the proof of the Lemma 4.1, we easily see that U'_n is also a locally finite Γ-cover.

Lemma 4.2. *There exists a C^∞ partition of unity e_n, such that*

- $e_n(x) = 0$ *for* $x \overline{\in} U'_n$,
- $e_n(x) > 0$ *for* $x \in U_n$ *and*
- *all functions* e_n *are* Γ*-invariant.*

Proof. Take "bump" functions b_n such that

- b_n is non-negative.
- $b_n(x) > 0$ for $x \in B(m_n, \epsilon)$,
- $b_n(x) = 0$ for $x \overline{\in} B(m_n, \epsilon_1)$.

Then, "spread" b_n. Namely, define

- $e'_n(x) = b_n(\gamma^{-1}x)$ for $x \in \gamma B(m_n, \epsilon_1)$,
- $e'_n(x) = 0$ otherwise.

Then e'_n are Γ-invariant C^∞ functions. Since U'_n is locally finite, $a(x) = \sum e'_n(x)$ is a well defined C^∞ function. Since U_n is also a cover, $a(x) > 0$ for all $x \in M$.

Set $e_n = e'_n/a(x)$ to prove the lemma. □

Now we are ready to prove Theorem 4.1.

Proof. Let $\alpha(\gamma, m)$ be a cocycle, $\gamma \in \Gamma, m \in M$.

Construct functions f_n in the following way:

First define

$$h_n(m) = \begin{cases} \alpha(\gamma, \gamma^{-1}(m)) & \text{if } m \in \gamma B(m_n, \epsilon_1), \\ 0 & \text{otherwise.} \end{cases}$$

Now, put $f_n = e_n h_n$. Then obviously f_n are C^∞ functions.

We have

$$f_n(\gamma m) - f_n(m) = e_n(m)\alpha(\gamma, m).$$

Put $f = \sum_{n=1}^{\infty} f_n$. Then since U_n' is locally finite, f is a well defined C^∞ function on M. Furthermore,

$$f(\gamma m) - f(m) = \sum_{n=1}^{\infty}(f_n(\gamma m) - f(m)) = \sum_{n=1}^{\infty} e_n(m)\alpha(\gamma, m) = \alpha(\gamma, m).$$

Therefore, $-f$ trivializes α. □

Remark. It is easy to see that the assumptions 1) and 2) of Theorem 4.1 are not really essential for it to be true. All we need, is the existence of the partition of unity by Γ-invariant functions subordinate to the locally finite cover $U_n = \bigcup_{\gamma \in \Gamma} \gamma B_n$, where B_n are open sets such that γB_n, $\gamma \in \Gamma$ are disjoint.

This observation leads to many analogs of the Theorem 4.1 for different classes of totally discontinuous actions. For example, the assumptions 1) and 2) can be substituted by an assumption that the factor M/Γ is compact.

Also, note that if M has transitive group of isometries, then assumption 2) follows from the other assumptions of Theorem 4.1.

We will not discuss the other versions of Theorem 4.1, since we only use it to get the following:

Theorem 4.2. *Let G be a Lie group, Γ a discrete subgroup, acting on G from the left. Then $H(\Gamma, G) = 0$.*

Proof. Put a left invariant Riemanian metric on G and apply Theorem 4.1. □

Remark. Another useful observation about trivializations of cocycles of subgroup actions on groups is the following: assume that G is a Lie group, P a Lie subgroup, that the orbit foliation for the left action of P on G admits a global transverse section T, which is a submanifold of G, and that every g in G decomposes as $p(g)t(g)$, where $p(g) \in P$, $t(g) \in T$, and $p(g), t(g) : G \to G$ are C^∞.

Let β be a cocycle, such that $\beta : P \times G \to \mathbb{R}$ is C^∞ (which, of course, is much more then is required in the definition of C^∞ cocycle), then $f(g) = -\beta(p(g), t(g))$ trivializes β. Indeed,

$$f(g) - f(pg) = \beta(pp(g), t(g)) - \beta(p(g), t(g)) = \beta(p, p(g)t(g)) = \beta(p, g).$$

Note that all elements of H^{tr}-subspaces have representatives that satisfy the above assumption.

5. Constant cocycles theorem

For the rest of the paper, G is a semi-simple connected Lie group of $\mathbb{R}$-rank $n \geq 2$, with finite center and without compact factors. Let $\mathcal{G}$ be its Lie algebra, A a connected component of the maximal split Cartan subgroup, $\mathcal{A}$ the corresponding commutative subalgebra, N a closed subgroup containing A, $\mathcal{N}$ the corresponding subalgebra, λ_i the roots with respect to $\mathcal{A}$ and V_{λ_i} the corresponding root spaces.

Let

$$W_{\lambda_i} = \bigoplus_{c \in \mathbb{R}} V_{c\lambda_i}.$$

The elements of W_{λ_i} we will denote by $\mathfrak{g}_{\lambda_i}$. Let Γ be an irreducible cocompact lattice in G.

We will prove the following:

Theorem 5.1. *Let G be a semi-simple connected Lie group of $\mathbb{R}$-rank $n \geq 2$, with finite center and without compact factors, A a connected component of the maximal split Cartan subgroup, N a closed subgroup containing A. Let Γ be an irreducible cocompact lattice in G.*

(*) Also we will assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m, 1)$ or $\mathfrak{su}(m, 1)$.

Then all C^∞ cocycles of the action of N on G/Γ are C^∞ cohomologous to constant cocycles.

Remark. The (*) condition is needed due to a non-uniformity in certain estimates on the decay of correlation coefficients for unitary representations of groups $\mathfrak{so}(m, 1)$ and $\mathfrak{su}(m, 1)$. It is quite possible that for the particular representations considered in the proof of Theorem 5.2, sufficient estimates still can be obtained and then the (*) condition will disappear from all our results.

First we will prove two lemmas which allow us to reduce the proof of Theorem 5.1 to the case when N is connected. Let N_0 be the connected component of identity.

Lemma 5.1. *If a cocycle α is trivial on N_0, it is constant on the whole N.*

Proof. Let $p \in N$ and $\overline{p}$ be the corresponding coset in $N_0 \backslash N$. Consider the right action of A on $N_0 \backslash N$. Since $N_0 \backslash N$ is not more then countable, the stabilizer of $\overline{p}$ is not empty, i.e., there exists $a \in A$ such that $\overline{pa} = \overline{p}$. In other words, there exists $p_1 \in N_0$ such that $p_1 p = pa$. Then we have

$$\alpha(pa, m) = \alpha(p, am) + \alpha(a, m) = \alpha(p_1 p, m) = \alpha(p_1, pm) + \alpha(p, m)$$

where $m \in G/\Gamma$; so $\alpha(p, am) = \alpha(p, m)$.

Since a has an infinite order, by Moore's ergodicity theorem (see [8]), it acts ergodicly on G/Γ. Thus, it acts topologically transitively (since every open set has positive measure), i.e., it has a dense orbit. Therefore $\alpha(p, \cdot)$ is constant on G/Γ. □

Lemma 5.2. *If $\alpha(p, \cdot)$ is constant for every $p \in N_0$, then it is constant for all $p \in N$.*

Proof. Let μ be the normalized Haar measure on G/Γ. Let

$$\beta(p) = \int_{G/\Gamma} \alpha(p, m) d\mu.$$

Then, due to the invariance of μ, we get

$$\begin{aligned} (1) \quad \beta(p_1 p_2) &= \int_{G/\Gamma} \alpha(p_1, p_2 m) d\mu + \int_{G/\Gamma} \alpha(p_2, m) d\mu \\ &= \int_{G/\Gamma} \alpha(p_1, m) d\mu + \int_{G/\Gamma} \alpha(p_2, m) d\mu = \beta(p_1) + \beta(p_2). \end{aligned}$$

So $\beta(p)$ is a one-dimensional representation of N, i.e., a constant cocycle.

Let us prove that $\beta(p) = \alpha(p, m)$. Consider $\theta(p, m) = \alpha(p, m) - \beta(p)$. Then θ is a cocycle, and for $p \in N_0$, $\theta(p, m) = 0$. So, by the Lemma 5.1, $\theta(p, m)$ is constant for all p. But from its definition as $\alpha(p, m) - \beta(p)$, we see that it is orthogonal to the constants. Thus, $\theta = 0$, and $\alpha(p, m) = \beta(p)$. □

To prove Theorem 5.1, it will be enough to prove it for N connected. Indeed, then we will know that every cocycle is cohomologous to a cocycle constant on N_0, and thus constant on the whole N by the Lemma 5.2.

We start by proving the following simple structural lemma:

Lemma 5.3. *If $\mathcal{N}$ is a subalgebra containing $\mathcal{A}$, then it is generated as a vector space (and thus as an algebra) by a set of $\mathfrak{g}_{\lambda_i} \in W_{\lambda_i}$.*

Proof. It will be enough to prove that if $v \in \mathcal{N}$ and

$$v = \mathfrak{g}_0 + \sum_{i=1}^{k} \mathfrak{g}_{\lambda_i},$$

with $\lambda_i \neq 0$, and λ_i and λ_j not proportional for $i \neq j$, then every $\mathfrak{g}_{\lambda_i} \in \mathcal{N}$. Let us prove that $\mathfrak{g}_{\lambda_k} \in \mathcal{N}$. Pick up $a_m \in \mathrm{Ker}(\lambda_m)$ and $a_m \overline{\in} \mathrm{Ker}(\lambda_i), i \neq m$. Then

$$[a_{k-1}, [a_{k-2}, [\dots [a_1, v] \dots]]] = \lambda_k(a_1)\lambda_k(a_2) \cdots \lambda_k(a_{k-1})\mathfrak{g}_{\lambda_k} \in \mathcal{N}.$$

So, $\mathfrak{g}_{\lambda_k} \in \mathcal{N}$. □

To prove Theorem 5.1 when N is connected, we will need the following Theorem due Katok and Spatzier ([2]):

Theorem 5.2. *For A, G and Γ as in Theorem 5.1, every C^∞ cocycle $\alpha : A \times G/\Gamma \to \mathbb{R}$ is C^∞ cohomologous to a constant cocycle.*

Now, we are ready to finish the proof of Theorem 5.1.

Proof. Assume that N is connected. By Theorem 5.2, α is C^∞ cohomologous to a cocycle constant on A. Let us prove that it will then be constant on the whole N. First of all, after subtracting integral α's like in the proof of the Lemma 5.2, we can assume that $\alpha(a, m) = 0, \forall a \in A, m \in G/\Gamma$.

Suppose that $p \in N$ commutes with some element $a \in A$. Then we have

$$\alpha(ap, m) = \alpha(a, pm) + \alpha(p, m) = \alpha(p, am) + \alpha(a, m).$$

Thus, $\alpha(p, m) = \alpha(p, am), m \in G/\Gamma$, i.e., $\alpha(p, \cdot)$ is constant on orbits of a on G/Γ .

By the same ergodicity argument as in the Lemma 5.1, we conclude that $\alpha(p, \cdot)$ is constant on G/Γ.

Let $p \in \mathrm{Exp}(W_{\lambda_i} \bigcap \mathcal{N})$. Take $a \in \mathrm{Exp}(\mathrm{Ker}\lambda_i)$. Then the previous argument applies, and thus, $\alpha(p, \cdot)$ is constant.

If $\alpha(p_1, \cdot)$ and $\alpha(p_2, \cdot)$ are constant, then $\alpha(p_1 p_2, \cdot)$ is also constant. Indeed,

$$\alpha(p_1 p_2, m) = \alpha(p_1, p_2 m) + \alpha(p_2, m) = \alpha(p_1, m) + \alpha(p_2, m).$$

Now, to conclude the proof, we note that due to Lemma 5.3, every p in a small enough neighborhood of the unit in N can be represented as a finite product of elements from $\mathrm{Exp}(W_{\lambda_i} \bigcap \mathcal{N})$, and every element of N can be represented as a finite product of elements from that neighborhood. □

6. Rigidity theorem

We are now ready to prove the following rigidity theorem for C^∞ cocycles of Γ action on $N\backslash G$:

Theorem 6.1. *Let G be a semi-simple connected Lie group of $\mathbb{R}$-rank $n \geq 2$, with finite center and without compact factors, A a connected component of the maximal split Cartan subgroup, N a closed subgroup containing A. Let Γ be an irreducible cocompact lattice in G.*

(*) Also we will assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m,1)$ or $\mathfrak{su}(m,1)$.

Then every C^∞ cocycle of the Γ action on $N\backslash G$ uniquely extends to a cocycle of the G action.

Proof. By the Theorem 4.2, $\mathrm{H}(\Gamma, G) = 0$, and so

$$\mathrm{H}(\Gamma, N\backslash G) = \mathrm{H}^{tr}(\Gamma, N\backslash G) \cong \mathrm{H}^{tr}(N, G/\Gamma).$$

On the other hand, $\mathrm{H}(G, G) = 0$ (every cocycle $\alpha(g_1, g_2)$ is trivialized by a function $\alpha(g, e)$, where e is the unit element in G), and thus,

$$\mathrm{H}(G, N\backslash G) = \mathrm{H}^{tr}(G, N\backslash G) \cong \mathrm{H}^{tr}(N, G/G).$$

However, by Theorem 5.1, $\mathrm{H}(N, G/\Gamma) = \widehat{N}$, where $\widehat{N}$ is the space of 1-dimensional real representations of N. Also, obviously, $\mathrm{H}(N, G/G) = \widehat{N}$.

So, both $\mathrm{H}^{tr}(N, G/G)$ and $\mathrm{H}^{tr}(N, G/\Gamma)$ are isomorphic to the space $\widehat{N}^{tr}_G$ of 1-dimensional representations of N which give rise to trivial cocycles of the action of N on G.

Also, it is easy to see that given an element $\pi \in \widehat{N}^{tr}_G$, the duality produces such cocycles $\alpha \in \mathrm{H}(\Gamma, N\backslash G)$ and $\beta \in \mathrm{H}(G, N\backslash G)$ that β is an extension of α. □

From the proof we easily get:

Corollary 6.1.

$$H(\Gamma, N\backslash G) = H(G, N\backslash G) = \widehat{N}^{tr}_G.$$

7. Application to the Γ action on the maximal boundary

Assume G is split. If P is a minimal parabolic subgroup, then $P\backslash G$ is the Furstenberg's maximal boundary of the symmetric space corresponding to G (see [6]). We can easily find explicit formulas for all the cocycles of the Γ action on it.

By the Iwasawa decomposition, G splits into a topological direct product $G = KP$, where K is the connected subgroup corresponding to the maximal compactly imbedded subalgebra in G. And, we can use $T = K$

as the cross section described in the remark at the end of Section 4. Since the remark is obviously applicable to constant cocycles, we get $\widehat{P} = \widehat{P}_G^{tr}$. So, by the Corollary 6.1, we have:

$$\mathrm{H}(\Gamma, P\backslash G) \cong \widehat{P}_G^{tr} = \widehat{P}.$$

Notice that $\widehat{P}$ is naturally identified with the $\widehat{A}$. Moreover, using our duality construction, we can find the explicit formulas for all Γ-cocycles on $P\backslash G$:

Theorem 7.1. *Let G be a semi-simple connected Lie group of $\mathbb{R}$-rank $n \geq 2$, with finite center and without compact factors. Assume G is split. Let P be a minimal parabolic subgroup. Let Γ be an irreducible cocompact lattice in G.*

(*) Also we will assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m,1)$ or $\mathfrak{su}(m,1)$.

Then, for C^∞ cocycles, we have:

(1) *$H(\Gamma, P\backslash G)$ is n dimensional.*
(2) *By the Iwasawa decomposition, G splits into a topological direct product $G = KP$, where K is the connected subgroup corresponding to the maximal compactly imbedded subalgebra in G. So, every $g \in G$ is uniquely represented as $g = p(g)t(g)$, where $p(g) \in P$ and $t(g) \in K$. Then $H(\Gamma, P\backslash G)$ is isomorphic to $\widehat{P}$, and the cocycle corresponding to the representation $\pi \in \widehat{P}$ is*

$$\alpha(\gamma, g) = \pi(p(g\gamma^{-1})) - \pi(p(g)).$$

Remark. We are abusing the notation writing $\alpha(\gamma, g)$ for a cocycle on $P\backslash G$, but it is easy to check that since π is a representation of P, $\alpha(\gamma, g)$ depends only on the coset of g in $P\backslash G$, and thus, really defines a cocycle on $P\backslash G$.

8. Hölder cocycles

If G is split, the A action on G/Γ is not just partially hyperbolic but Anosov. This allows Katok and Spatzier to get the Hölder analog of the Theorem 5.2 (see [1]).

Then, for split G, we can get the Hölder analogs of Theorems 5.1, 6.1 and 7.1. In particular, we get the following regularity result:

Theorem 8.1. *If G is split, then every Hölder cocycle of the Γ action on $N\backslash G$ is Hölder cohomologous to a C^∞ cocycle.*

The proofs are completely parallel to the proofs of the corresponding C^∞ results, except for the fact that in the appropriate moment we should use the Hölder analog of Theorem 5.2.

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