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A CONJECTURED ANALOGUE OF DEDEKIND'S ETA FUNCTION FOR $K3$ SURFACES

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Abstract. A fundamental formula in the study of elliptic functions is the product formula for Dedekind's eta function or, equivalently, for the holomorphic cusp form on the upper half plane **h** which is of weight 12 with respect to the action by $PSL(2,\mathbf{Z})$. A related formula expresses the determinant of the Laplacian which acts on the space of smooth functions on an elliptic curve with a period of the elliptic curve and the Dedekind eta function. In [JT 94a], we constructed a holomorphic function on the moduli space of marked, polarized, algebraic $K3$ surfaces of fixed degree using determinants of Laplacians. The aim of this article is to state a conjecture which expresses a product formula for this holomorphic form. In addition, we will present speculative relations with the representation theory of the Mathieu group M_{24}, as well as state many other problems currently under investigation.

1. Determinants and the eta function for elliptic curves

Given a complex number $\tau = a + ib$ with $b > 0$, let Λ_τ be the 2-dimensional lattice in $\mathbf{C}$

$$\Lambda_\tau = \{n + m\tau \mid n, m \in \mathbf{Z}\},$$

and let E_τ be the elliptic curve $E_\tau = \mathbf{C}/\Lambda_\tau$. View E_τ as a Riemannian manifold with flat metric of area one. If $\{\lambda_n\}$ denotes the sequence of positive eigenvalues of the Laplacian which acts on smooth functions on E_τ, then the associated spectral zeta function is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_\tau(s) = \sum \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum e^{-\lambda_n t} t^s \frac{dt}{t}.$$

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If the metric is represented by $\mu(z) = \frac{i}{2b} dz \wedge d\bar{z}$, then the associated Laplacian is

$$\Delta = -4b \frac{\partial^2}{\partial z \partial \bar{z}}.$$

The eigenvalues are explicitly computable, and the spectral zeta function is

$$\zeta_\tau(s) = (2\pi)^{-2s} \sum_{(n,m)\neq(0,0)} \frac{b^s}{|n\tau + m|^{2s}}.$$

The function $\zeta_\tau(s)$ is the classical non-holomorphic Eisenstein series on the hyperbolic upper half plane $\mathbf{h}$ with respect to the group $PSL(2, \mathbf{Z})$, and its analytic continuation and special values are well known (see, for example, [La 87] or [W 76]). Let $q_\tau = \exp(2\pi i \tau)$, and let $\eta(\tau)$ denote the Dedekind eta function

$$\eta(\tau) = q_\tau^{1/24} \prod_{n=1}^{\infty} (1 - q_\tau^n).$$

A direct application of Kronecker's first limit formula yields the equation

$$\exp(-\zeta_\tau'(0)) = b|\eta(\tau)|^4.$$

Following standard conventions, the special value $\exp(-\zeta_\tau'(0))$ of the spectral zeta function is known as the determinant of the Laplacian, and we shall write

$$\exp(-\zeta_\tau'(0)) = \det {}^*\Delta.$$

The asterisk reflects the fact that the zero eigenvalue has been omitted in the definition of the spectral zeta function. Let $\langle dz, dz \rangle$ denote the L^2 norm squared of the holomorphic 1-form dz on E_τ. With this, we can write the above formula as

$$\frac{\det^* \Delta}{\langle dz, dz \rangle} = |\eta(\tau)|^4.$$

This expression lead us to seek and obtain in [JT 94a] and [JT 94b] similar formulae in the setting of polarized, algebraic $K3$ surfaces, which we now shall briefly recall.

2. Basic properties of $K3$ surfaces

Let us review some basic properties of $K3$ surfaces. For a more general and complete discussion, the reader is referred to [Ast 85].

A $K3$ surface X is a compact, complex 2-dimensional manifold with the following properties:

a) There exists a nonzero holomorphic 2-form ω;
b) $H^1(X, \mathcal{O}_X) = 0$.

For the purposes of this article, we will assume that all surfaces are projective varieties. From the defining properties, one can prove that the canonical bundle on X is trivial. In [Sh 67], the following properties of $K3$ surfaces are proved. The surface X is simply connected, and the homology group $H_2(X, \mathbf{Z})$ is a torsion free abelian group of rank 22. The intersection form $\langle\ ,\rangle$ on $H_2(X, \mathbf{Z})$ has the properties:

a) $\langle u, u\rangle = 0 \mod (2)$;
b) $\det(\langle e_i, e_j\rangle) = -1$, where $\{e_i\}$ is a basis of $H_2(X, \mathbf{Z})$;
c) The symmetric form $\langle\ ,\ \rangle$ has signature (3,19).

Theorem 5 from page 54 of [Ser 76] implies that the Euclidean lattice $H_2(X, \mathbf{Z})$ is isomorphic to the $K3$ lattice Λ, i.e.,

$$H_2(X, \mathbf{Z}) \cong \Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 \oplus (-E_8)^2.$$

Let X be a $K3$ surface, and let $\alpha = \{\alpha_i\}$ be a basis of $H_2(X, \mathbf{Z})$ with intersection matrix Λ. The pair (X, α) is called a marked $K3$ surface. Let $e \in H^{1,1}(X, \mathbf{R})$ be the class of a hyperplane section. The triple (X, α, e) is called a marked, polarized $K3$ surface.

The period map π for a marked $K3$ surface (X, α) is defined by integrating the holomorphic 2-form ω along the basis α of $H_2(X, \mathbf{Z})$, meaning

$$\pi(X, \alpha) = (\ldots, \int_{\alpha_i} \omega, \ldots).$$

The Riemann bilinear relations hold for $\pi(X, \alpha)$, that is

$$\langle \pi(X, \alpha), \pi(X, \alpha)\rangle = 0 \quad \text{and} \quad \langle \pi(X, \alpha), \overline{\pi(X, \alpha)}\rangle > 0.$$

The subvariety in $\mathbf{P}(H^2(X, \mathbf{Z}) \otimes \mathbf{C})$ described by the Riemann bilinear relations is isomorphic to $SO_0(3, 19)/SO(2) \times SO(1, 19)$. Following Piatetski-Shapiro and Shafarevich [PSS 71], the results from Burns and Rapoport [BR 75] and Todorov [To 80] combine to prove that the space of all isomorphism classes of marked, polarized $K3$ surfaces (algebraic or not) is in one-to-one correspondence with $SO_0(3,19)/SO(2)\times SO(1,19)$. If a $K3$ surface is not algebraic, one defines a polarization as a class in $H^{1,1}(X,\mathbf{R})$ lying in the Kähler cone of X (see [To 80]).

If (X,e,α) is an algebraic, marked, polarized $K3$ surface, the degree of the polarization is the integer d such that $2d-2=\langle e,e\rangle$. From rsults in [PSS 71] and [Ku 77] we have that the moduli space of isomorphism classes of marked, polarized, algebraic $K3$ surfaces of a fixed degree is equal to an open dense set in the symmetric space

$$\mathbf{h}_{2,19}=SO_0(2,19)/SO(2)\times SO(19).$$

Let

$$\Gamma_e=\{\phi\in \ \mathrm{Aut}(H^2(X,\mathbf{Z}))\mid\langle\phi(u),\phi(v)\rangle=\langle u,v\rangle \text{ and } \phi(e)=e\}.$$

The moduli space of isomorphism classes of polarized, algebraic $K3$ surfaces of a fixed degree, which we denote by $\mathcal{M}_{(X,e)}$, is isomorphic to a Zariski open set in the quasi-projective variety $\Gamma_e\backslash\mathbf{h}_{2,19}$. If we allow our surfaces to have singularities which are at most double rational points, then the corresponding moduli space of isomorphism classes of marked, polarized, algebraic surfaces is equal to the entire symmetric space $\mathbf{h}_{2,19}$.

Using the period map, one can give the following description of $\mathbf{h}_{2,19}$. Let $\langle\ ,\rangle$ denote the bilinear form defined by the cup product on the second cohomology group $H^2(X,\mathbf{Z})$. Then $\mathbf{h}_{2,19}$ is described by

$$\mathbf{h}_{2,19}=\{\langle u,u\rangle=0,\langle u,\bar{u}\rangle>0 \text{ and } \langle u,e\rangle=0\}\subset\mathbf{P}(H^2(X,\mathbf{Z})\otimes\mathbf{C}).$$

Let $\pi_{\mathrm{mar},e}:M_{(X,\alpha,e)}\to M_{(X,e)}$ be the natural map which forgets the marking. From the surjectivity of the period map, it follows that $\pi_{\mathrm{mar},e}$ coincides with the action of Γ_e on $M_{(X,\alpha,e)}$.

3. A canonical family of holomorphic 2-forms

In the case of elliptic curves, one has a specific family of holomorphically varying holomorphic 1-forms, namely the family $\{dz\}$. The construction of a canonical family of holomorphically varying holomorphic 2-forms can be described as follows:

1. Any marked, polarized, algebraic $K3$ surface is an element of a semi-stable family of $K3$ surfaces $\mathcal{E}\to D$, where D is the unit disc, such that the monodromy has a Jordan cell of dimension 3, i.e., if

T is the monodromy operator, on $H_2(X, \mathbf{Z})$, then $(T - \mathrm{id})^3 = 0$ and $(T - \mathrm{id})^2 \neq 0$ (see [To 76] and [JT 94a] for details).

2. On the generic fibre X_t of this family, we have, up to sign, a unique cycle γ such that $T\gamma = \gamma$ and any other T-invariant cycle is an integer multiple of γ. Further, there exists a cycle μ such that $T\mu = \gamma + \mu$.
3. Since $\mathbf{h}_{2,19}$ is contractable, there exists a globally defined, non-vanishing, holomorphically varying family of holomorphic 2-forms, say

$$\omega_t \in H^0(\mathbf{h}_{2,19}, \pi_* \mathcal{K}_{\mathcal{E}_{(X,\alpha)}/\mathbf{h}_{2,19}}).$$

4. In [JT 94a], it is shown that the function

$$\phi(t) = \int_\gamma \omega_t$$

is non-vanishing on $\mathbf{h}_{2,19}$.

To prove (4), one can argue as follows. The points in the space of all marked $K3$ surfaces for which ϕ vanishes is a hyperplane on which the set of Kummer surfaces is an everywhere dense set (see, for example, page 256 of [BPV 84]). Hence, it suffices to show that ϕ is nonzero for any Kummer surface. This computation is elementary and is presented in section 5 of [JT 94a], thus completing the proof of (4).

We define $\{\omega_t/\phi(t)\}$ to be the canonical family of holomorphic 2-forms. We remark that when following the identical steps in the case of elliptic curves, one constructs the family of holomorphic 1-forms $\{dz\}$.

4. Definition of the analytic discriminant for $K3$ surfaces

Let $\mathcal{T}_{(X,e)}$ be the sheaf of holomorphic vector fields on (X, e). By Kodaira-Spencer deformation theory, we identify the tangent space $T_{\mathcal{M}_{(X,e)}}$ of the moduli space of $\mathcal{M}_{(X,e)}$ at the point (X, e) with $H^1(X, \mathcal{T}_{(X,e)})$. The existence of the nonzero holomorphic 2-form ω on X implies that we can identify $H^1(X, \mathcal{T}_{(X,e)})$ with $H^1(X, \Omega^1)$. One can then deduce that the tangent space $T_{\mathcal{M}_{(X,\alpha,e)}}$ to the moduli space $\mathcal{M}_{(X,\alpha,e)}$ at the point (X, α, e) can be identified with the space

$$H^1(X, \Omega^1)_0 = \{u \in H^1(X, \Omega^1) \mid \langle u, e \rangle = 0\}.$$

We view any $\phi \in H^1(X, \mathcal{T}_{(X,e)})$ as a linear map from $\Omega^{1,0}$ to $\Omega^{0,1}$ pointwise on X. Given ϕ_1 and ϕ_2 in $H^1(X, T_{(X,e)})$, the trace of the map

$$\phi_1 \overline{\phi_2} : \Omega^{0,1} \to \Omega^{0,1}$$

at a point $x \in X$ with respect to the unit volume Calabi-Yau metric g (meaning a Kähler-Einstein metric compatible with the given polarization class e) is simply

$$\mathrm{Tr}(\phi_1\overline{\phi_2})(x) = \sum_{k,l,m,n} (\phi_1)_l^k(\overline{(\phi_2)_n^m})g^{n\bar{l}}g_{k\bar{m}}.$$

We define the Weil-Petersson metric on $\mathcal{M}_{(X,\alpha,e)}$ via the inner product

$$\langle\phi_1, \phi_2\rangle = \int_X \mathrm{Tr}(\phi_1\overline{\phi_2})\mathrm{vol}_g$$

on the tangent space $H^1(X, T_{(X,\alpha,e)})$ of $M_{(X,\alpha,e)}$ at (X, α, e). It is shown in [To 89] that the Weil-Petersson metric on $M_{X,\alpha,e}$ is equal to the restriction of the Bergman metric on $\mathbf{h}_{2,19}$. Therefore, the Weil-Petersson metric is a Kähler metric with Kähler form μ_{WP}.

For any holomorphic 2-form ω on X, let

$$\|\omega\|_{L^2}^2 = \langle\omega, \omega\rangle = \int\limits_X \omega \wedge \bar{\omega}.$$

In [To 89] and [Ti 88] it was proved that $\log \|\omega\|_{L^2}^2$ is a potential for the Weil-Petersson metric. The following theorem from [JT 94a] proves the existence of a second potential for the Weil-Petersson metric.

Theorem 4.1. *Let (X, e) be a polarized, algebraic $K3$ surface of degree d, and let μ denote the unit volume Kähler-Einstein form compatible with the given polarization. Let $\det^* \Delta_{(X,e)}^{(0,1)}$ be the determinant of the Laplacian which acts on the space of smooth $(0,1)$-forms on X. Let $\{\omega_{(X,e,\alpha)}\}$ be the normalized family of holomorphically varying $\Omega^{(0,1)}$ 2-forms on the moduli space $\mathcal{M}_{(X,\alpha,e)}$. Then*

$$dd^c \log \left(\frac{\det^* \Delta_{(X,e)}^{(0,1)}}{\|\omega_{(X,\alpha,e)}\|_{L^2}^2} \right) = 0,$$

or, equivalently

$$-dd^c \log \det{}^* \Delta_{(X,e)}^{(0,1)} = -dd^c \log \|\omega_{(X,\alpha,e)}\|_{L^2}^2 = \mu_{\mathrm{WP}}.$$

In other words, $-\det^* \Delta_{(X,e)}^{(0,1)}$ is a potential for the Weil-Petersson metric on $\mathcal{M}_{(X,\alpha,e)}$.

From results in [Ko 88], we have that $N_e = \#(\Gamma_e/[\Gamma_e, \Gamma_e])$ is finite. With this, we can follow the pattern observed for elliptic curves to define an analytic discriminant for polarized, algebraic $K3$ surfaces. In [JT 94a] and [JT 94b] we proved the following theorem.

Theorem 4.2. *Let* $\mathcal{D}_e = (\Gamma_e \backslash \mathbf{h}_{2,19}) \backslash \mathcal{M}_{(X,e)}$. *Then there is a holomorphic function* f_e *on* $\mathbf{h}_{2,19}$ *which vanishes on* $\mathcal{D}_e$ *such that*

$$|f_e([X,\alpha,e])| = \left(\frac{\det^* \Delta^{(0,1)}_{(X,e)}}{\|\omega_{(X,\alpha,e)}\|^2_{L^2}} \right);$$

whence f_e *does not vanish on* $\mathcal{M}_{(X,\alpha,e)}$. *Moreover,* $f_e^{N_e}$ *is an automorphic form on* $\mathbf{h}_{2,19}$ *with respect to the action by* Γ_e.

In section 8 below we shall discuss further properties of f_e as an automorphic form.

By Theorem 4.2, we can view $f_e^{N_e}$ as a section of the line sheaf

$$(\pi_* \mathcal{K}_{\mathcal{X}/\mathcal{M}_{(X,e)}})^{N_e}$$

which vanishes precisely on $\mathcal{D}_e$. The holomorphic function f_e defined as in Theorem 4.2 will be called the analogue of a Dedekind η function for polarized, algebraic $K3$ surfaces.

To conclude, let us remark that $\mathcal{D}_e$ can be realized as the moduli space of algebraic $K3$ surfaces whose polarization e is of degree d and is such that any associated projective embedding has singular double rational points. Further discussion of this point, together with interpretations in terms of singularities of the Calabi-Yau metrics, is given in [KT 87].

5. Realization of $\Gamma_e \backslash \mathbf{h}_{2,19}$ as a tube domain

In order to state a product formula for the analytic discriminant defined in Theorem 4.2, we need a specific realization of the symmetric space $\mathbf{h}_{2,19}$ as a tube domain; that is, we need to define a convex cone V^+ in $\mathbf{R}^{19}$ and represent $\mathbf{h}_{2,19}$ as $\mathbf{R}^{19} + \sqrt{-1}V^+$. For this, we will follow the approach in [In 82], [PS 69], and [To 94]. To begin, we need two elementary lemmas in linear algebra. Throughout this section, we use the notation

$$\mathbf{h}_{2,n} = SO_0(2,n)/SO(2) \times SO(n).$$

Lemma 5.1. *Let* $\langle\ ,\ \rangle$ *be a symmetric bilinear form on* $\mathbf{R}^{n+2}$ *which has signature* $(2,n)$. *Let*

$$A = \{u \in \mathbf{P}(\mathbf{R}^{n+2} \otimes \mathbf{C}) \mid \langle u,u \rangle = 0 \ \text{ and } \ \langle u,\bar{u} \rangle > 0\}.$$

Then A *is isomorphic to*

(a) $\mathbf{h}_{2,n}$;

(b) $\{E \subset \mathbf{R}^{n+2} \mid \dim E = 2,\ E \text{ is oriented, and } \langle\ ,\ \rangle|_E > 0\}$.

Proof. For the proof see [To 80]. □

Lemma 5.2. *Let $Q = [\ ,\ ]$ be a bilinear form on $\mathbf{R}^n (n > 1)$ of signature $(1, n-1)$. Let*

$$V = \{v \in \mathbf{R}^n \mid [v, v] > 0\},$$

and let V^+ be one of the components of V. Then $\mathbf{R}^n + \sqrt{-1}V^+$ is isomorphic to $\mathbf{h}_{2,n}$.

Proof. Let $H(V) = \mathbf{R}^n + \sqrt{-1}\mathbf{V}^+ \subset \mathbf{R}^n + \sqrt{-1}\mathbf{R}^n = \mathbf{C}^n$. Define the map

$$\psi : H(V) \to \mathbf{CP}^{n+1} \quad \text{by} \quad u \mapsto (u_1, \dots, u_n, -1/2[u, u], 1) \in \mathbf{CP}^{n+1}.$$

Consider a new symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbf{R}^{n+2}$ by

$$\langle\ ,\ \rangle = (\ ,\ ) \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which naturally extends to a hermitian form over $\mathbf{C}^{n+2} = \mathbf{R}^{n+2} \otimes \mathbf{C}$. It is immediate that the signature of $\langle\ ,\ \rangle$ is $(2, n)$. Moreover, we have

$$\langle \psi(u), \psi(u) \rangle = 0 \quad \text{and} \quad \langle \psi(u), \overline{\psi(u)} \rangle > 0.$$

By Lemma 5.1(a), ψ is an embedding of $H(V)$ into $\mathbf{h}_{2,n}$. It remains to prove that ψ is surjective. For this purpose, it suffices to establish the following observation. For any $g \in SO_0(2, n)$, the last coordinate of $g(\psi(u))$ is different from zero.

Assume $g(\psi(u)) = (v, v_{n+1}, 0)$. Since

$$\langle g\psi(u), g\psi(u) \rangle = 0 \quad \text{and} \quad \langle g\psi(u), \overline{g\psi(u)} \rangle > 0,$$

we then conclude

$$[v, v] = 0 \text{ and } [v, \bar{v}] > 0.$$

From Lemma 5.1(b) it follows that the 2-dimensional subspace in $\mathbf{R}^n$ spanned by $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ is such that $[\ ,\ ]$ restricted to this space is strictly positive. However, this is impossible since $[\ ,\ ]$ has a signature $(1, n-1)$. Therefore, the last component of $g(\psi(u))$ is never zero, and the proof of Lemma 5.2 is complete. □

Recall that the lattice of primitive cohomology classes of degree d is defined by

$$H_2^d = (-E_8)^2 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \oplus e\mathbf{Z}, \text{ where } \langle e, e \rangle = 2d - 2.$$

Choose a point in $\bar{\mathcal{M}}_{X,a,e} \setminus \mathcal{M}_{X,a,e}$ with maximal unipotent stablizer (see [JT 94a]). Such a point defines three cycles in H_2^d, say γ_0, γ_1 and γ_2, with the property that if T is the monodromy operator, then

$$T(\gamma_0) = \gamma_0, \quad T(\gamma_1) = \gamma_1 + \gamma_0, \quad T(\gamma_2) = \gamma_2 + \gamma_1 + 1/2\gamma_0.$$

Now let us choose cycles γ_0' and γ_2' such that

$$\langle \gamma_0', \gamma_0' \rangle = \langle \gamma_2', \gamma_2' \rangle = 0$$

and

$$\langle \gamma_0', \gamma_2' \rangle = 1.$$

Specifically, one has, by direct calculation, that $\gamma_0' = 1/2\gamma_0$ and $\gamma_2' = \gamma_2$, and that the two cycles γ_0' and γ_2' span a hyperbolic lattice in H_2^d. Define L_d as the orthogonal complement of $\{\gamma_0', \gamma_2'\}$ in H_2^d, so then

$$L_d = (-E_8)^2 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus e\mathbf{Z}, \text{ where } \langle e, e \rangle = 2d - 2.$$

Since L_d has signature $(1, 18)$, Lemma 5.2 applies. Let $\mathcal{O}_+(L_d)$ be the group of automorphisms of the lattice L_d which preserve the cone V^+. Directly from [To 94], we have the following theorem.

Theorem 5.3. *Set*

$$\mathcal{M}_d = \mathcal{O}_+(L_d) \backslash (L_d \otimes \mathbf{R} + \sqrt{-1}V^+)$$

Then $\Gamma_e \backslash \mathbf{h}_{2,19} \cong \mathcal{M}_d$.

We shall denote the isomorphism given in Theorem 5.3 by

$$\pi_e : \Gamma_e \backslash \mathbf{h}_{2,19} \to \mathcal{O}_+(L_d) \backslash (L_d \otimes \mathbf{R} + \sqrt{-1}V^+).$$

6. Realization of $\mathcal{D}_e$ in the tube domain

With Theorem 5.3, we need to identity the subset in the quotient of the tube domain corresponding to $\mathcal{D}_e$. To begin, let us write a canonically defined divisor in the quotient of the tube domain, and then we will prove that this subset corresponds to the divisor $\mathcal{D}_e$.

Recall that $\mathcal{D}_e \subset \Gamma_e \backslash \mathbf{h}_{2,19}$ can be realized as the set of algebraic $K3$ surfaces whose polarization e is of degree d and is such that any associated projective embedding has singular double rational points. Let

$$\Delta_e = \{ l \in H^2(X, \mathbf{Z}) \mid \langle l, l \rangle = -2, \langle l, e \rangle = 0 \}.$$

In [PSS 71], it is proved that there is a decomposition $\Delta_e = \Delta_e^+ \cup (-\Delta_e^+)$ where $\Delta_e^+ \cap (-\Delta_e^+)$ is empty. The main result of this section is the following theorem.

Theorem 6.1. *For each $l \in \Delta_e^+$, let*

$$H_{e,l} = \{u \in \mathbf{P}(H^2(X,\mathbf{Z}) \otimes \mathbf{C}) \mid \langle u, l\rangle = 0 \text{ and } \langle u, e\rangle = 0\}.$$

Then $\pi_{\mathrm{mar},e}^{-1}(\mathcal{D}_e) = \bigcup_{l \in \Delta_e^+} \left(H_{e,l} \cap \mathcal{M}_{(X,\alpha,e)}\right)$.

In order to prove Theorem 6.1 we need the following lemma.

Lemma 6.2. *Let $\mathcal{L} = \mathcal{O}_X(D)$ be a line sheaf on a $K3$ surface X such that $c_1(\mathcal{L}) = 1$ and $\langle l, l\rangle = -2$. Then either*

$$H^0(X, \mathcal{O}(D)) \neq 0 \quad \textit{or} \quad H^0(X, \mathcal{O}(-D)) \neq 0.$$

Proof. From the Riemann-Roch theorem we have

$$\begin{aligned} &\chi(\mathcal{O}_X(D)) \\ &\quad = \dim_{\mathbf{C}} H^0(X, \mathcal{O}(D)) - \dim_{\mathbf{C}} H^1(X, \mathcal{O}(D)) + \dim_{\mathbf{C}} H^2(X, \mathcal{O}(D)) \\ &\quad = -\frac{\langle l, l\rangle}{2} + 2 = 1, \end{aligned}$$

hence

$$\dim_{\mathbf{C}} H^0(X, \mathcal{O}(D)) + \dim_{\mathbf{C}} H^2(X, \mathcal{O}(D)) \geq 1.$$

It follows from Serre duality and the triviality of the canonical class that $H^2(X, \mathcal{O}(D))$ and the dual of $H^0(X, \mathcal{O}(-D))$ are isomorphic. Therefore,

$$\dim_{\mathbf{C}} H^0(X, \mathcal{O}(D)) + \dim_{\mathbf{C}} H^0(X, \mathcal{O}(-D)) \geq 1$$

from which the result follows. □

Lemma 6.3. *Let $\mathcal{L} \simeq \mathcal{O}(D)$ be a line bundle on a $K3$ surface X such that $H^0(X, \mathcal{L}) > 0$ and $\langle D, D\rangle = -2$. Then D is a union of nonsingular rational curves. Further, D can be described by Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8.*

Proof. This corollary follows directly from Lemma 6.2, Theorem 2.7 and figure 2.8 of [A 62]. □

Proof of Theorem 6.1. Let

$$\mathcal{D} = \bigcup_{l \in \Delta_e^+} \left(H_{e,l} \cap \mathcal{M}_{(X,\alpha,e)}\right).$$

Let $\tau \in \pi^{-1}_{\mathrm{mar},e}(\mathcal{D}_e)$. From the surjectivity of the period map, there exists a polarized $K3$ surface (X_τ, e_τ) such that for any projective embedding associated to any power of the given polarization, the image of X_τ has at least one double rational singular point. Take a minimal resolution of these double rational points to obtain a minimal $K3$ surface $\tilde{X}_\tau$. The preimage of each double rational point from $\tilde{X}_\tau$ onto X_τ will define a divisor D such that the Chern class $c_1(\mathcal{O}(D))$ will be a vector $l \in H^2(\bar{X}_\tau, \mathbf{Z})$ for which $\langle l, e\rangle = 0$, $\langle l, \tau\rangle = 0$, and $\langle l, l\rangle = -2$. Therefore, $\pi^{-1}_{\mathrm{mar},e}(\mathcal{D}_e) \subset \mathcal{D}$.

Conversely, for each $l \in H^2(X, \mathbf{Z})$ such that $\langle l, l\rangle = -2$ and $\langle l, e\rangle = 0$, there is a hyperplane

$$H_l = \{\tau \in \mathbf{h}_{2,19} \subset \mathbf{P}(H^2(X, \mathbf{Z}) \otimes \mathbf{C}) \mid \langle \tau, l\rangle = 0\}.$$

From Lemma 6.3 and the surjectivity of the period map, we conclude that for $\tau \in H_l$ we have $\pi_e^{-1}(\tau) \in \pi^{-1}_{\mathrm{mar},e}(\mathcal{D}_e)$, which completes the proof of Theorem 6.1. □

We now will translate Theorem 6.1 into the language of the tube domain description of $\mathbf{h}_{2,19}$. Let us write

$$H_2^d = L_d \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and any element $l \in \Delta_e^+ \subset H_2^d$ as (c, a, b) with $c = (l_1, \ldots, l_{19}) \in \mathbf{Z}^{19}$ and $a, b \in \mathbf{Z}$. Let

$$\Omega_d = \{u \in \mathbf{P}(H_2^d \otimes \mathbf{C}) : \langle u, u\rangle = 0 \text{ and } \langle u, \bar{u}\rangle > 0\}.$$

By Lemma 5.1(a), Ω_d is isomorphic to $\mathbf{h}_{2,19}$. Let ψ be as defined in the proof of Lemma 5.2.

Lemma 6.4.

(a) *If either a or b is equal to* 0*, then $\langle c, c\rangle = -2$ so $c \in L_d$ and $\psi^{-1}(H_{e,l})$ is the hyperplane*

$$\psi^{-1}(H_{e,l}) = H_c = \{\tau \in \mathbf{R} \otimes L_d + \sqrt{-1}V^+ \mid \langle \tau, c\rangle = 0\}.$$

(b) *If neither a nor b is equal to* 0*, then $\psi^{-1}(H_{e,l})$ is the hyperboloid*

$$\psi^{-1}(H_{e,l}) = \{\tau \in \mathbf{R} \otimes L_d + \sqrt{-1}V^+ \mid \langle \tau, c\rangle - \frac{1}{2}a\langle \tau, \tau\rangle + b = 0\}.$$

The proof of Lemma 6.4 follows directly from the definition of ψ. As a corollary of Theorem 6.1, Lemma 6.4, and the existence of the isomorphism π_e (see §5), we have the following result.

Theorem 6.5. *Let*

$$\pi_d : L_d \otimes \mathbf{R} + \sqrt{-1}V^+ \to \mathcal{O}_+(L_d)\backslash L_d \otimes \mathbf{R} + \sqrt{-1}V^+$$

be the canonical projection map. Then the set $\pi_d^{-1}(\pi_e(\mathcal{D}_e))$ *in the tube domain* $L_d \otimes \mathbf{R} + \sqrt{-1}V^+$ *is the union of the hyperplanes and hyperboloids* $\psi^{-1}(H_{e,l})$*, as described above.*

7. Conjectured product formula

For each $l \in L_d$ such that $\langle l, l\rangle = -2$, we define the linear map

$$s_1 : V^+ \to V^+ \quad \text{by} \quad s_l(v) = v + \langle v, l\rangle l.$$

It is immediate that $s_l \in \mathcal{O}_+(L_d)$. Let Γ_{L_d} be the subgroup generated by s_l. Then, Γ_{L_d} is a normal subgroup in $\mathcal{O}_+(L_d)$, and Γ_{L_d} acts properly and discontinuously on V^+. Let $\mathcal{F}(L_d)$ denote the fundamental domain of Γ_{L_d} such that $\mathcal{F}(L_d)$ is a convex polyhedron whose walls are defined by the hyperplanes H_l. For a proof of these facts, see [Bour 89].

Let $\Delta_d = \{l \in L_d \mid \langle l, l\rangle = -2\}$. Each choice of $\mathcal{F}(L_d)$ defines a splitting of $\Delta_d = \Delta_d^+ \cup (-\Delta_d^+)$. Let us fix $\mathcal{F}(L_d)$ and let $e_1, \ldots, e_{19}$ be vectors in L_d which lie on the walls of $\mathcal{F}(L_d)$ and span L_d. Then the family $\{e_i\}$ defines a flat coordinate system $\tau_1, \ldots, \tau_{19}$ in $L_d \otimes \mathbf{R} + \sqrt{-1}V^+$ (see [COGP 92] and [To 94]).

Conjecture 7.1. Consider f_e as a function on $\mathcal{M}_d$. Let

$$N_d = \#(\mathcal{O}_+(L_d)/[\mathcal{O}_+(L_d), \mathcal{O}_+(L_d)]$$

and set $\tau = \sum_{i=1}^{19} \tau_i e_i$. Set $q_0(\tau) = \exp(2\pi\sqrt{-1}\sum \tau_i)$. For each $l \in \Delta^+ \subset L_d$, let $q_l(\tau) = \exp(2\pi\sqrt{-1}\langle \tau, l\rangle)$. For each $l' = (c, a, b) \in H_d = L_d \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $ab \neq 0$ and $\langle l', l'\rangle = -2$, define

$$q_{l'}(\tau) = \exp\left(2\pi i(\langle c, \tau\rangle - \frac{a}{2}\langle \tau, \tau\rangle + b)\right).$$

Then there is a constant c_d and a set of positive integers $\{m(l), m(l')\}$ such that

$$f_e(\tau) = c_d q_0(\tau)^{1/N_d} \prod_l (1 - q_l(\tau))^{m(l)} \prod_{l'} (1 - q_{l'}(\tau))^{m(l')}.$$

For a detailed discussion concerning the integers $\{m(l), m(l')\}$ and their structure, the reader is referred to the preprint [Bor 94].

Remark 7.2. One can prove easily that the product defined in Conjecture 7.1 convergences for all vectors $\tau \in L_d \otimes \mathbf{R} + \sqrt{-1}V^+$ with $\mathrm{Im}(\tau)$ sufficiently large. In the remarkable paper [Bo 94], Borcherds proves that one can analytically continue the product to all τ. In [JT 94a] and [JT 94b], we prove that f_e vanishes precisely for those τ for which the above product (formally) vanishes. Our method of proof of the vanishing of f_e follows from estimates of heat kernels (due to Li and Yau) and from the fact that f_e is a section of the line sheaf $\pi_* \mathcal{K}_{\mathcal{X}/\mathcal{M}_{X,e}}$, where $\mathcal{X}$ is the versal family of polarized, algebraic $K3$ surfaces.

Remark 7.3. One can prove that $\Delta_d^+ \subset H_d^2$ has a finite number of orbits under the action of $\mathcal{O}_+(H_d^2) = \Gamma_d$. The multiplity $m(l)$ is a constant for each orbit.

When expanding the product in Conjecture 7.1, one obtains a Fourier series with integer coefficients. The integers are related to the number of -2 curves on the $K3$ surface. In this way, our discriminant can be viewed as a type of generating function. Recursive relations between the coefficients can be established and suggest that a connection with Hecke theory on $\mathbf{h}_{2,19}$ should be studied. Further investigation into these questions is under consideration (see [JT 94d]).

In the case of elliptic curves, one can prove that the holomorphic function constructed from the determinant of the Laplacian has a product formula by proving that the product expansion has the same automorphic behavior as the holomorphic function. This fact can be verified through the Kronecker limit formula. An analogue of the Kronecker limit formula in the setting of algebraic, polarized $K3$ surfaces is currently under investigation (see the preprint [JT 95b]).

Remark 7.4. It was shown in [JT 94b] that $\det^* \Delta_{(X,e)}^{(0,1)} < 1$. Therefore, Conjecture 7.1 would then give an upper bound for the product in terms of the L^2 norm of the image of the period point in the tube domain.

Remark 7.5. One can associate a generalized Borcherds-Kac-Moody algebra to $\Delta_d^+ \subset L_d$. We speculate that our discriminant is related to the denominator of the Weyl character formula of the above defined algebra.

8. An analogue of the elliptic q-parameter and j-function

In [JT 94a], we constructed, following [COGP 92], an analogue of the q-parameter which exists in the setting of elliptic curves. Briefly, the construction is as follows.

Let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathcal{D}$ be a semi-stable degenerating family of polarized $K3$ surfaces such that the monodromy operator $T \in \mathrm{Aut}(H_2(X_t, \mathbf{Z}), L_t)$ has a single Jordan cell of dimension 3 (see [To 76]). In the language of [Ku 77], such a family is of type III. Then there is a free, 3-dimensional submodule $W(X_t, L_t) \subset H_2(X_t, \mathbf{Z})$ for which the action of the monodromy operator is unipotent. That is, with respect to a continuously varying basis $\{A_t, B_t, C_t\}$ of $W(X_t, L_t)$ over $\mathbf{Q}$, the action of the monodromy is by the matrix

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, there is a unique invariant 1-dimensional submodule, generated by $\pm A_t$ for $t \in \mathcal{D}$. Let ω_t be as in section 3, so

$$\int\limits_{A_t} \omega_t = \pm 1.$$

The vanishing cycle is the cycle A_t such that the above integral is equal to 1. An element B_t in $W(X_t, L_t)$ for which $T(B_t) = B_t + A_t$ will be called a transverse cycle. Note that two transverse cycles differ by an additive factor of the form nA_t, where n is an integer. The $K3$ modular parameter associated to the above degenerating family is defined by

$$q_\pi(t) = \exp\left(2\pi i \int\limits_{B_t} \eta_t\right).$$

Since $\mathbf{h}_{2,19}$ is simply connected, one can use deformation arguments to show that q_π is independent of the family and depends solely on (X, α, e); hence, we shall write $q_{(X,\alpha,e)}$. Using results from [JT 94a], we can prove the following: If (X, α, e) is a Kummer surface associated to the abelian surface $\mathbf{C}^2/L(\Omega)$, where $L(\Omega)$ is the lattice associated to I_2, the two by two identity matrix, and Ω, a matrix in the Siegel upper half space of dimension two, then $q_{(X,\alpha,e)} = \exp(\pi i \mathrm{Tr}(\Omega))$.

Using the asymptotics of the periods as given in [Gr 70], we can prove

$$q_{(X,\alpha,e)}(0) = 0 \quad \text{and} \quad q'_{(X,\alpha,e)}(0) \neq 0.$$

An interesting question is to relate the above q parameter and the parameter defined in Conjecture 7.1. It can be shown that when one restricts τ to a line in the tube domain generated by a rational direction (meaning a line such that some integer multiple lies in the lattice), then the two q parameters coincide (see [JT 95b]).

Let us now define the $K3$ analogue of the elliptic j-function, together with a conjecture concerning its Fourier expansion. As above, let us view $f_e^{N_e}$ as a holomorphic modular form on the tube domain constructed in section 5. The main result from [Ba 70], as stated on page 141, asserts that $f_e^{N_e}$ can be written as an isobaric polynomial involving Eisenstein series defined in [Ba 70]. Let h_e be the weight of the polynomial. Let S_h denote the set of isobaric polynomials in the Eisenstein series of weight h with integer coefficients. Further work in [Ba 70] and [Ba 73] asserts that for sufficiently large h, a basis of S_h provides a map defined over $\mathbf{Q}$ of the quotient of the tube domain into projective space such that the image is birationally equivalent to the Satake compactification of the quotient of the tube domain. Let $h = c_e h_e$ be the smallest weight divisible by h_e, so c_e is an integer, such that S_h provides such a map, and let $\{E_{h,k}\}$ with $k = 1, \ldots, N$ be a $\mathbf{Q}$ basis of S_h. The set of modular functions $\{E_{h,k}/f_e^{c_e}\}$ with $k = 1, \ldots, N$ is the $K3$ analogue of the elliptic j-function, which is known to be expressible as the quotient of a weight twelve Eisenstein series divided by the Dedekind delta function. In the case of elliptic curves, one scales the elliptic j-function so that the lead coefficient in a q expansion is one.

Conjecture 8.1. For every k, there is a $c_k \in \mathbf{Q}$ such that for every type III degenerating family, the q expansion of $c_k E_{h,k}/f_e^{c_e}$ has positive integer coefficients.

Going beyond Conjecture 8.1, we speculate, assuming the validity of Conjecture 8.1, that the coefficients are related to the dimensions of irreducible representations of the Mathieu group M_{24}, in a manner similar to that which relates dimensions of the irreducible representations of the Fischer-Griess monster simple group and the q expansion of the elliptic j-function. We base this speculation on two facts. The first observation is a combination of the connection with M_{24} and the monster, as discussed in

[CN 79], together with the connection between elliptic curves and boundary components in the moduli space of polarized, algebraic $K3$ surfaces (see [Ku 77]). The second observation is a result of Mukai [Mu 88], which states that any automorphism group of a polarized, algebraic $K3$ surfaces is a certain subgroup of M_{24}. The reader is referred to [Mu 88] for further details of his proof. In ongoing work, we are investigating the following so far heuristic approach to Mukai's theorem.

As described in [T 85], one can embed M_{24} into the automorphism group, modulo reflections, of a 26-dimensional $\mathbf{Z}$ lattice of signature $(25, 1)$. More specifically, let L_{26} be the set of vectors in $u \in \mathbf{R}^{26}$ with coefficients in $\frac{1}{2}\mathbf{Z}$ and such that $(u, f) \in 2\mathbf{Z}$, where f is the vector in $\mathbf{R}^{26}$ with all entries equal to 1/2, and $(,)$ is an inner product of signature $(25, 1)$. Let W be the subgroup of $\mathrm{Aut}(L_{26})$ generated by reflections of the form

$$s_l(u) = u + (u, l)l \quad \text{where} \quad (l, l) = -2.$$

Then one can embed M_{24} into $\mathrm{Aut}(L_{26})/W$. On the other hand, it is shown in [PSS 71] that $\mathrm{Aut}(X, L)$, the automorphism group of a polarized, algebraic $K3$ surface, is equal to $\mathcal{O}^+(\mathrm{Pic(X)})/W$, where $\mathrm{Pic}(X)$ is the Picard group $H^2(X, \mathbf{Z}) \cap H^{1,1}(X, \mathbf{R})$ and W is a group of reflections. Using the Hodge decomposition theorem, one can show that the signature of the inner product on $H^2(X, \mathbf{Z})$ has signature $(n, 1)$ on $\mathrm{Pic}(X)$, where $n = \mathrm{rk}(\mathrm{Pic}(X) - 1$. So, one can embed $\mathrm{Pic}(X)$ into L_{26}. Now, we need to consider subgroups in $\mathcal{O}^+(L_{26})$ which stabilize $\mathrm{Pic}(X)$. Such subgroups will define subgroups of M_{24} in a natural way via intersection.

In [JT 95a], we are investigating the continuation of these ideas to the setting of Enriques surfaces. The corresponding symmetric space is $SO(2, 10)$, and the corresponding simple group is M_{12}.

In [LY 94], the authors are using ideas from mirror symmetry to study a connection between the Fischer-Griess monster simple group and $K3$ surfaces which are complete intersections in weighted projective spaces.

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References

[A 62] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496.

[Ast 85] *Géométrie des surfaces $K3$: modules et périodes*, Astérisque **126** (1985).

[Ba 70] W. L. Baily, Jr., *Eisenstein series on tube domains*, in Problems in nalysis: a symposium in honor of Salomon Bochner, R. C. Gunning, ed., Princeton University Press, Princeton, 1970, 139–156.

[Ba 73] ———, *On the Fourier coefficients of certain Eisenstein series on the adéle group*, in Number theory, algebraic geometry, and commutative algebra: in honor of Y. Akizuki, Y. Kusunoki, et. al., ed., Kinokuniya, Tokoyo, 1973, 23–43.

[BPV 84] W. Barth, C. Peters and A. Van De Ven, *Compact complex surfaces*, Ergebnisse der Math. **4** (1984).

[Bor 94] R. Borcherds, *Automorphic forms on $O_{s+2,2}(\mathbf{R})$ and infinite products*, preprint (1994).

[Bour 89] N. Bourbaki, *Lie Groups and Lie Algebras*, Chapters 1–3, Springer-Verlag, New York, 1989.

[BuR 75] D. Burns, Jr. and M. Rapoport, *On the Torelli problem for Kählerian $K3$ surfaces*, Ann. scient. Ëc. Norm. Sup. 4^e série, t. **8** (1975), 235–274.

[COGP 92] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theorem*, in Essays on mirror manifolds, S. T. Yau, ed., International Press Co., Hong Kong, 1992,, 31–95.

[CN 79] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), 308–339.

[CS 93] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices, and groups*, Grundlehren der mathematischen Wissenschaften, vol. 290, Springer-Verlag, New York, 1993.

[Gr 70] P. Griffiths, *Periods of integrals on algebraic manifolds: summary of main results and discussion of open problems*, Bull. AMS. **75** (1970), 228–296.

[In 82] R. Indik, *Fourier coefficients of non-holomorphic Eisenstein series on a tube domain associated to an orthogonal group*, Princeton University thesis (1982).

[JT 94a] J. Jorgenson and A. Todorov, *An analytic discriminant for polarized algebraic $K3$ surfaces*, Yale University preprint (1994).

[JT 94b] ———, *Analytic discriminants for manifolds with canonical class zero*, Yale University preprint (1994).

[JT 95a] ———, *Spectral theory and holomorphic forms on Enriques surfaces*, in preparation.

[JT 95b] ———, *A Kronecker limit formula associated to the moduli space of algebraic $K3$ surfaces*, in preparation.

[KT 87] R. Kobayashi and A. Todorov, *Polarized period map for generalized $K3$ surfaces and the moduli of Einstein metrics*, Tôhoku Math. Journ. **39** (1987), 341–363.

[Ko 88] S. Kondo̱, *On the Albanese variety of the moduli space of polarized $K3$ surfaces*, Invent. Math. **91** (1988), 587–593.

[Ku 77] V. Kulikov, *Degenerations of $K3$ surfaces and Enriques surfaces*, Math. USSR Izv. **11** (1977), 957–989.

[La 87] S. Lang, *Elliptic functions, second edition*, Graduate texts in mathematics, vol. 112 Springer-Verlag, New York, 1987.

[LY 94] B. Lian and S. T. Yau, *Arithmetic properties of mirror and quantum coupling*, preprint (1994).

[Mu 88] S. Mukai, *Finite groups of automorphisms of $K3$ surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.

[PS 69] I. I. Piatetski-Shapiro, *Automorphic functions and the geometry of classical domains*, Gordon and Breach, New York, 1969.

[PSS 71] I. I. Piatetski-Shapiro and I. Shafarevich, *A Torelli theorem for algebraic surfaces of type $K3$*, (also in Collected mathematical papers, Springer-Verlag, New York, 1989, 516–557), Math USSR Izv. **5** (1971), 547–588.

[Ser 73] J. P. Serre, *A course in arithmetic*, Graduate texts in mathematics, vol. 7, Springer-Verlag, New York, 1973.

[Sh 67] I. Shafarevich, et. al., *Algebraic Surfaces. Proc. of the Steklov Institut*, translated by the AMS (1967) **75** (1965).

[Ti 88] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, in Math. Aspects of String Theory, S. T. Yau, ed., World Scientific, Singapore (1988), 629–646.

[T 85] J. Tits, *Le monstre*, Astérisque **121–122** (1985), 105–122.

[To 76] A. Todorov, *Finiteness conditions for monodromy of families of curves and surfaces*, Izv. Akad Nauk USSR **10** (1974), 749–762.

[To 80] ———, *Applications of Kähler-Einstein-Calabi-Yau metric to moduli of $K3$ surfaces*, Invent. Math. **61** (1980), 251–265.

[To 89] ———, *The Weil-Petersson geometry of the moduli space of $SU(n \geq 3)$ (Calabi-Yau) manifolds I.*, Commun. Math. Phys. **126** (1989), 325–346.

[To 94] ———, *Applications of ideas from mirror symmetry to the moduli of $K3$ surfaces*, preprint (1994).

[Y 78] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I.*, Commun. Pure Appl. Math **31** (1978), 339–411.

[W 76] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer-Verlag, New York, 1976.

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