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TANGENT SPACES TO SCHUBERT VARIETIES

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In this note, we announce a criterion for smoothness of a Schubert variety in the flag variety G/B. Let G be a semi-simple, simply connected algebraic group, which we assume for simplicity to be defined over an algebraically closed field k of characteristic 0. (The following discussion is valid in any characteristic, in fact even over $\mathbf{Z}$.) Let T be a maximal torus in G, and W the Weyl group. Let R be the system of roots of G relative to T. Let B be a Borel subgroup of G, where $B \supset T$. Let S (resp. R^+) be the set of simple (resp. positive) roots of R relative to B. For $\alpha \in R$, let s_α be the reflection. For $\alpha \in R^+$, let $E_\alpha,\ F_\alpha$ denote the elements of the Chevalley basis for $\mathfrak{g}(= \text{Lie}G)$, corresponding to α. For $w \in W$, let e_w be the point in G/B, and $X(w)$ $(= \overline{BwB} \pmod B)$, the Schubert variety, associated to w. Let $T(w, e_{id})$ be the the tangent space to $X(w)$ at e_{id}. Let

$$(1) \qquad N_w = \{\beta \in R^+ \mid F_\beta \in T(w, e_{id})\}.$$

Now $T(w, e_{id})$ being a T-submodule of the T-module $\sum_{\alpha\in R^-} \mathfrak{g}_\alpha$ (here R^- denotes the set of negative roots in R, and $\mathfrak{g}_\alpha$ denotes the root space kF_α), we have

$$(2) \qquad T(w, e_{id}) \ = \ \text{the span of } \{F_\beta,\ \beta \in N_w\}.$$

For a dominant weight λ, let $V(\lambda)$ be the irreducible G-module (over $\mathbf{C}$) with highest weight λ. Let us fix a highest weight vector u in $V(\lambda)$. For $w \in W$, let $u_w = w \cdot u$, and $V_w = U^+(\mathfrak{g})u_w$ (here $U^+(\mathfrak{g})$ is the subalgebra of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, generated by $\{E_\alpha, \alpha \in S\}$). For a weight μ in $V(\lambda)$, let $m(\mu)$ (resp. $m_w(\mu)$) denote the multiplicity of μ in $V(\lambda)$ (resp. V_w).

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Theorem 1. *Let $\beta \in R^+$, and $\rho = \frac{1}{2}$ the sum of positive roots. Then $\beta \in N_w$ if and only if $m_w(\rho - \beta) \;=\; m(\rho - \beta)$.*

As a consequence, we obtain a criterion for the smoothness of a Schubert variety as given by the following

Corollary. *Let $w \in W$, and $M_w = \{\beta \in R^+ \mid m_w(\rho - \beta) \;=\; m(\rho - \beta)\}$. Then $X(w)$ is smooth if and only if $\# M_w = l(w)$, where l_w denotes the length of w (= dim $X(w)$).*

The proof is immediate, since $X(w)$ is smooth if and only if it is smooth at e_{id}.

Outline of a proof of Theorem 1

For generalities on algebraic groups, one may refer to [B].

Let us fix a dominant, regular weight λ. Let $V(\lambda)$, u, u_w, V_w be as above. We have (cf. [P])

$$T(w, e_{id}) \;=\; \text{the span of } \{F_\beta,\ \beta \in R^+ \mid F_\beta u \in V_w\}. \tag{3}$$

From (2) and (3), we obtain

$$N_w = \{\beta \in R^+ \mid F_\beta u \in V_w\}. \tag{4}$$

In $[\mathrm{L}]_3$, we constructed a basis $\mathcal{B}(\lambda)$ for $V(\lambda)$ which is compatible with the Bruhat order, i.e., $V_w \cap \mathcal{B}(\lambda) = \mathcal{B}_w(\lambda)$, say, is a basis for V_w. Further, this basis consists of elements of the form Du, where D is either 1 or $F_{\gamma_1}^{(n_1)} \cdots F_{\gamma_r}^{(n_r)}$, γ_i simple, $n_i > 0$ (for some suitable n_i's), and $s_{\gamma_r} \cdots s_{\gamma_1}$ is reduced (here $F_\gamma^{(n)} = \frac{F_\gamma^n}{n!}$). To be more precise, let

$$I = \{\text{Lakshmibai-Seshadri paths of shape } \lambda\}, \tag{5}$$

$$I_w = \{\pi \in I \mid w \geq \phi(\pi)\} \tag{6}$$

notations being as in [Li]. Then it is shown in [Li],

$$\mathrm{char} V(\lambda) = \sum_{\pi \in I} e^{\nu(\pi)} \tag{7}$$

$$\mathrm{char} V_w = \sum_{\pi \in I_w} e^{\nu(\pi)} \tag{8}$$

In particular, using (7), we obtain a formula for $m(\mu),\ \mu \in X$, the weight lattice, namely

$$m(\mu) = \#\{\pi \in I \mid \nu(\pi) = \mu\} \tag{9}$$

Fixing a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_r}$ for w, we have (cf. $[\mathrm{L}]_3$)

$$I_w = \{f_{i_1}^{n_1} \cdots f_{i_r}^{n_r} \pi_0,\ \text{for suitable}\ n_i \in \mathbf{Z}^+\}, \tag{10}$$

$$\mathcal{B}_w(\lambda) = \{F_{i_1}^{(n_1)} \cdots F_{i_r}^{(n_r)} u \mid f_{i_1}^{n_1} \cdots f_{i_r}^{n_r} \pi_0 \in I_w\}. \tag{11}$$

Here, π_0 is the Lakshmibai-Seshadri path given by the line segment in $X \otimes \mathbf{R}$ joining the origin and λ, and f_i are the operators on I as defined in [Li]).

Let us write $\mathcal{B}(\lambda) = \{Q_\pi,\ \pi \in I\}$. For $\lambda = \rho$, we are able to write down (cf.$[\mathrm{L}]_4$) very precisely the expression for $F_\beta u$ as a linear combination of the elements in $\mathcal{B}(\lambda)$, namely,

$$F_\beta u = \sum_{I^{\rho,\beta}} c_\pi Q_\pi, \quad c_\pi \in k^* \tag{12}$$

where

$$I^{\rho,\beta} = \{\pi \in I \mid \nu(\pi) = \rho - \beta\} \tag{13}$$

We have (cf.(9))

$$m(\rho - \beta) = \# I^{\rho,\beta} \tag{14}$$

By the Bruhat order compatibility of $\mathcal{B}(\lambda)$, we have (cf.(12))

$$F_\beta u \in V_w \Longleftrightarrow Q_\pi \in V_w,\ \forall \pi \in I^{\rho,\beta} \tag{15}$$

Now (14) and (15) imply that

$$F_\beta u \in V_w \Longleftrightarrow m_w(\rho - \beta) = m(\rho - \beta) \tag{16}$$

Hence, from (4) and (16), we obtain

$$\beta \in N_w \Longleftrightarrow m_w(\rho - \beta) = m(\rho - \beta)$$

Further consequences

Given $\beta \in R^+$, using the expression for $F_\beta u$ as a linear combination of the elements in $\mathcal{B}(\rho)$ as given by (12) above, we are able to describe N_w in a very elegant form for classical groups as described in Theorem 2 below. We shall follow the notation in [Bou] to denote the elements of R^+. We shall denote the Bruhat order in W by $\geq$.

Theorem 2. *Let $\beta \in R^+$.*

(*a*) *Let G be of type $\mathbf{A}_n$. Then $\beta \in N_w \iff w \geq s_\beta$.*

(*b*) *Let G be of type $\mathbf{C}_n$.*

(1) *Let $\beta = \epsilon_i - \epsilon_j$, or $2\epsilon_i$. Then $\beta \in N_w \iff w \geq s_\beta$.*
(2) *Let $\beta = \epsilon_i + \epsilon_j$. Then $\beta \in N_w \iff w \geq$ either $s_{\epsilon_i+\epsilon_j}$ or $s_{2\epsilon_i}$.*

(*c*) *Let G be of type $\mathbf{B}_n$.*

(1) *Let $\beta = \epsilon_i - \epsilon_j,\ \epsilon_n$, or $\epsilon_i + \epsilon_n$. Then $\beta \in N_w \iff w \geq s_\beta$.*
(2) *Let $\beta = \epsilon_i,\ i < n$. Then $\beta \in N_w \iff w \geq$ either s_{ϵ_i} or $s_{\epsilon_i+\epsilon_n}$.*
(3) *Let $\beta = \epsilon_i + \epsilon_j,\ j < n$. Then $\beta \in N_w \iff$*
$$w \geq \text{ either } s_{\epsilon_i+\epsilon_j} \text{ or } s_{\epsilon_i}s_{\epsilon_j+\epsilon_n}.$$

(*d*) *Let G be of type $\mathbf{D}_n$.*

(1) *Let $\beta = \epsilon_k - \epsilon_l$, or $\epsilon_i + \epsilon_j,\ j = n-1, n$. Then $\beta \in N_w \iff w \geq s_\beta$.*
(2) *Let $\beta = \epsilon_i + \epsilon_j,\ j < n-1$. Then $\beta \in N_w \iff$*
$$w \geq \text{ either } s_{\epsilon_i+\epsilon_j} \text{ or } s_{\epsilon_i+\epsilon_n}s_{\epsilon_i-\epsilon_n}s_{\epsilon_j+\epsilon_{n-1}}.$$

Remark 1. The result in Theorem 2 for type $\mathbf{A}_n$ is contained in [L-S] also. In $[\mathrm{L}]_1$, $[\mathrm{L}]_2$, [L-R], results were obtained towards the determination of $T(w, e_{id})$ for types $\mathbf{C}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, respectively. The formulation of $T(w, e_{id})$ as given in Theorem 2 is a nice refinement of the formulations in loc.cit. Moreover, the method of proof outlined above is much more simple and straightforward than the proofs in loc.cit.

Remark 2. In $[\mathrm{S}]_2$ (see also $[\mathrm{S}]_1$), the authour gives a criterion for smoothness of Schubert varieties in terms of the nil Hecke ring.

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