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COHOMOLOGY OF ARTIN GROUPS

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Introduction

This short note is an addendum to the paper ([S]). If $\mathbf{W}$ is a *Coxeter group*, acting on $\mathbb{C}^n$ as a reflection group, and $G_{\mathbf{W}}$ is the associated *Artin group* (see [B], [B-S]) then (using [S_0]) a combinatorial complex $\mathbf{X_W}$ was constructed (obtained very naturally from the Coxeter complex of $\mathbf{W}$ by identifications on the faces) which is homotopy equivalent to the orbit space

$$\mathbf{Y_W} \;=\; \left[\mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H_{\mathbb{C}}\right] / \;\mathbf{W}$$

where $\mathcal{A}$ is the arrangement of reflection hyperplanes of $\mathbf{W}$, and $H_{\mathbb{C}}$ is the complexification of H. So, when $\mathbf{Y_W}$ is a space of type $k(\pi,1)$ (for example, when $\mathbf{W}$ is finite ([D])), $\mathbf{X_W}$ describes the homotopy type of $k(G_{\mathbf{W}},1)$.

This topological construction was used to produce an algebraic complex which computes rank-1 local systems on $\mathbf{X_W}$ ([S, theorem 1.10]).

After looking for possible generalizations, we could almost immediately recognize that theorem 1.10 of [S] is a specialization of the general theorem below which produces very natural combinatorial formulas computing the cohomology of *any* local system for *any* Artin group (when $\mathbf{Y_W}$ is a $k(\pi,1)-$space). In fact, we obtain coboundary formulas which hold in the group algebra $\mathbb{Z}[G_{\mathbf{W}}]$.

1.

The notations are similar to that of [S]. So, let $(\mathbf{W}, S)$ be a Coxeter system, realized as an irreducible reflection group in $\mathbb{R}^n$: if $\mathcal{A}$ is the arrangement of reflection hyperplanes of $\mathbf{W}$, then S will be the set of reflections with respect to the walls of a fixed chamber $\mathcal{C}_0$.

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Let

$$\mathbf{Y} \ = \ \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$$

(where $H_{\mathbb{C}}$ is the complexification of H). Then $\mathbf{W}$ acts freely on $\mathbf{Y}$: let $\mathbf{Y}_{\mathbf{W}}$ be the quotient space (see the introduction). The fundamental group $G_{\mathbf{W}} \ = \ \pi_1(\mathbf{Y}_{\mathbf{W}})$ is the *Artin group* associated to $\mathbf{W}$: a presentation of it is obtained by the standard presentation of $\mathbf{W}$ by deleting the relations of kind $s^2 = 1$ (see [B], [B-S], [D], [V]). So, if

$$\mathbf{W} \ = < s \in S \mid (ss')^{m(s,s')} = 1 >$$

then
$G_{\mathbf{W}} =$

$$< G_s, \ s \in S \mid G_s G_{s'} G_s \cdots = \ G_{s'} G_s G_{s'} \ldots \ (s \neq s', \ m(s,s') \text{ factors}) > .$$

The "forgetting" homomorphism $G_{\mathbf{W}} \to \mathbf{W}$, which takes G_s into $s, s \in S$, has a section $G : \mathbf{W} \to G_{\mathbf{W}}$ (which is not a homomorphism):

$$w = s_{i_1} \ldots s_{i_l} \mapsto G_w \ := G(w) := G_{s_{i_1}} \ldots G_{s_{i_l}}$$

where $s_{i_1} \ldots s_{i_l}$ is a reduced decomposition of w. The fact that G is well defined is a well known theorem by Matsumoto [M]. By linearity one gets a linear map $G : \mathbb{Z}[\mathbf{W}] \to \mathbb{Z}[G_{\mathbf{W}}]$.

For any subsets $\Gamma, \Gamma' \subset S, \Gamma \subset \Gamma'$, let us introduce the subset of $\mathbf{W}$ (see [H;§1.10]):

$$\mathbf{W}^{\Gamma}_{\Gamma'} = \{ w \in \mathbf{W}_{\Gamma'} : \ l(ws) > l(w), \text{for all } s \in \Gamma \}.$$

Therefore $\mathbf{W}^{\Gamma}_{\Gamma'}$ is the set of *minimal coset representatives* for the parabolic subgroup $\mathbf{W}_{\Gamma}$ in $\mathbf{W}_{\Gamma'}$. Let $S^{\Gamma}_{\Gamma'} \in \mathbb{Z}[\mathbf{W}]$ be the sum element

$$S^{\Gamma}_{\Gamma'} = \sum_{w \in \mathbf{W}^{\Gamma}_{\Gamma'}} w.$$

Similarly set

$$T^{\Gamma}_{\Gamma'} := G(S^{\Gamma}_{\Gamma'}) = \sum_{w \in \mathbf{W}^{\Gamma}_{\Gamma'}} G_w \in \mathbb{Z}[G_{\mathbf{W}}].$$

If $\rho : G_{\mathbf{W}} \to Aut(R)$ is a representation of $G_{\mathbf{W}}$ into some module R, we also indicate by

$$T^{\Gamma}_{\Gamma'}(\rho) := \rho(T^{\Gamma}_{\Gamma'}) = \sum_{w \in \mathbf{W}^{\Gamma}_{\Gamma'}} \rho(G_w).$$

Set also

$$T^{\Gamma}_{\Gamma'}(-\rho) := T^{\Gamma}_{\Gamma'}((-1)^l \otimes \rho),$$

where $(-1)^l$ is the 1-dimensional representation (over $\mathbb{Z}$) taking G_w into $(-1)^{l(w)}$; so

$$T^{\Gamma}_{\Gamma'}(-\rho) = \sum_{w \in \mathbf{W}^{\Gamma}_{\Gamma'}} (-1)^{l(w)} \rho(G_w).$$

We can now formulate the theorem. Notations are similar to those of [S; theorem 1.10]: in particular, S is finite and we fix a total order on S. We now set $\Lambda^k S$ equal to the family of k-subsets of S.

Theorem. *Let $\rho : G_{\mathbf{W}} \to Aut(R)$ be a representation of $G_{\mathbf{W}} = \pi_1(\mathbf{X_W})$, let $\mathcal{L}_\rho$ be the associated local system on $\mathbf{X_W}$. Then*

$$H^*(\mathbf{X_W}; \mathcal{L}_\rho) \cong H^*(\mathcal{C}^*, \delta^*_\rho)$$

where $(\mathcal{C}^*, \delta^*_\rho)$ is the algebraic complex given by:

$$\mathcal{C}^k = R^{\Lambda^k S}$$

and, given $f \in R^{\Lambda^k S}$, $\Gamma = \{i_1 < \cdots < i_{k+1}\} \in \Lambda^{k+1} S$,

$$(*) \qquad \delta^k_\rho(f)(\Gamma) = \sum_{r=1}^{k+1} (-1)^r [T^{\Gamma - \{i_r\}}_{\Gamma}(-\rho)(f)]\,(\Gamma - i_r)\,.$$

Remark. If R is a unitary ring, q is a unit of R and ρ is the representation given by $\rho(G_s)(x) = qx,\ s \in S$, then one obtains theorem 1.10, (i), of [S]. Part (ii) of such theorem is obtained in a similar way.

Proof. Let us briefly recall the construction given in [S] of the complex $\mathbf{X_W}$. We denote by $\mathbf{Q}$ the cellular complex which is *dual* to the stratification induced by the arrangement $\mathcal{A}$ ($\mathbf{Q}$ is isomorphic to the Coxeter complex of $\mathbf{W}$). The unique cell of $\mathbf{Q}$ which is dual to the facet F will be denoted by $e(F)$. One gives a realization of $\mathbf{Q}$ as a convex polytope in $\mathbb{R}^n$, stable under the $\mathbf{W}-$action which permutes the cells of the same dimension with $w(e(F)) = e(w(F))$ for all facets F and $w \in \mathbf{W}$. Thus, since any facet F is $\mathbf{W}$-conjugate to a unique facet F_0 in a fundamental chamber $\mathcal{C}_0$, we can identify $e(F)$ with $e(F_0)$ using the unique element $\gamma_F \in \mathbf{W}$ of minimal length with the property that $\gamma_F(F_0) = F$.

Of course, γ_F is the element of minimal length in the coset $\gamma_F.\mathbf{W}_\Gamma$. ($\Gamma \subset S$ being the *type* of F_0 and F). Equivalently, γ_F takes the $0-$cell

$v_0 = \mathbf{Q}_0 \cap \mathcal{C}_0 = e(\mathcal{C}_0)$ into that $0-$cell $w_0(F) \in \mathbf{Q}_0 \cap e(F)$ which has the minimal distance from v_0 (here we use the combinatorial distance in the graph $\mathbf{Q}_1$).

Then $\mathbf{X_W}$ is the quotient complex under these identifications. So each $k-$cell of $\mathbf{X_W}$ corresponds to a unique $k-$subset $\Gamma \subset S$, and also to a unique k-codimensional facet F (the facet of type Γ) of C_0.

Set $F(\Gamma)$ as the facet of $\mathcal{C}_0$ of type Γ and $\Gamma(F)$ as the type of the facet F.

Let also $\pi_{\mathbf{W}} : \mathbf{Q} \to \mathbf{X_W}$ be the projection onto the quotient.

By using the above description of $\mathbf{X_W}$ and standard computation of the cohomology of local systems over a cellular complex, one gets (see the proof of [S; thm.1.10]) a coboundary formula

$$\delta^k_\rho(f)(\Gamma) = \sum_{\substack{F(\Gamma)\subset cl(F') \\ codim(F')=k}} \sum_{h\in \mathbf{W}^{\Gamma(F)}_{\Gamma(F')}} [e(F) : e(h(F'))]\pi_{\mathbf{W}}\big(u(v_0, w_0(h(F')))\big)_*(f(\Gamma(F'))),$$

where $[e(F') : e(h(F))]$ is the incidence number between two cells, which, from [S; lemma 1.7], equals $(-1)^{l(h)+r}$. The term $u(v_0, w_0(h(F')))$ is a path in $e(F)$ between v_0 and $w_0(h(F'))$, along whose $\pi_{\mathbf{W}}-$image the section $\alpha \in R$ has to be transported. It can be realized as a combinatorial path in the $1-$skeleton of $e(F)$ which is of minimal combinatorial length, i.e. it crosses each hyperplane of $\mathcal{A}$ at most once. If

$$h = s_{i_1} \dots s_{i_l}, \quad (l = l(h))$$

is a reduced decomposition of h then a minimal path $u(v_0, w_0(h(F')))$ crosses facets each of codimension 1 and of type successively $i_1, \dots, i_l$, and its $\pi_{\mathbf{W}}-$image runs along the sense which is compatible with the positive orientation of the $1-$cells of $\mathbf{X_W}$. Therefore

$$(\pi_{\mathbf{W}}(u(v_0), w_0(h(F')))_*(\alpha) = \rho(G_{i_1} \cdots G_{i_l})(\alpha), \quad \alpha \in R$$

This is formula (*). $\square$

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