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MODULES WITH EXTREMAL RESOLUTIONS

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Introduction

Let R be a commutative noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The size of a minimal free resolution of a finite R–module M is given by its *Betti numbers* $\beta_n^R(M) = \operatorname{rank}_k \operatorname{Ext}_R^*(M,k)$. Dually, that of a minimal injective resolution is measured by the *Bass numbers* $\mu_R^n(M) = \operatorname{rank}_k \operatorname{Ext}_R^*(k,M)$.

These sizes may be estimated asymptotically on a natural, a polynomial, and an exponential scale. The first scale yields the classical homological dimensions. The second produces known notions of complexity, which distinguish between modules of infinite homological dimensions. The third leads to new concepts of homological curvatures, which discriminate among modules with infinite complexities.

It is well known that k has maximal homological dimensions among all finite R–modules. An elementary computation shows that its complexities and curvatures are maximal as well. By analyzing the *representations* on $\operatorname{Ext}_R^*(M,k)$ and $\operatorname{Ext}_R^*(k,M)$ *of the homotopy Lie algebra* π^* of the local ring R, we show that modules with extremal homological invariants occur with unexpected frequency.

1. Asymptotic invariants

The *projective complexity* $\operatorname{proj\,cx}_R M$ is equal to d if $d-1$ is the smallest degree of a polynomial $p(t)$ such that $\beta_n^R(M) \le p(n)$ for all n; if no such d exists one sets $\operatorname{proj\,cx}_R M = \infty$. Under the name of complexity and the notation $\operatorname{cx}_R M$, this concept was introduced in representation theory in [1], and transplanted to local algebra in [4], [5]. Replacing Betti numbers by Bass numbers, one gets a notion of *injective complexity* $\operatorname{inj\,cx}_R M$; this is the plexity $\operatorname{px}_R M$ considered in [5].

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We define the *projective curvature* and the *injective curvature* of M by

$$\operatorname{proj\,curv}_R M = \limsup_{n\to\infty} \sqrt[n]{\beta^n_R(M)} \text{ and } \operatorname{inj\,curv}_R M = \limsup_{n\to\infty} \sqrt[n]{\mu^n_R(M)}\,.$$

The names of the last two numbers reflect the fact that they are reciprocal values of the radii of convergence of the *Poincaré series* $\mathrm{P}^R_M(t) = \sum_n \beta^R_n(M)t^n$ and of the *Bass series* $\mathrm{I}^M_R(t) = \sum_n \mu^n_R(M)t^n$, respectively.

For background and further terminology from commutative algebra we refer to the beautiful exposition of Matsumura [13].

Remarks. We list some relations among the various invariants, letting the prefix 'hom' stand for either 'proj' or 'inj'.

(1) $\operatorname{hom\,dim}_R M < \infty \iff \operatorname{hom\,curv}_R M = 0 \iff \operatorname{hom\,cx}_R M = 0\,.$

(2) $\operatorname{hom\,dim}_R M = \infty \iff \operatorname{hom\,curv}_R M \geq 1.$

(3) $\operatorname{hom\,cx}_R M < \infty \implies \operatorname{hom\,curv}_R M \leq 1\,$; the converse is known for $M = k$ by [3], but is open in general, cf. the discussion for projective invariants in [6].

(4) $\operatorname{hom\,curv}_R M < \infty$ for all M: cf. [9;(1.1)] or [4;(2.5)], and Lemma 1.

(5) $\text{f-hom}\operatorname{cx} R = \sup_M\{\operatorname{hom\,cx}_R M < \infty\}$ define the *finitistic complexities* of R. It is not known if they are finite. In the known cases $\text{f-proj}\operatorname{cx} R \leq \operatorname{edim} R - \operatorname{depth} R$, where $\operatorname{edim} R = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2$ is the *embedding dimension* of R, cf. [6]. Furthermore, by [7; (3.10)] for each finite R–module M there exists an $\widehat{R}$–module N such that $\mu^R_n(M) = \beta^R_{n-d}(N)$ for $n > d = \dim M$, and hence $\text{f-inj}\operatorname{cx} R \leq \text{f-proj}\operatorname{cx} \widehat{R}$.

We use $\preccurlyeq$ and $\succcurlyeq$ to denote coefficientwise inequalities of formal power series.

Lemma 1. *For each R–module M there exists an integer $\ell \geq 1$ such that*

$$\mathrm{P}^R_M(t) \preccurlyeq \ell \cdot \mathrm{P}^R_k(t) = \ell \cdot \mathrm{I}^k_R(t) \succcurlyeq \mathrm{I}^M_R(t)$$

The first inequality is proved in [9; Lemma, p. 290]. Before giving a different (and elementary) proof at the end of the section, we note an immediate consequence.

Proposition 2. *If R is a local ring with residue field k, then*

$$\begin{aligned}
\operatorname{proj\,dim}_R k &= \sup_M\{\operatorname{proj\,dim}_R M\} = \sup_M\{\operatorname{inj\,dim}_R M\} = \operatorname{inj\,dim}_R k\,;\\
\operatorname{proj\,cx}_R k &= \sup_M\{\operatorname{proj\,cx}_R M\} = \sup_M\{\operatorname{inj\,cx}_R M\} = \operatorname{inj\,cx}_R k\,;\\
\operatorname{proj\,curv}_R k &= \sup_M\{\operatorname{proj\,curv}_R M\} = \sup_M\{\operatorname{inj\,curv}_R M\} = \operatorname{inj\,curv}_R k\,,
\end{aligned}$$

where M ranges over all finite R–modules.

The first row is classical, and its middle equality defines the *global dimension* $\operatorname{gl\,dim} R$. Accordingly, we use the second row to define the *global complexity* $\operatorname{gl\,cx} R$ of R, and the third row to introduce its *global curvature* $\operatorname{gl\,curv} R$.

By the Auslander-Buchsbaum-Serre theorem, $\operatorname{gl\,dim} R = d < \infty$ precisely when R is regular of dimension d. To formulate similar statements for the asymptotic global invariants, we recall some earlier results. By Tate [18; Theorem 6], if R is a complete intersection, then $\mathrm{P}_k^R(t) = (1+t)^e/(1-t^2)^c$, where $e = \operatorname{edim} R$ and $c = \operatorname{edim} R - \dim R$ is the *codimension* of R. On the other hand, if $\operatorname{cx}_R k < \infty$, then R is a complete intersection by Gulliksen [8; (2.3)]. The same conclusion is derived in [3; (6.2)] from the much weaker condition $\operatorname{curv}_R k \leq 1$.

In the language introduced here, these results may be restated as:

Theorem 3. *The following conditions on a local ring R are equivalent.*

(i) *R is a complete intersection (respectively, of codimension c).*
(ii) *$\operatorname{gl\,cx} R$ is finite (respectively, and equal to c).*
(iii) *$\operatorname{gl\,curv} R$ is at most 1.*

Proof of Lemma 1. The proof is by induction on the Krull dimension $\dim M$. When $\dim M = 0$ a secondary induction on the length of M, using the cohomology exact sequence associated with a proper submodule, yields $\mathrm{P}_M^R(t) \preccurlyeq (\operatorname{length}_R M)\,\mathrm{P}_k^R(t)$.

If $\dim M > 0$ and $L = \bigcup_{i\geqslant 0}(0 :_M \mathfrak{m}^i)$, then the cohomology exact sequence associated with $0 \to L \to M \to N \to 0$ yields $\mathrm{P}_M^R(t) \preccurlyeq \mathrm{P}_L^R(t) + \mathrm{P}_N^R(t)$, hence it suffices to prove the inequality for N. As $\operatorname{depth}_R N > 0$, there is an exact sequence $0 \to N \xrightarrow{x} N \to P \to 0$ for some $x \in \mathfrak{m}$. The first homomorphism induces the zero map in cohomology, so $\mathrm{P}_N^R(t) \preccurlyeq \mathrm{P}_P^R(t)$. Since $\dim P < \dim N = \dim M$, the induction hypothesis applies to P, and we are done.

A parallel argument deals with $\mathrm{I}_M^R(t)$. Finally, $\mathrm{P}_k^R(t) = \mathrm{I}_k^R(t)$. □

2. Extremal submodules

The main result of this section implies that each non-zero R–module may be obtained as an extension of a module of maximal complexity and curvature by a second module with the same characteristics. In particular, the Grothendieck group of finite R–modules is generated by modules with extremal behavior.

Theorem 4. *If M is a finite module and L a submodule such that $L \supseteq \mathfrak{m}M \neq \mathfrak{m}L$, then the following hold:*

$$\begin{aligned} \operatorname{proj}\operatorname{cx}_R L &= \operatorname{inj}\operatorname{cx}_R L = \operatorname{gl}\operatorname{cx} R\,; \\ \operatorname{proj}\operatorname{curv}_R L &= \operatorname{inj}\operatorname{curv}_R L = \operatorname{gl}\operatorname{curv} R\,. \end{aligned}$$

The *proof* is immediate from Proposition 2 and Lemma 6 below. □

Corollary 5. *For each $i \geq 1$, the R–module $\mathfrak{m}^i M$ is either zero, or has extremal complexities and curvatures.*

Remark. The corollary implies that if $L = \mathfrak{m}M \neq 0$ and $\operatorname{proj}\dim_R L$ or $\operatorname{inj}\dim_R L$ is finite, then R is regular: the first result is due to Levin [10], cf. also [11; (1.1)]; the second is a new characterization of regularity. In conjunction with Theorem 3, the corollary also produces several criteria for complete intersections.

Our arguments use cohomology representations of the *homotopy Lie algebra* π^* of the local ring R, discussed in more detail in [3].

Remark. By Levin [10] and Schoeller [16], the Yoneda algebra $E^* = \operatorname{Ext}^*_R(k,k)$ is a graded Hopf algebra over k, whose dual $\operatorname{Tor}^R_*(k,k)$ is a skew-commutative graded Hopf algebra with a system of divided powers compatible with the coproduct. A theorem of Milnor and Moore [14; (5.18)] in characteristic 0 and of André [2; Theorem 17] in characteristic $p > 0$ (adjusted by Sjödin [17; Theorem 2] for $p = 2$) shows that E^* is the universal enveloping algebra $\mathrm{U}(\pi^*)$ of a graded Lie algebra π^*.

Lemma 6. *If $\operatorname{edim} R = e$, then for L as in Theorem 4 there are inequalities*

$$\begin{aligned} \mathrm{P}^R_L(t) \cdot (1+t)^e &\succcurlyeq b \cdot \mathrm{P}^R_k(t) \\ \big(\mathrm{I}^L_R(t) - s\big) \cdot (1+t)^e &\succcurlyeq m \cdot \mathrm{P}^R_k(t) \cdot t \end{aligned}$$

with $b = \operatorname{rank}_k M/L > 0$, $s = \operatorname{rank}_k (0 :_L \mathfrak{m})$, and $m = \operatorname{rank}_k M/(\mathfrak{m}L :_M \mathfrak{m}) > 0$.

Proof. The commutative diagram of R–modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & L/\mathfrak{m}L & \longrightarrow & M/\mathfrak{m}L & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

induces a commutative square of homomorphisms of graded *right* E^*–modules

$$\begin{array}{ccc} \mathrm{Ext}_R^*(k,N) & \xrightarrow{\eth'} & \mathrm{Ext}_R^*(k,L) \\ \Big\| & & \Big\downarrow \\ \mathrm{Ext}_R^*(k,N) & \xrightarrow{\eth} & \mathrm{Ext}_R^*(k,L/\mathfrak{m}L) \end{array}$$

where $\eth'$ and $\eth$ are connecting maps of degree 1. As $\mathfrak{m}$ annihilates N and $L/\mathfrak{m}L$, the modules in the lower row are isomorphic to $N \otimes_k E^*$ and $(L/\mathfrak{m}L) \otimes_k E^*$, respectively.

By the Poincaré-Birkhoff-Witt theorem, $E^* \cong \bigwedge^*(\pi^1) \otimes_k U^*$ as graded right modules over the subalgebra $U^* = \mathrm{U}(\pi^{>1})$. Noting that $\pi^1 = E^1$, we include $\eth$ in a commutative square of homomorphisms of graded U^*–modules

$$\begin{array}{ccc} N \otimes_k E^* & \xrightarrow{\eth} & (L/\mathfrak{m}L) \otimes_k \bigwedge^*(E^1) \otimes_k U^* \\ \Big\uparrow & & \Big\uparrow {\scriptstyle \zeta \otimes_k U^*} \\ N \otimes_k U^* & \xrightarrow{\eth^0 \otimes_k U^*} & \mathrm{Ext}_R^1(k, L/\mathfrak{m}L) \otimes_k U^* \end{array}$$

in which ζ is the composition of the canonical isomorphism $\mathrm{Ext}_R^1(k, L/\mathfrak{m}L) \cong (L/\mathfrak{m}L) \otimes_k E^1$ with the obvious inclusion map.

By combining the two squares, we locate in $\mathrm{Ext}_R^{\geqslant 1}(k,L)$ a copy of $\mathrm{Im}(\eth^0) \otimes_k U^*$. As $E^1 \cong \mathrm{Hom}_k\big(\mathfrak{m}/\mathfrak{m}^2, k\big)$, we have $(1+t)^e \cdot \sum_i \mathrm{rank}_k\, U^i t^i = \mathrm{P}_k^R(t)$, and thus

$$\mathrm{I}_R^L(t) \succcurlyeq\; s + \mathrm{rank}_k \,\mathrm{Im}(\eth^0) \cdot \frac{\mathrm{P}_k^R(t)}{(1+t)^e} \cdot t\ .$$

After rewriting the beginning of the cohomology exact sequence induced by the lower row of the initial diagram of R–modules in the form

$$0 \longrightarrow L/\mathfrak{m}L \longrightarrow (\mathfrak{m}L :_M \mathfrak{m})/\mathfrak{m}L \longrightarrow N \xrightarrow{\eth^0} \mathrm{Ext}_R^1(k, L/\mathfrak{m}L)$$

we see by a dimension count that $\mathrm{rank}_k \,\mathrm{Im}(\eth^0) = \mathrm{rank}_k\, M/(\mathfrak{m}L :_M \mathfrak{m}) = m$. To finish with Bass series, it remains to note that $\mathfrak{m}M \neq \mathfrak{m}L$ implies $m > 0$.

The proof of the inequality for Poincaré series proceeds along similar lines; it is simplified by observing that the relevant connecting homomorphisms of graded left E^*–modules $\eth'^0$ and $\eth^0$ are injective. □

3. Extremal quotients

Each finite R–module $M \neq 0$ has an obvious quotient with extremal invariants, namely $M/\mathfrak{m}M$. Another class of extremal quotients is obtained by dualizing the arguments of the preceding section:

Proposition 7. *If M is a finite R–module, and L is a submodule such that $L \subseteq (0 :_M \mathfrak{m}) \neq (L :_M \mathfrak{m})$, then the module $N = M/L$ satisfies*

$$\begin{aligned} \operatorname{proj\,cx}_R N \;&=\; \operatorname{inj\,cx}_R N \;=\; \operatorname{gl\,cx} R\,; \\ \operatorname{proj\,curv}_R N \;&=\; \operatorname{inj\,curv}_R N \;=\; \operatorname{gl\,curv} R\,. \end{aligned}$$

In particular, if $(0 :_M \mathfrak{m}^i) \neq (0 :_M \mathfrak{m}^{i+1})$, then $M/(0 :_M \mathfrak{m}^i)$ has extremal curvature ans complexity. A third family of extremal quotient modules is given by

Theorem 8. *Let M be a syzygy of k. If $\beta\colon M \to N$ is a homomorphism to a finite R–module, and the induced map $\beta^n\colon \operatorname{Ext}^n_R(N,k) \to \operatorname{Ext}^n_R(M,k)$ is non-zero for some $n \geq 0$, then* $\operatorname{proj\,cx}_R N = \operatorname{gl\,cx} R$ *and* $\operatorname{proj\,curv}_R N = \operatorname{gl\,curv} R$.

Corollary 9. *Each non-zero homomorphic image of a finite direct sum of syzygies of k has maximal projective complexity and curvature.*

Remark. The theorem implies that if R is not regular and $\beta(M) \nsubseteq \mathfrak{m}N$, then $\operatorname{proj\,dim}_R N = \infty$: this is a recent result of Martsinkovsky [12; Proposition 8]. The corollary strengthens [12; Proposition 7] in a similar way.

Proof of Theorem 8. Let $M = \operatorname{Syz}^R_m(k)$, let β' be a homomorphism of $M' = \operatorname{Syz}^R_{m+n}(k)$ to $N' = \operatorname{Syz}^R_n(N)$ induced by some extension of β to a morphism of minimal free resolutions, and denote by γ the natural projection $N' \xrightarrow{\gamma} N'/\mathfrak{m}N'$. In cohomology we get a composition

$$\xi^*\colon E^* \otimes_k \operatorname{Hom}(N'/\mathfrak{m}N',k) \xrightarrow{\gamma^*} \operatorname{Ext}^*_R(N',k) \xrightarrow{\beta'^*} \operatorname{Ext}^*_R(M',k) \xrightarrow{\alpha^*} E^{\geqslant m+n}$$

where α^* is an iterated connected homomorphism, and thus bijective. Since γ^0 is an isomorphism and $\beta'^0 = \beta^n$ is not trivial, we have $\xi^0 \neq 0$.

We reintroduce the notation of the Remark preceding Lemma 6.

If R is a complete intersection of codimension c, then $\pi^j = 0$ for $j > 2$, hence E^* is a finite free module over the symmetric algebra $U^* = \mathrm{U}(\pi^2)$ of the rank c vector space π^2. Thus, ξ^* is a non-zero map of a free U^*–module into a torsion-free one, and hence $\operatorname{Im}\xi^*$ contains a non-zero free graded U^*–module generated in degree 0. This yields (in)equalities of formal power series $\mathrm{P}^R_{N'}(t) \succcurlyeq \sum_i \operatorname{rank} U^i t^i = 1/(1-t^2)^c$. Together with Theorem 3 they imply numerical relations

$$\operatorname{proj}\operatorname{cx}_R N = \operatorname{proj}\operatorname{cx}_R N' \geq c = \operatorname{gl}\operatorname{cx} R$$

so $\operatorname{proj}\operatorname{cx}_R N = \operatorname{gl}\operatorname{cx} R$ is finite, hence $\operatorname{proj}\operatorname{curv}_R N$ and $\operatorname{gl}\operatorname{curv} R$ are both 0 or 1.

If R is not a complete intersection, then consider the graded subalgebra $V^* = \mathrm{U}(\pi^{>m+n})$ of E^*. By Poincaré-Birkhoff-Witt $E^* \cong V^* \otimes W^*$ as graded left V^*–modules, where W^* is the tensor product of the exterior algebra of the odd dimensional subspace of $\pi^{\leqslant s}$ with with the symmetric algebra on its even dimensional subspace. This implies injectivity for the right hand vertical map in the commutative diagram of graded *left* V^*–modules

$$\begin{array}{ccc} E^* \otimes_k \operatorname{Hom}_R(N'/\mathfrak{m}N', k) & \xrightarrow{\ \xi^*\ } & E^{\geqslant m+n} \\ \uparrow & & \uparrow \\ V^* \otimes_k \operatorname{Hom}_R(N'/\mathfrak{m}N', k) & \xrightarrow{V^*\otimes_k \xi^0} & V^* \otimes_k E^{m+n}\ . \end{array}$$

Thus, $\operatorname{Im}(\xi^*)$ contains a copy of V^*. Since ξ^* factors over β'^*, so does $\operatorname{Ext}^*_R(N', k)$.

We now have an inequality $\mathrm{P}^R_{N'}(t) \succcurlyeq \sum_i \operatorname{rank}_k V^i t^i$. On the other hand, the series $W(t) = \sum_i \operatorname{rank} W^i t^i$ is the expansion around $t = 0$ of the product

$$\frac{\prod_{2i<m+n}(1+t^{2i+1})^{e_{2i+1}}}{\prod_{2i\leqslant m+n}(1-t^{2i})^{e_{2i}}} \qquad \text{where} \quad e_j = \operatorname{rank}_k \pi^j\ .$$

Thus, the preceding inequality and the Poincaré-Birkhoff-Witt theorem yield

$$\mathrm{P}^R_{N'}(t)\cdot W(t) \succcurlyeq \left(\sum_i \operatorname{rank}_k V^i t^i\right)\cdot W(t) = \mathrm{P}^R_k(t)\,.$$

With $\rho\big(A\big)$ denoting the radius of convergence of a power series $A(t)$, we get

$$\min\{\rho\big(\mathrm{P}^R_{N'}\big), 1\} = \min\{\rho\big(\mathrm{P}^R_{N'}\big), \rho\big(W\big)\} \leq \rho\big(\mathrm{P}^R_{N'}\cdot W\big) \leq \rho\big(\mathrm{P}^R_k\big)\,.$$

Taking inverses, and referring to Proposition 2, we conclude that

$$\max\{\operatorname{proj}\operatorname{curv}_R N, 1\} = \max\{\operatorname{proj}\operatorname{curv}_R N', 1\} \geq \operatorname{proj}\operatorname{curv}_R k = \operatorname{gl}\operatorname{curv} R\,.$$

As $\operatorname{gl}\operatorname{curv} R > 1$ by Theorem 3, we see that $\operatorname{proj}\operatorname{curv}_R N = \operatorname{gl}\operatorname{curv} R$, and hence $\operatorname{proj}\operatorname{cx}_R N = \operatorname{gl}\operatorname{cx} R = \infty$. □

4. Extremal algebras

The following result exhibits a new reason for the appearance of extremal resolutions: the passage from a ring with a bad singularity to a ring with a well behaved singularity creates complications in the module structure.

Proposition 10. *Let $R \to S$ be a finite local homomorphism of local rings.*
If S is regular, then $\operatorname{proj}\operatorname{cx}_R S = \operatorname{gl}\operatorname{cx} R$.
If S is a complete intersection, then $\operatorname{proj}\operatorname{curv}_R S = \operatorname{gl}\operatorname{curv} R$.

The *proof* follows from Theorem 3 and the simple result below. □

Lemma 11. *If $R \to S$ is a finite homomorphism of local rings, then*

$$\begin{aligned}\operatorname{gl}\operatorname{cx} R \ &\leq \ \operatorname{gl}\operatorname{cx} S + \operatorname{proj}\operatorname{cx}_R S\,;\\ \operatorname{gl}\operatorname{curv} R \ &\leq \ \max\{\operatorname{gl}\operatorname{curv} S, \operatorname{proj}\operatorname{curv}_R S\}\,.\end{aligned}$$

Proof. The second page (or: term) of the standard change of rings spectral sequence

$$^2\mathrm{E}_{p,q} = \operatorname{Ext}_S^p(N, \operatorname{Ext}_R^q(S,k)) \implies \operatorname{Ext}_R^{p+q}(N,k)$$

for a finite S–module N can be rewritten as $\operatorname{Ext}_S^*(N,l)\otimes_k \operatorname{Ext}_R^*(S,k)$, where l is the residue field of S. The convergence of the spectral sequence implies an inequality of power series $\mathrm{P}_N^S(t)\cdot\mathrm{P}_S^R(t) \succcurlyeq \mathrm{P}_N^R(t)$, from which one gets numerical inequalities

$$\begin{aligned}\operatorname{proj}\operatorname{cx}_R N \ &\leq \ \operatorname{proj}\operatorname{cx}_S N + \operatorname{proj}\operatorname{cx}_R S\,;\\ \operatorname{proj}\operatorname{curv}_R N \ &\leq \ \max\{\operatorname{proj}\operatorname{curv}_S N, \operatorname{proj}\operatorname{curv}_R S\}\,.\end{aligned}$$

Setting $N = l$, and remarking that as an R–module l is a finite direct sum of copies of k, one obtains the desired inequalities by applying Proposition 2. □

The homological invariants of the unique R–module of length 1 are listed in Proposition 2. Our last result provides an equally complete story in length 2.

Proposition 12. *If S is a module of length 2 over a local ring R, then*

$$\begin{aligned}\operatorname{proj}\operatorname{cx}_R S \ &= \ \operatorname{inj}\operatorname{cx}_R S \ \geq \ \operatorname{gl}\operatorname{cx} R - 1\,;\\ \operatorname{proj}\operatorname{curv}_R S \ &= \ \operatorname{inj}\operatorname{curv}_R S \ = \ \operatorname{gl}\operatorname{curv} R\,.\end{aligned}$$

If, furthermore, $\operatorname{proj\,dim}_R S < \infty$ *or* $\operatorname{inj\,dim}_R S < \infty$*, then* R *is a hypersurface of multiplicity* ≤ 2*. Conversely, each hypersurface of multiplicity at most* 2 *has modules of length* 2 *with finite projective and injective dimensions.*

Proof. For $S \cong k \oplus k$ all the assertions are clear. Else, $S \cong R/(x^2, x_1, \dots, x_d)$ for some minimal set of generators $x, x_1, \dots, x_d$ of $\mathfrak{m}$. This R–module is isomorphic to its Matlis dual, so $\mu_n^R(S) = \beta_n^R(S)$ each n, and the injective invariants of S are equal to its projective one. The map $R \longrightarrow S$ is a surjective ring homomorphism. The periodic resolution $\dots \longrightarrow S \xrightarrow{x} S \xrightarrow{x} \dots$ of k over S yields $\operatorname{gl\,cx} S = \operatorname{gl\,curv} S = 1$, so the desired relations for $\operatorname{proj\,cx}_R S$ and $\operatorname{proj\,curv}_R S$ follow from Lemma 11.

When $\operatorname{proj\,dim}_R S < \infty$ the first inequality becomes $\operatorname{gl\,cx} R \leq 1$, so R is a codimension 1 complete intersection by Theorem 3. Completing R if necessary, we may further assume that $R = Q/(f)$, where $(Q, \mathfrak{n})$ is a $(d+1)$-dimensional regular local ring and $f \in \mathfrak{n}^2$. Then $S = Q/\mathfrak{a}$, with $\mathfrak{a}$ generated by a maximal Q–regular sequence. Tate [18; Theorem 4] provides an infinite free resolution of the R–module S, which is minimal if $f \in \mathfrak{n}\mathfrak{a}$. Thus, $f \notin \mathfrak{n}\mathfrak{a}$ by the finiteness of $\operatorname{proj\,dim}_R S$. But as $\mathfrak{n}^2 S = 0$, we have $\mathfrak{n}^2 \subseteq \mathfrak{a}$, hence $f \notin \mathfrak{n}^3$, so the multiplicity of R is at most 2.

Conversely, let R be a hypersurface of multiplicity at most 2 and dimension d. There is nothing to prove when R is regular. Otherwise, the associated graded ring of R for the $\mathfrak{m}$-adic filtration is of the form $G/(q)$, where G is a polynomial ring in $d+1$ variables and q is a quadratic form. The set $\mathcal{L}$ of linear forms dividing q is either empty, or consists of the scalar multiples of at most two forms. Thus, $\mathcal{L} \neq G_1$ as long as $d > 1$, so by induction on d, we can find a G–regular sequence $g_1, \dots, g_d \subseteq G_1$. If $x_i \in \mathfrak{m}$ has g_i as its initial form, then the sequence $x_1, \dots, x_d$ is R–regular, and $\operatorname{length}_R R/(x_1, \dots, x_d) = \operatorname{rank}_k G/(q, g_1, \dots, g_d) = 2$. □

To finish, we note that the proposition cannot be extended to length 3 or higher.

Examples 13.
(1) In the ring $R = k[[x,y,z]]/(x^2, xy, y^2 - xz)$ the element z is regular and $S = R/(z) \cong k[[x,y]]/(x^2, xy, y^2)$, so $\operatorname{proj\,dim}_R S = 1$ and $\operatorname{length}_R S = 3$. A well known change of rings theorem [15; (27.5)] gives the first equality below

$$\mathrm{P}_k^R(t) = (1+t)\,\mathrm{P}_k^S(t) = \frac{1+t}{1-2t}\,.$$

The second one follows from the recurrence relation $\mathrm{P}_k^S(t) - 1 = 2t \cdot \mathrm{P}_k^S(t)$, obtained from the cohomology exact sequence associated with

$0 \to \mathfrak{m}S \to S \to k \to 0$, and the isomorphism $\mathfrak{m}S \cong k^2$. Thus, $\operatorname{gl}\operatorname{cx} R = \infty$ and $\operatorname{gl}\operatorname{curv} R = 2$.

(2) The hypersurface $R = \mathbb{F}_2[[x,y]]/\big(xy(x+y)\big)$ has multiplicity 3, and for $g = x + x^2 + xy + y^2$ the R–module $R/(g)$ has projective dimension 1 and length 4, but R possesses no length 3 module of finite projective dimension.

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