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THE SEIBERG-WITTEN EQUATIONS AND FOUR-MANIFOLDS WITH BOUNDARY

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0. Introduction

Let Z be a smooth, compact, oriented 4-manifold. The intersection form of Z is a symmetric bilinear pairing

$$J_Z = H_2(Z;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}.$$

An early result by Donaldson says that if Z is closed and J_Z is negative definite then J_Z is isomorphic to some diagonal form $\langle -1\rangle \oplus \cdots \oplus \langle -1\rangle$. More generally one may ask which negative definite forms can occur if Z is allowed to have some fixed oriented rational homology sphere Y as boundary. The main purpose of the present paper is to apply the equations recently introduced by Seiberg and Witten [W] to prove a finiteness result about the definite forms associated to an arbitrary Y.

It is useful to consider the more general situation where the boundary of Z is a disjoint union of rational homology spheres: $\partial Z = Y_1 \cup \cdots \cup Y_l$. (Of course, $\cup_j Y_j$ and $\#_j Y_j$ bound the same intersection forms, since the standard cobordism connecting them has no rational homology in dimension 2.) Let

$$J_Z = m\langle -1\rangle \oplus \tilde{J}_Z,$$

where $\tilde{J}_Z$ has no elements of square -1. Note that

$$|\det(\tilde{J}_Z)| = |\det(J_Z)| \leq \text{order}(H_1(\partial Z;\mathbb{Z})),$$

with equality if $H_1(Z;\mathbb{Z})$ is free.

Let $J_Z^{\#} = H_2(Z,\partial Z)/\text{Torsion}$ be the lattice dual to J_Z. A characteristic vector for J_Z is an element $\xi \in J_Z^{\#}$ such that $\xi\cdot x \equiv x\cdot x \mod 2$ for all $x \in J_Z$. If J_Z is *unimodular* then $|\xi|^2 \equiv \text{rk}(J_Z) \mod 8$ for any characteristic

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vector ξ. If J_Z is unimodular and not diagonal then by a theorem of Elkies [E1] it has a characteristic vector ξ satisfying

$$\mathrm{rk}(J_Z) - |\xi|^2 \geq 8.$$

Moreover, according to [E2] it has a characteristic vector ξ satisfying $\mathrm{rk}(J_Z) - |\xi|^2 \geq 16$ unless $\tilde{J}_Z$ is among the finite number of forms listed in [E2].

Our main result is the following theorem:

Theorem 1. *Let Z be a smooth, compact, oriented 4-manifold whose boundary is a disjoint union of rational homology spheres: $\partial Z = Y_1 \cup \cdots \cup Y_l$. Suppose the intersection form J_Z is negative definite. Then for any characteristic vector $\xi \in J_Z^{\#}$ we have*

$$\mathrm{rk}(J_Z) - |\xi|^2 \leq \sum_j \gamma(Y_j),$$

where $\gamma(\cdot) \in \mathbb{Q}$ is a certain invariant of oriented rational homology 3-spheres. In particular, if we assume $\tilde{J}_Z$ is even then for a fixed boundary $\cup Y_j$ there are only finitely many possibilities for $\tilde{J}_Z$.

The author does not know whether the finiteness assertion is true in general for odd $\tilde{J}_Z$ (compare the concluding remarks in [E2]).

We will derive Theorem 1 from a slightly stronger result involving the spin^c-structures on Z; this is given in section 3.

For a *closed* 4-manifold Z Theorem 1 simply says that $\mathrm{rk}(J_Z) - |\xi|^2 \leq 0$ for any characteristic vector. The proof of this inequality by means of the Seiberg-Witten equations was first found by Kronheimer. When combined with Elkies' result it provides a new proof of Donaldson's theorem.

The invariant γ is defined in section 3. For the standard 3-sphere we have $\gamma(S^3) = 0$. In section 4 we use Kronheimer's construction of ALE-spaces as hyperkähler quotients to show that for the Poincaré homology sphere P, oriented as the link of the E_8 singularity, $\gamma(P) = -\gamma(\bar{P}) = 8$. Since E_8 is the only unimodular even positive definite form of rank ≤ 8 we deduce

Proposition 2. *Let Z be a smooth, compact, oriented 4-manifold whose boundary is the Poincaré sphere oriented as above. If the intersection form J_Z is negative definite and $\tilde{J}_Z$ is even then either $\tilde{J}_Z = 0$ or $\tilde{J}_Z = -E_8$.*

In [F2] we will show that in Proposition 2, $\tilde{J}_Z$ cannot be odd if Z is simply-connected.

1. The Seiberg-Witten equations on a cylinder

Let X be a smooth, oriented, Riemannian 4-manifold. Choose a spin^c-structure for X, and let W^+ and W^- be the associated spin^c-bundles. Clifford multiplication defines a (fibrewise) map

$$T^*X \times W^{\pm} \to W^{\mp}$$

and a linear map

$$\rho : i\Lambda^2 \to \mathrm{End}_{\mathbb{C}}(W^+),$$

whose kernel is $i\Lambda^-$. Here $\Lambda^- \subset \Lambda^2$ is the subbundle of anti-selfdual 2-forms and $i = \sqrt{-1}$. For $x \in X$ the image $\rho(i\Lambda^2_x)$ is the subspace of trace-free Hermitian endomorphisms of W^+. We write $L = \det(W^+)$ and let $\mathcal{A}(L)$ denote the affine space of connections in L.

Let $\mu \in \Omega^2_X$ be a parameter. We will study the perturbed Seiberg-Witten equations for a pair $(A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$ given by the following pair of equations:

$$\begin{aligned} D_A\phi &= 0 \\ \rho(F_A + i\mu) &= \frac{1}{2}\phi \otimes \phi^* - \frac{1}{4}|\phi|^2 \cdot \mathbf{1} \end{aligned} \tag{1.1}$$

where $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator associated to A. The group $\mathcal{G} = \mathrm{Map}(X, U(1))$ acts on $\mathcal{A}(L) \times \Gamma(W^+)$ by

$$u(A, \phi) = (u^2(A), u\phi).$$

This action preserves the set of solutions to (1.1).

Now consider the special case where $X = \mathbb{R} \times Y$ for some oriented, closed Riemannian 3-manifold Y, and assume μ is the pull-back of a closed form on Y, also denoted μ. Given a connection A in L we can use holonomy in the $\mathbb{R}$-direction to identify $L = \mathbb{R} \times L_0$, where L_0 is a complex line bundle over Y. Similarly, $W^+ = \mathbb{R} \times W_0$, where W_0 is the spin^c-bundle over Y with respect to the spin^c-structure inherited from $\mathbb{R} \times Y$. Clifford multiplication with dt gives an isomorphism $W^+ \approx W^-$ and an action of the Clifford bundle of Y on W_0. As noted in [KM] the equations (1.1) now become the downward gradient flow equation for the functional $C : \mathcal{A}(L_0) \times \Gamma(W_0) \to \mathbb{R}$ determined up to a constant by the formula

$$C(A + a, \phi) = C(A, 0) - \int_Y (F_\mu(A) + \frac{1}{2}da) \wedge a - \frac{1}{2}\int_Y \langle D_{A+a}(\phi), \phi\rangle,$$

where $a \in i\Omega^1_Y$ and $F_\mu(A) = F(A) + i\mu$. Explicitly,

$$\mathrm{grad}(C)_{(A,\phi)} = (*F_\mu(A) - \frac{1}{4}\sigma(\phi,\phi), -D_A\phi),$$

where $\sigma : W_0 \times W_0 \to iT^*Y$ is the symmetric bilinear pairing satisfying

$$\langle a, \sigma(\phi,\psi)\rangle = \langle a\phi, \psi\rangle$$

for all $a \in i\Omega^1_Y$, $\phi, \psi \in \Gamma(W_0)$. In general we write $\langle\ ,\ \rangle$ for Euclidean inner products and $\langle\ ,\ \rangle_{\mathbb{C}}$ for Hermitian inner products. If $u : Y \to U(1)$ is a map and $z = u^*[U(1)] \in H^1(Y;\mathbb{Z})$ then we find

$$C(u(A,\phi)) - C(A,\phi) = -(2\pi c_1(L_0) + [\mu]) \cup 4\pi z.$$

In the remainder of this section let Y be a *rational homology sphere*. In this case C descends to a map $\mathcal{B} = (\mathcal{A}(L_0) \times \Gamma(W_0))/\mathcal{G} \to \mathbb{R}$. Also, we can take $\mu = d\nu$ for some $\nu \in \Omega^1$ and we write C_ν for the corresponding functional on $\mathcal{B}$. Let $\mathcal{R} \in \mathcal{B}$ be the set of (equivalence classes of) critical points of C, which is a compact set (this follows for instance from Corollary 3 in [KM]). Let $\theta \in \mathcal{R}$ be the unique critical point with zero spinor field, and set $\mathcal{R}^* = \mathcal{R}\backslash\{\theta\}$. When convenient we will consider the completions of $\mathcal{A}(L_0) \times \Gamma(W_0)$ and $\mathcal{G}$ in the L^2_1 and L^2_2-metrics, respectively. Then $\mathcal{B}^* \subset \mathcal{B}$, the subset of elements with spinor field not identically zero, is a Hilbert manifold.

For any critical point (A,ϕ) of C let $H_{(A,\phi)} : i\Omega^1 \times \Gamma(W_0) \to i\Omega^1 \times \Gamma(W_0)$ be the derivative of $\mathrm{grad}(C)$ at (A,ϕ), i.e.

$$H_{(A,\phi)}(a,\psi) = (*da - \frac{1}{2}\sigma(\phi,\psi), -\frac{1}{2}a\phi - D_A\psi).$$

Also, let $\lambda_\phi : i\Omega^0 \to i\Omega^1 \times \Gamma(W_0)$ be the infinitesimal action of $\mathcal{G}$ on (A,ϕ), namely

$$\lambda_\phi(\zeta) = (-2d\zeta, \zeta\phi).$$

We say (A,ϕ) is a *non-degenerate* critical point if the middle cohomology group of the following complex is zero:

$$i\Omega^0 \xrightarrow{\lambda_\phi} i\Omega^1 \times \Gamma(W_0) \xrightarrow{H_{(A,\phi)}} i\Omega^1 \times \Gamma(W_0). \tag{1.2}$$

A standard elliptic argument shows that a non-degenerate critical point is isolated in $\mathcal{R}$.

Proposition 3. *There is a Baire set of forms $\nu \in \Omega^1_{C^{k+1}}$ for which all critical points of C_ν are non-degenerate.*

Proof. Let

$$\tilde{E} = \{(A, \phi, a, \psi) \in \mathcal{A}(L_0) \times \Gamma(W_0)\backslash 0 \times i\Omega^1 \times \Gamma(W_0) \mid \lambda^*_\phi(a, \psi) = 0\}.$$

Then $E = \tilde{E}/\mathcal{G}$ is a vector-bundle over $\mathcal{B}^*$ and $\operatorname{grad} C_\nu$ defines a smooth section s_ν. As we will see in a moment, a critical point of C_ν is non-degenerate precisely when it is a regular zero of s_ν.

We can regard the family of sections $s = \{s_\nu\}$ as a section in the bundle $\Omega^1 \times E \to \Omega^1 \times \mathcal{B}^*$. Writing $V = \ker(\lambda^*_\phi) \subset i\Omega^1 \times \Gamma(W_0)$ the intrinsic derivative of s at a zero-point $[\nu, A, \phi]$ can be identified with

$$Ds : \Omega^1 \times V \to V, \quad (\nu, a, \psi) \mapsto (i * d\nu, 0) + H_{(A,\phi)}(a, \psi).$$

We will show that Ds is surjective. Suppose $(a, \psi) \in V$ is orthogonal to $\operatorname{im}(Ds)$. This means the following four equations are satisfied:

$$(i)\ da = 0; \quad (ii)\ \sigma(\phi, \psi) = 0$$
$$(iii)\ -2d^*a + \langle i\phi, \psi\rangle i = 0; \quad (iv)\ \frac{1}{2}a\phi + D_A\psi = 0.$$

Note first that these equations imply that a and ψ are smooth. From (ii) we see that on the comlement of $\phi^{-1}(0)$ we have

$$\psi = ir\phi \tag{1.3}$$

for some smooth function $r : Y\backslash\phi^{-1}(0) \to \mathbb{R}$. Combining this with (iv) we deduce

$$-\frac{1}{2}a\phi = D_A(\psi) = idr \cdot \phi,$$

which implies $-\frac{1}{2}a = idr$ where $\phi \neq 0$. Now, since ϕ satisfies the Dirac equation we can argue as in [FU, pp. 57-58] to show that $Y \setminus \phi^{-1}(0)$ is connected. By (i) and the assumption $H^1(Y; \mathbb{R}) = 0$ it then follows that r can be smoothly extended to all of Y. Inserting (1.3) into (iii) we get

$$4d^*dr + r|\phi|^2 = 0, \tag{1.4}$$

so

$$0 \geq \int_Y \langle d^*dr, r\rangle = \int_Y |dr|^2,$$

whence $a = -2idr = 0$. (1.4) and (1.3) then gives $\psi = 0$. Therefore the intrinsic derivative Ds is surjective at any zero.

Since Ds_ν is Fredholm at every zero of s_ν it follows from the Sard-Smale theorem that for a Baire set of parameters $\nu \in \Omega^1_{C^{k+1}}$, the zero-set of s_ν is regular.

Finally, we must show that the critical point with zero spinor field is generically non-degenerate. Let A be a smooth flat connection in L_0. Then $(A - i\nu, 0)$ is a critical point of C_ν which is non-degenerate if and only if $\ker(D_{A-i\nu}) = 0$. Consider the vector bundle $E \to \Omega^1 \times \Gamma(W_0)\backslash 0$ whose fibre over (ν, ϕ) is the (real) L^2-orthogonal complement $(i\phi)^\perp \subset \Gamma(W_0)$. Then $s(\nu, \phi) = D_{A-i\nu}(\phi)$ is a smooth section $C^{k+1} \times L^2_1 \to L^2$. Arguing as above one finds that the the zero-set of s is regular. Hence for a Baire set of parameters ν the real dimension of $\ker(D_{A-i\nu})\backslash 0$ is equal to

$$\mathrm{ind}_{\mathbb{R}}(p_\phi \circ D_{A-i\nu}) = 1,$$

where $p_\phi : L^2(W_0) \to L^2(W_0)$ is projection onto $(i\phi)^\perp$. Since $D_{A-i\nu}$ is complex linear we must have $\ker(D_{A-i\nu}) = 0$ for such ν. $\square$

For any pair (A, ϕ) consider the self-adjoint elliptic operator

$$P_{(A,\phi)} = \begin{pmatrix} 0 & \frac{1}{2}\lambda^*_\phi \\ \frac{1}{2}\lambda_\phi & H_{(A,\phi)} \end{pmatrix}$$

acting on sections of $E_0 = i\Lambda^0 \oplus (i\Lambda^1 \oplus W_0)$. If $D_A\phi = 0$ then (1.2) is a complex, with cohomology group $\mathcal{H}^*_{(A,\phi)}$ say, and

$$\ker(P_{(A,\phi)}) = \mathcal{H}^0_{(A,\phi)} \oplus \mathcal{H}^1_{(A,\phi)}.$$

For any non-degenerate critical points α, β we define a *relative index* $i(\alpha, \beta)$ as follows. Let $(A_t, \phi_t) \subset \mathcal{A}(L_0) \times \Gamma(W_0)$ be a smooth path which is constant in t outside some interval (t_-, t_+), and represents α for $t \le t_-$ and β for $t \ge t_+$. Choose $\lambda > 0$ such that neither of the operators P_α and P_β has any eigenvalue in the interval $(0, \lambda)$. Let $E = \pi_2^*(E_0)$, a bundle over $\mathbb{R} \times Y$, and define $i(\alpha, \beta)$ to be the index of the Fredholm operator

$$\frac{\partial}{\partial t} + P_{(A_t,\phi_t)} + \lambda : L^2_1(E) \to L^2(E).$$

This index is the same as the spectral flow of the family of operators $\{P_{(A_t,\phi_t)} + \lambda\}$, see [APS].

2. Moduli spaces

Consider again the set-up of the previous section, with $X = \mathbb{R} \times Y$ and Y an oriented rational homology 3-sphere. Fix $\nu \in \Omega^1(Y)$ such that all critical points of $C = C_\nu$ are non-degenerate. For each pair α, β of critical points we shall define a moduli space $M(\alpha, \beta) = M(\mathbb{R} \times Y; \alpha, \beta)$ which will be a perturbed version of the space of gradient lines connecting α and β. In the language of finite-dimensional Morse theory our approach is somewhat analogous to perturbing the gradient vector field away from the critical points.

To define these perturbations let $\eta_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth function supported in $[-1, 1]$ and satisfying $\int \eta_1 = 1$, and let $\eta_2 : \mathbb{R} \to \mathbb{R}$ be a smooth, compactly supported function such that $\eta_2(t) = t$ on some interval containing all critical values of C. If A is any connection in L and ϕ a section of W^+ let $S = (A, \phi)$ and define a smooth function $h_S : \mathbb{R} \to \mathbb{R}$ by

$$h_S(T) = \int_{\mathbb{R}} \eta_1(t_1 - T)\, \eta_2\Big(\int_{\mathbb{R}} \eta_1(t_2 - t_1) C(S_{t_2})\, dt_2\Big)\, dt_1,$$

where $S_t = S(t)$ is the restriction of S to $\{t\} \times Y$. Note that h_S is bounded with all its derivatives independently of S, and the association $S \mapsto h_S$ is a C^∞-map $L^p_1 \to C^k$ for all k (and $p \geq 2$, say).

Let the parameter $\omega \in \Omega^2(\mathbb{R} \times Y)$ have compact support contained in a set $\Xi \times Y$, where Ξ is the result of removing from $\mathbb{R}$ a small open interval around each critical value of C. Let $h_S^*(\omega)$ denote the pull-back of ω by the smooth map $h_S \times \mathrm{id}_Y : \mathbb{R} \times Y \to \mathbb{R} \times Y$. We will use the following translationary invariant equations for $S = (A, \phi)$:

$$\begin{aligned} &D_A \phi = 0 \\ &\rho(F_A + i\pi_2^*(d\nu) + i\, h^*_{(A,\phi)}(\omega)) = \frac{1}{2}\phi \otimes \phi^* - \frac{1}{4}|\phi|^2 \cdot \mathbf{1}, \end{aligned} \tag{2.1}$$

where $\pi_2 : \mathbb{R} \times Y \to Y$ is the projection. If $S = (A, \phi)$ is in temporal gauge then the equations take the form

$$\frac{\partial S_t}{\partial t} = -\mathrm{grad}(C_\nu)_{S_t} + E_S(t), \tag{2.2}$$

where the perturbation term $E_S(t)$ depends on $S|_{(t-1,t+1)\times Y}$ and vanishes if $h_S(t) \notin \Xi$.

Lemma 4. *If b is any constant then for $\|\omega\|_{C^k}$ sufficiently small the following holds. Let $S = (A, \phi)$ be any smooth solution to the equations* (2.1) *satisfying a pointwise bound $|\phi| \leq b$. Then*

(1) *Either $\frac{\partial}{\partial t} C(S_t) < 0$ for all t, or $[S_t] \equiv \alpha$ for some critical point α.*

(2) *If $C(S_t)$ is bounded in t then there are critical points α_+, α_- of C such that the gauge equivalence class $[S_t]$ converges in C^k to $\alpha_\pm$ as $t \to \pm\infty$.*

The proof is a compactness and unique continuation argument and is deferred to an appendix.

For any critical points α, β of C we define, as a set,

$$M(\alpha,\beta)=\{S=(A,\phi) \text{ satisfying (2.1)} \mid \lim_{t\to-\infty}[S_t]=\alpha;\ \lim_{t\to\infty}[S_t]=\beta\}/\text{Gauge},$$

where the limits refer to the C^k-topology. Note that the elements of $M(\alpha,\beta)$ satisfy the gradient flow equation for C outside a compact subset of $\mathbb{R}\times Y$.

It was proved in [KM] that if (A,ϕ) is a solution to (1.1) then at points where $|\phi|$ has a local maximum one has

$$|\phi|^2 \le \max(0, 4|\rho(i\mu)| - 2s),$$

where s is the scalar curvature of the underlying 4-manifold. It follows that if $\|\omega\|_{C^0} \le 1$, say, there is a constant b such that for any critical points α, β and any $[A,\phi] \in M(\alpha,\beta)$ we have a pointwise bound $|\phi| \le b$. In the following we will assume ω chosen so that the conclusions of Lemma 4 hold for this constant b.

It is convenient at this stage to introduce suitable function spaces. Let α, β be critical points of C. Let $v > 0$ be smaller than the first positive eigenvalue of P_α, and let $w > 0$ be smaller than the first positive eigenvalue of P_β. Choose a connection A_0 in L and a section ϕ_0 of W^+ such that $S_0 = (A_0, \phi_0)$ is in temporal gauge outside a compact set and satisfies $[S_t] = \alpha$ for $t \ll 0$ and $[S_t] = \beta$ for $t \gg 0$. Let $2 < p < 4$ and set

$$\mathcal{C} = \mathcal{C}(\alpha,\beta) = S_0 + L_1^{p;v,w}(i\Lambda^1 \times W^+).$$

Here $L_m^{p;v,w}$ is the L_m^p Sobolev space defined using a weight $\exp(-vt)$ on the negative end and $\exp(wt)$ on the positive end.

For the group of gauge transformations we take

$$\mathcal{G} = \{u : \mathbb{R}\times Y \to U(1) \mid u \in L^p_{2,\mathrm{loc}};\ du\cdot u^{-1} \in L_1^{p;v,w}\}.$$

Then $\mathcal{G}$ acts on $\mathcal{C}$, and if $\mathcal{C}^* \subset \mathcal{C}$ is the subset of elements with spinor field not identically zero then $\mathcal{C}^*/\mathcal{G}$ is a Banach manifold.

There is now a natural identification

$$M(\alpha,\beta) = \{(A,\phi) \in \mathcal{C}(\alpha,\beta) \text{ satisfying (2.1)}\}/\mathcal{G}. \tag{2.3}$$

The existence of a natural injective map from right to left in (2.3) follows from the usual elliptic techniques. To see that this map is onto one can use the non-degeneracy of the critical points of C to prove an exponential decay result for solutions in temporal gauge.

Let $M(\alpha, \beta) \subset \mathcal{C}/\mathcal{G}$ have the subspace topology.

The moduli spaces $M(\alpha, \beta)$ have compactness properties familiar from finite-dimensional Morse theory. If $S_n = (A_n, \phi_n) \in M(\alpha, \beta)$ is any sequence then we can fix gauge by requiring $d^*(A_n - A_1) = 0$. Since $|\phi_n|$ is uniformly bounded we find as in [KM] that a subsequence of $\{S_n\}$ converges in C^k over compact subsets to some solution S_∞ of the equations (2.1). By Lemma 4 this limit lies in some moduli space $M(\alpha', \beta')$ with $C(\alpha) \geq C(\alpha') \geq C(\beta') \geq C(\beta)$. (If $[S_\infty] \in M(\alpha, \beta)$ then the convergence is global.) Now let $C(\alpha) = x_0 > x_1 > \cdots > x_m = C(\beta)$ be the critical values of C in the interval $[C(\beta), C(\alpha)]$ and choose $y_j \in (x_j, x_{j-1})$. Let S_n^j be the translate of S_n with $C(S_n^j) = y_j$. Then a subsequence of $\{S_n^j\}$ converges over compacta to an element of some moduli space $M(\alpha_j, \beta_j)$, and by continuity we must have $\beta_{j-1} = \alpha_j$.

Proposition 5. *For generic small $\omega \in C^k$, each moduli space $M(\alpha, \beta)$ is a smooth finite-dimensional manifold of dimension*

$$\dim(M(\alpha, \beta)) = -d_\alpha + i(\alpha, \beta),$$

where $d_\alpha = 1$ if $\alpha = \theta$ and $d_\alpha = 0$ otherwise.

Proof. By Lemma 4, for small ω all moduli spaces $M(\alpha, \alpha)$ will consist of a single point, so we may assume $\alpha \neq \beta$. Let Ω^2_Ξ be the space of parameters ω, i.e. the space of C^k 2-forms on $\mathbb{R} \times Y$ with compact support contained in $\Xi \times Y$. We assume ω is small in the sense spelled out in the previous section. The linearization of the equations (2.1) at a point (ω, A, ϕ), taking into account the Coulomb gauge fixing condition, is the operator

$$T = T_{(\omega, A, \phi)} : \Omega^2_\Xi \times L_1^{p;v,w}(i\Lambda^1 \oplus W^+) \to L^{p;v,w}(i\Lambda^0 \oplus i\Lambda^+ \oplus W^-)$$

which takes (κ, a, ψ) to

$$(\lambda^*_\phi(a, \psi), d^+a + i(h^*_{(A,\phi)}(\kappa))^+ + K(a, \psi) - \frac{1}{2}(dt \wedge \sigma(\phi, \psi))^+, \frac{1}{2}a\phi + D_A(\psi)).$$

Here $K = K_{(\omega, A, \phi)}$ is a bounded operator $L_1^p \to C^k$ satisfying

$$\operatorname{supp}(K(a, \psi)) \subset h^{-1}_{(A,\phi)}(\Xi) \times Y.$$

We must first verify that if we fix ω then the linearization at a point (A, ϕ) is Fredholm. For this purpose we may assume (A, ϕ) is smooth and in temporal gauge. Identify $\pi_2^*(\Lambda^0_Y \oplus \Lambda^1_Y) = \Lambda^1_{\mathbb{R}\times Y}$ and $\pi_2^*(\Lambda^1_Y) = \Lambda^+_{\mathbb{R}\times Y}$, the latter by means of the map $\omega \mapsto dt \wedge \omega + *_Y \omega$. Let

$$E = \pi_2^*(i\Lambda^0_Y \oplus i\Lambda^1_Y \oplus W_0).$$

Then up to scalar factors the operator $T' : (a, \psi) \mapsto T(0, a, \psi)$ can be written as

$$T' = \frac{\partial}{\partial t} + P_t + K : L^{p;v,w}_1(E) \to L^{p;v,w}(E),$$

where $P_t = P_{(A_t,\phi_t)}$ is the operator over Y defined in the previous section. Our choice of weights for the Sobolev spaces then implies that T' is Fredholm of index $-d_\alpha + i(\alpha, \beta)$.

Secondly, we must show that $T = T_{(\omega,A,\phi)}$ is surjective whenever (ω, A, ϕ) is a solution to the equations (2.1). Let $(\zeta, a', \psi') \in L^{q;-v,-w}$ be L^2-orthogonal to $\mathrm{im}(T)$, where q is the exponent conjugate to p. Because either α or β must have non-zero spinor field, ϕ is not identically zero. Hence $\lambda^*_\phi \lambda_\phi$ is an isomorphism $L^{p;v,w}_2 \to L^{p;v,w}$, so $\zeta = 0$. By Lemma 4 we have

$$\frac{\partial}{\partial t} h_{(A,\phi)}(t) < 0$$

for all t, so by varying κ alone we see that a' vanishes on $h^{-1}_{(A,\phi)}(\Xi) \times Y$. By varying a alone we conclude as in [KM] that ψ' must also vanish on $h^{-1}_{(A,\phi)}(\Xi) \times Y$. But then (a', ψ') lies in the kernel of the operator

$$\left(\frac{\partial}{\partial t} + P_t\right)^* = -\frac{\partial}{\partial t} + P_t,$$

so by unique continuation, (a', ψ') is identically zero. This shows T is surjective. $\square$

The discussion above carries over to 4-manifolds with tubular ends in a natural way. Let $Y_1, \dots, Y_l$ be oriented rational homology spheres, each with a Riemannian metric, a spin^c-structure, a good perturbation of the functional C_j (as in Proposition 3), and some form ω_j defining a perturbation of the equations over $\mathbb{R} \times Y_j$. Let X be an oriented Riemannian 4-manifold with an open subset isometric to $\mathbb{R}_+ \times Y_j$ for each j, such that the interior $X_0 = X \backslash \cup_j (\mathbb{R}_+ \times Y_j)$ is compact. Let X have a spin^c-structure compatible with those on the ends. For any form $\mu \in \Omega^2_c(X)$ we obtain equations on X for a pair $(A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$ by combining the equations (1.1) on X_0 with (2.1) on the ends, using a suitable cut-off function. For any vector $\underline{\alpha} = \{\alpha_j\}$, with α_j a critical point of C_j, let

$$M(X; \underline{\alpha})$$

be the space of solutions to these equations over X which are asymptotic to α_j on the j'th end, modulo gauge. For generic μ all moduli spaces $M(X;\underline{\alpha})$ will be smooth away from points with zero spinor field.

There is also an analogue of Lemma 4 for solutions to the perturbed equations over X. In particular, for small ω_j we may assume that for every $[S] \in M(X;\underline{\alpha})$ and each end of X, either $\frac{\partial}{\partial t} C_j(S_t) < 0$ for all (non-negative) t, or $[S_t] \equiv \alpha_j$.

3. The main result

We first define the invariant $\gamma(Y,c)$ for an oriented rational homology 3-sphere Y with spin^c-structure c. Choose a Riemannian metric g on Y and a perturbation ν for which the functional $C = C_\nu$ has only non-degenerate critical points. Let A be a connection in L_0 satisfying $F_A + id\nu = 0$, i.e. $(A,0)$ is a critical point of C_ν. The non-degeneracy condition for $(A,0)$ means that the Dirac operator D_A has zero kernel. Let V be any simply-connected oriented Riemannian 4-manifold with one tubular end $\mathbb{R}_+ \times Y$. Choose an extension of the spin^c-structure on $\mathbb{R}_+ \times Y$ to all of V and let $L \to V$ be the associated $U(1)$-bundle. Let $\mathbf{A}$ be a smooth connection in L such that on the end, $\mathbf{A} = \pi_2^*(A)$. The Dirac operator $D_{\mathbf{A}}$ defines a Fredholm operator $L^2_1(W^+) \to L^2(W^-)$.

Let m be the smallest non-negative integer such that C has no critical points α with $i(\alpha,\theta) = 2m+1$, where θ is the critical point with zero spinor field. Then define

$$\gamma(Y,c,g,\nu) = 8m + c_1(L)^2 - \sigma(V) - 8\,\mathrm{ind}_{\mathbb{C}}(D_{\mathbf{A}}),$$

where $\sigma(V)$ is the signature of V. Note that on a *closed* spin^c 4-manifold $\tilde{V}$ the index of the Dirac operator is precisely $\frac{1}{8}(c_1(\tilde{L})^2 - \sigma(\tilde{V}))$. To see that $\gamma(Y,c,g,\nu)$ depends only on (Y,c,g,ν), use the additivity of the index of the Dirac operator over $\tilde{V} = V_0 \cup_Y V_0'$.

Now define

$$\gamma(Y,c) = \inf_{g,\nu} \gamma(Y,c,g,\nu); \quad \gamma(Y) = \max_c \gamma(Y,c).$$

If Y is an *integral* homology sphere then $c_1(L)^2 \equiv \sigma(V)$ (8), whence $\gamma(Y) \in 8\mathbb{Z}$.

Theorem 6. *Let Z be any smooth, compact, oriented 4-manifold whose boundary is a disjoint union of rational homology spheres: $\partial Z = Y_1 \cup \cdots \cup Y_l$. Suppose the intersection form of Z is negative definite and $b_1(Z) = 0$.*

Given any spin^c-structure on Z, let L be the associated $U(1)$-bundle on Z and $\mathbf{c}_j$ the induced spin^c-structure on Y_j. Then

$$c_1(L)^2 - \sigma(Z) \leq \sum_j \gamma(Y_j, \mathbf{c}_j),$$

where $\sigma(Z)$ is the signature of Z.

Note that applying Theorem 6 with $Z = [0,1] \times Y$ gives

$$0 \leq \gamma(Y, c) + \gamma(\bar{Y}, \bar{c}), \tag{3.1}$$

so $\gamma(Y, c)$ is always finite.

Proposition 7. *Let Y be any oriented rational homology 3-sphere which admits a metric with positive scalar curvature. If c is any spin^c- structure on Y and $\bar{c}$ the corresponding spin^c-structure on $\bar{Y}$ then*

$$\gamma(\bar{Y}, \bar{c}) = -\gamma(Y, c).$$

Proof. If Y has a metric g with positive scalar curvature then for $\nu = 0$ the functional C_ν has only one critical point, namely the one with zero spinor field, and this is non-degenerate. Choose an embedding of Y in a smooth, compact, simply-connected, oriented 4-manifold: $\tilde{V} = V_1 \cup_Y V_2$. Stretching $\tilde{V}$ along Y we obtain

$$0 = \gamma(Y, c, g, \nu) + \gamma(\bar{Y}, \bar{c}, g, \nu),$$

so

$$\gamma(Y, c) \leq \gamma(Y, c, g, \nu) = -\gamma(\bar{Y}, \bar{c}, g, \nu) \leq -\gamma(\bar{Y}, \bar{c}).$$

Combining this with (3.1) proves the proposition. $\square$

Proof of Theorem 1 assuming Theorem 6. Let Z be as in Theorem 1. As noticed in [FS], by performing surgery on a set of generators for the free part of $H_1(Z; \mathbb{Z})$ we obtain a new 4-manifold Z' with $b^1 = 0$ and with the same intersection form. If $\xi \in J^{\#}_{Z'}$ is any characteristic element then we can find a spin^c-structure on Z' with associated $U(1)$-bundle L such that $c_1(L)$ is Poincaré dual to ξ modulo torsion. Since $\gamma(Y) = \max_c \gamma(Y, c)$, Theorem 1 follows from Theorem 6. $\square$

Proof of Theorem 6. Choose a metric g_j and a good perturbation ν_j on Y_j. We will show that the assumption

$$c_1(L)^2 - \sigma(Z) > \sum_j \gamma(Y_j, \mathbf{c}_j, g_j, \nu_j) \tag{3.2}$$

leads to a contradiction. Let X be the manifold obtained by adding a half-tube $\mathbb{R}_+ \times Y_j$ to each boundary component of Z. Extend the product metrics on the ends to all of X. Let $M = M(X; \underline{\theta})$, where $\underline{\theta} = \{\theta_j\}$. If $D_{\mathbf{A}}$ is the Dirac operator on X then dim $M = \mathrm{ind}_{\mathbb{R}}(D_{\mathbf{A}}) - 1$. The assumption (3.2) is therefore equivalent to

$$\dim M + 1 > \sum 2m_j,$$

where m_j is the integer entering in the definition of $\gamma(Y_j)$. Let $2m = \dim M - 1 - \sum 2m_j$, which is non-negative since M has odd dimension.

If the forms ω_j defining perturbations on the tubes are zero then M has a unique reducible point P (i.e. one with zero spinor field), since X is negative definite. Moreover, as a *reducible* solution, P is regular. So for small non-zero ω_j, M will still contain a unique reducible point P. After perturbing M in a neighbourhood of P (along the lines of [D1]) if necessary, we may assume P is regular as a point in M. P will then have an open neighbourhood $U \subset M$ homeomorphic to an open cone on $\mathbb{CP}^d$ for some $d \geq 0$. Let $M^* = M \backslash \{P\}$ and $M^0 = M \backslash U$.

Let $B \subset X$ be a compact ball and let $\mathcal{B}_B^*$ be the orbit space of pairs (A, ϕ) defined over B with $\phi \neq 0$, modulo gauge. (Thus the A's are connections in $L|_B$ and the ϕ's are sections in $W^+|_B$.) Suitably completed, $\mathcal{B}_B^*$ is a Hilbert manifold. Let $\mathbb{L}_B \to \mathcal{B}_B^*$ be the canonical $U(1)$-bundle (i.e. the base-point fibration).

For any $\underline{\alpha}$ let

$$r : M^*(X; \underline{\alpha}) \to \mathcal{B}_B^*$$

be the restriction map, where the star in M^* means we remove the singular point if $\underline{\alpha} = \underline{\theta}$. Thus the domain of r is the union of all moduli spaces $M^*(X; \underline{\alpha})$. Note that r really maps into $\mathcal{B}_B^*$: for a solution to the Dirac equation cannot vanish on an open set of X unless it is identically zero.

Let s be a generic section of the m-fold direct sum $m\mathbb{L}$. By [DK, Lemma 5.2.9] we may assume each restriction map r is transverse (on each moduli space) to $s^{-1}(0)$, so each stratum of $V = r^{-1}s^{-1}(0)$ is smooth of the correct dimension.

In a moduli space $M^*(X; \underline{\alpha})$ with $\alpha_j = \theta_j$ there is also another source of cohomology. To explain this, let $\mathcal{B}_j^*$ be the orbit space of pairs (A, ϕ) (with $\phi \neq 0$) defined over $I_j = [0, 1] \times Y_j$. We will define a smooth map $r_j : M^*(X; \underline{\alpha}) \to \mathcal{B}_j^*$. Choose ϵ such that for each j, C_j has no critical value in the interval $(C_j(\theta_j), \epsilon]$. Fix $T \gg 0$ and define $\bar{t}_j : M(X; \underline{\alpha}) \to \mathbb{R}$ implicitly by

$$\begin{aligned} \bar{t}_j(S) &= T \quad \text{if } C_j(S_T) \leq \epsilon \\ C_j(S_{\bar{t}_j(S)}) &= \epsilon \quad \text{if } C_j(S_T) \geq \epsilon, \end{aligned}$$

where $S_t = S|_{\{t\}\times Y_j}$. Let $t_j = f\circ\bar{t}_j$, where f is a smooth function satisfying $f(x) = T$ for $x \leq T+1$, $f(x) = x$ for $x \geq T+2$, and $f' \geq 0$. Then t_j is smooth. We now define r_j by

$$r_j([S]) = [S|_{[t_j(S),t_j(S)+1]\times Y_j}].$$

Similarly, for any critical point $\alpha \neq \theta_j$ set

$$M^\epsilon(\alpha,\theta_j) = \{[S] \in M(\alpha,\theta_j) \mid C_j(S|_{\{0\}\times Y_j}) = \epsilon\}$$

and define a smooth map $r_j : M^\epsilon \to \mathcal{B}_j^*$ by

$$r_j([S]) = [S|_{[0,1]\times Y_j}].$$

Now let $\mathbb{L}_j \to \mathcal{B}_j^*$ be the canonical complex line bundle and s_j a generic section of $m_j\mathbb{L}_j$. We may assume r_j is transverse (on each moduli space where it is defined) to $s_j^{-1}(0)$. Let $V_j = r_j^{-1}s_j^{-1}(0)$. We can arrange that all intersections of strata in $V, V_1, \dots, V_l$ are transverse, and similarly for the intersection $\mathbb{CP}^d \cap V \cap (\cap V_j)$ in $M^* = M^*(X;\underline{\theta})$.

Now set

$$U = M^0 \cap V \cap (\cap V_j).$$

We claim U is compact. For U has dimension 1, so the standard compactification of U consists of U itself together with "broken solutions" with exactly two factors. The codimension 1 strata resulting from factorization on the j'th end have the form

$$(M(X;\underline{\alpha}) \cap V \cap (\bigcap_{i\neq j} V_i)) \times (M^\epsilon(\alpha_j,\theta_j) \cap V_j),$$

with $\alpha_j \neq \theta_j$ and $\alpha_i = \theta_i$ for $i \neq j$. For reasons of dimension we must have $i(\alpha,\theta_j) = 2m_j + 1$. However, by definition of m_j there is no such α_j, hence U is compact.

Now, it is easy to see that $r^*(\mathbb{L})$ and $r_j^*(\mathbb{L}_j)$ restrict to the tautological line bundle on $\mathbb{CP}^d$. But then $\partial U = \mathbb{CP}^d \cap V \cap (\cap V_j)$ has one point counted mod 2, contradicting the fact that U is a compact 1-manifold-with-boundary. Therefore (3.2) is impossible, so the theorem is proved. $\square$

Remark. Our main object in this paper is to give a simple proof of Theorem 1. However, when studying concrete examples it may be of interest to examine the proof above to see if one can obtain a lower bound on $\mathrm{rk}(J_Z) - |\xi|^2$. We will do this in [F2] and show that Theorems 1 and 6 hold for an invariant γ defined in terms of certain distinguished elements of the (metric dependent) Seiberg-Witten-Floer cohomology groups (with coefficients in some fixed field) rather than the critical points of C. These distinguished elements measure interaction with the critical point θ and can be identified with the differentials in a spectral sequence for computing the equivariant Seiberg-Witten-Floer group ([AB1], [AB2], [F1]).

4. Binary polyhedral spaces

Let $\Gamma \subset SU(2)$ be a finite subgroup and $D \subset \mathbb{C}^2$ the closed unit ball. Consider the rational homology sphere $Y = \partial D/\Gamma$. Let X be the smooth 4-manifold underlying the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$. The portion $Z \subset X$ lying above D/Γ is a compact 4-manifold with boundary Y. Recall that the intersection form J_Z is a root lattice. In particular, if Γ is the binary icosahedral group, in which case Y is the Poincaré sphere, then $J_Z = -E_8$.

Proposition 8. *Let c be the spin-structure that Y inherits from the unique spin-structure on Z. Then $\gamma(Y, c) = \mathrm{rk}(J_Z)$.*

Proof. Theorem 6 gives $\mathrm{rk}(J_Z) \leq \gamma(Y, c)$. We must prove that $\mathrm{rk}(J_Z) \geq \gamma(Y, c)$. The universal covering space $\tilde{V}$ of $V = X \setminus Z$ can be naturally identified with $\mathbb{C}^2 \backslash D$ as a smooth manifold. Let x_i be euclidean coordinates on $\tilde{V}$ and $r = \sum x_i^2$. By [K1] there is a hyperkähler structure on X whose metric g satisfies

$$\partial^p(g^{ij} - \delta^{ij}) = o(r^{-4-p}), \quad p \geq 0 \tag{4.1}$$

on V, where δ is the euclidean metric and ∂ denotes differentiation with respect to the coordinates x_i. Extend the metric δ to all of X, let $u : X \to \mathbb{R}_+$ be a smooth function equal to r^{-1} outside Z, and set $\tilde{g} = u^2 g$, $\tilde{\delta} = u^2\delta$. Then $(V, \tilde{\delta}|_V)$ is isometric to $\mathbb{R}_+ \times Y$, with $t = \log(r)$ the coordinate on $\mathbb{R}_+$.

Since X is simply-connected and has even intersection form it has a unique spin structure. Let $D^{\tilde{\delta}} : \Gamma(W^+) \to \Gamma(W^-)$ be the Dirac operator on $(X, \tilde{\delta})$. Recall that Y has positive scalar curvature, so $D^{\tilde{\delta}}$ defines a Fredholm operator $L^2_1 \to L^2$. We will show that $\mathrm{ind}(D^{\tilde{\delta}}) = 0$. By definition of $\gamma(Y, c)$ this will imply

$$\gamma(Y, c) \leq -\sigma(Z) = \mathrm{rk}(J_Z)$$

and the proposition will be proven.

We wish to compare the Dirac operators $D^{\tilde{g}}$ and $D^{\tilde{\delta}}$. Since taking the square root of a positive real symmetric matrix is a smooth operation, there is a natural choice of an endomorphism of TX which intertwines $\tilde{g}$ and $\tilde{\delta}$. This allows us to identify the frame bundles of $(X, \tilde{g})$ and $(X, \tilde{\delta})$, hence also their spin bundles. We choose $\tilde{\delta}$ as reference metric and use the corresponding Levi-Civita connection $\nabla^{\tilde{\delta}}$ to define the Sobolev spaces $L^2_1(W^{\pm})$.

This being said it is standard fare to deduce from (4.1) that $D^{\tilde{g}}$ defines a Fredholm operator $L^2_1(W^+) \to L^2(W^-)$ of the same index as $D^{\tilde{\delta}}$. Now let $\tilde{\psi} \in L^2_1(W^+ \oplus W^-)$ be any section satisfying $D^{\tilde{g}}(\tilde{\psi}) = 0$. Then $\psi = u^{\frac{3}{2}}\tilde{\psi}$ satisfies

$$D^g(\psi) = 0; \quad |\psi| = o(r^{-\frac{3}{2}}); \quad |\nabla^g(\psi)| = o(r^{-\frac{5}{2}}).$$

(See [H].) In particular, $\langle \nabla^g(\psi), \psi \rangle_g = o(r^{-4})$. We can now copy the argument in the proof of [K2, Lemma 2.2], using the Weitzenböck formula for the Dirac operator and the fact that g has zero scalar curvature, to conclude that $\nabla^g(\psi) = 0$. Hence $\psi = 0$, so $\mathrm{ind}(D^{\tilde{g}}) = 0$. This completes the proof of Proposition 8. $\square$

Appendix A. Proof of Lemma 4

(0) We begin with a remark on unique continuation. Let $S = \{S_t\}$ be any smooth solution to the gradient flow equation $\frac{\partial S_t}{\partial t} = -\mathrm{grad}(C)_{S_t}$. Then if $\frac{\partial S_t}{\partial t} = 0$ at some point $t = t_0$ we must have $S_t = S_{t_0}$ for all t. This follows from the unique continuation argument in [DK, section 4.3.4] since we can define another solution $\tilde{S}$, of class C^1 on $\mathbb{R} \times Y$, by

$$\tilde{S}_t = \begin{cases} S_t \text{ for } t \leq t_0 \\ S_{t_0} \text{ for } t \geq t_0. \end{cases}$$

(1) Fix $b > 0$ and let $\{\omega_n\}$ be any sequence with $\|\omega_n\|_{C^k} \to 0$. Let $\{S_n = (A_n, \phi_n)\}$ be a smooth solution to (2.1) with $\omega = \omega_n$ and suppose $\|\phi_n\|_{C^0} \leq b$. Then $\|h^*_{S_n}(\omega_n)\|_{C^k} \to 0$. If B is any smooth connection in L then over any bounded open subset V (containing $[-1, 1]$, say) we can find gauge transformations u_n such that

$$d^*(u_n^2(A_n) - B) = 0.$$

Arguing as in [KM] we see that there is a subsequence of $\{u_n(S_n)\}$ which converges in C^k over compact subsets of V to some solution S_∞ of the equations (2.1) with $\omega = 0$.

Now let $C_n'(t) = \frac{\partial}{\partial t} C(S_n(t))$ and suppose $C_n'(t_n) \geq 0$ for some t_n. Since the equations (2.1) are translationary invariant we may assume $t_n = 0$. Equation (2.2) and the Cauchy-Schwartz inequality gives

$$\|\mathrm{grad}(C)_{S_n(0)}\|_{L^2} \leq \|E_{S_n}(0)\|_{L^2} \leq \mathrm{const} \cdot \|\omega_n\|_{C^0}. \tag{A.1}$$

Therefore $\mathrm{grad}(C)_{S_\infty(0)} = 0$, so the argument in (0) shows that in a temporal gauge for S_∞, $S_\infty(t)$ must be independent of t. Therefore

$$\lim_n h_{S_n}(0) = h_{S_\infty}(0) = C(S_\infty(0)) \notin \Xi.$$

But Ξ is closed, so for sufficiently large n we must have $h_{S_n}(0) \notin \Xi$, whence $E_{S_n}(0) = 0$ by definition. Equation (2.2) then gives

$$C_n'(0) = -\|\mathrm{grad}(C)_{S_n(0)}\|_{L^2}^2 \leq 0,$$

so $S_n(0)$ is a critical point. By (0) above, $[S_n(t)]$ is constant in t on the maximal interval containing 0 where $h_{S_n} \notin \Xi$. Since Ξ is closed and does not contain any critical points this interval must be all of $\mathbb{R}$. This proves statement (1) of the lemma.

(2) Now let ω be so small that statement (1) holds, and such that any $P \in \mathcal{A}(L_0) \times \Gamma(W_0)$ satisfying

$$\|\mathrm{grad}(C)_P\|_{L^2} \leq \mathrm{const} \cdot \|\omega\|_{C^0} \tag{A.2}$$

is close to some critical point of C, where the constant is the same as in (A.1). More precisely, we require $C(P) \notin \Xi$ whenever (A.2) holds.

Let $S = (A, \phi)$ be a smooth solution to the equations (2.1) satisfying $\|\phi\|_{C^0} \leq b$, and let $C(S_t)$ be bounded in t. Then if $\{t_n\}$ is any sequence with $t_n \to \infty$ we can find gauge transformations u_n such that a subsequence of $u_n(S|_{(t_n-2,t_n+2)\times Y})$ converges in C^k to a point $\tilde{S}$ which is defined on $(-2, 2) \times Y$, say, and provides a solution to (2.1) on $(-1, 1) \times Y$. Clearly, $\frac{\partial}{\partial t} C(\tilde{S}_t) = 0$, so our assumption on ω implies that for $t \in [-1, 1]$, $\tilde{S}_t$ is a critical point of C. Since the critical points of C are isolated, the gauge equivalence class $[\tilde{S}_t]$ must be independent of t, and we see that as $t \to \infty$, $[S_t]$ converges in C^k to some critical point α_+. Similarly, we obtain a limit α_- as $t \to -\infty$.

Added in proof: At the end of Section 2, in order to obtain transversality in moduli spaces over X it is useful to let the forms ω_j vary as well and argue as in the proof of Propostion 5.

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