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# LENGTHS OF PERIODS AND SESHADRI CONSTANTS OF ABELIAN VARIETIES

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## Introduction

The purpose of this note is to point out an elementary but somewhat surprising connection between the work of Buser and Sarnak [BS] on lengths of periods of abelian varieties and the Seshadri constants measuring the local positivity of theta divisors. The link is established via symplectic blowing up, in the spirit of [McDP]. As an application, we get a simple new proof of a statement of Buser-Sarnak type to the effect that a Jacobian has a period of unusually short length.

We start by recalling the definition of Seshadri constants. Let $X$ be a smooth complex projective variety, let $L$ be an ample line bundle on $X$, and fix a point $x \in X$. Consider the blowing-up

$$f : Y = \mathrm{Bl}_x(X) \longrightarrow X$$

of $X$ at $x$, with exceptional divisor $E = f^{-1}(x) \subset Y$. Then for $0 < \epsilon \ll 1$ the cohomology class $f^*c_1(L) - \epsilon \cdot [E]$ will lie in the Kähler cone of $Y$. As a measure of how positive $L$ is locally near $x$ we ask in effect how large we can take $\epsilon$ to be while keeping the class in question positive. More precisely, set

$$\epsilon(L, x) = \sup \{ \ \epsilon \geq 0 \mid f^*c_1(L) - \epsilon \cdot [E] \ \text{ is nef } \} .$$

Here $f^*c_1(L) - \epsilon \cdot [E]$ is considered as an $\mathbb{R}$-divisor class on $Y$, and to say that it is nef means that $\int_{C'} f^*c_1(L) \geq \epsilon(E \cdot C')$ for every irreducible algebraic curve $C' \subset Y$.[1] We refer to [Dem, §6] or [EKL, §1] for further discussion and alternative characterizations. Introduced by Demailly in [Dem], these Seshadri constants have attracted considerable interest in recent years. The main result of [EKL] states that if $X$ has dimension $n$, then at a very general

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[1] Recall that a theorem of Kleiman characterizes the nef cone as the closure of the ample cone.

point $x \in X$ one has the universal lower bound $\epsilon(L,x) \geq \frac{1}{n}$ (cf. also [KS]). Some more refined results when $X$ is a surface appear in [EL], [S] and [Xu2], but except in the simplest examples Seshadri constants have proven very difficult to control with any precision. We propose here to study these invariants when the ambient manifold is an abelian variety.

Suppose then that $(A,\Theta)$ is a principally polarized abelian variety of dimension $g$, i.e. that $A$ is a complex torus, and that $\Theta \subset A$ is an ample divisor with $h^0(A, \mathcal{O}_A(\Theta)) = 1$. Since $A$ is homogeneous, the Seshadri constants $\epsilon(\mathcal{O}_A(\Theta), x)$ are independent of $x \in A$, and we denote their common value by $\epsilon(A,\Theta)$ or simply $\epsilon(A)$. One has the elementary upper bound

$$\epsilon(A) \leq \sqrt[g]{g!}$$

(cf. [EKL, 1.8]). Nakamaye [N] has shown that $\epsilon(A,\Theta) \geq 1$, with equality iff $(A,\Theta)$ is the product of an elliptic curve and an abelian variety of dimension $g-1$.

Our goal is to relate the Seshadri constant $\epsilon(A)$ to a metric invariant of $(A,\Theta)$. As usual, write $A$ as a quotient

$$A = V/\Lambda$$

of its universal covering, so that $V \cong \mathbb{C}^g$, and $\Lambda \subset V$ is a lattice in $V$. The principal polarization $\Theta$ determines a positive definite Hermitian form $H$ on $V$ (cf. [LB,Chapter 2]), and following [BS] we define

$$m(A) = m(A,\Theta) = \min_{x \in \Lambda - \{0\}} H(x,x).$$

Thus $m(A)$ is the square of the minimal length (with respect to $H$) of a non-zero lattice vector. This is the analogoue for abelian variety period lattices of an invariant familiar in connection with sphere packings and the geometry of numbers (cf. [O]). Buser and Sarnak study the maximum value of $m(A)$ as $A$ varies over the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, and they show ([BS, §2]) that there exist p.p.a.v.'s $(A,\Theta)$ for which

$$\text{(BS1)} \qquad m(A) \geq \frac{1}{\pi}\,(2g!)^{1/g}\,.$$

The most surprising result of [BS] is that if $C$ is a smooth projective algebraic curve of genus $g \geq 2$, and $(J(C), \Theta_C)$ is its polarized Jacobian, then one has the upper bound

$$\text{(BS2)} \qquad m(J(C)) \leq \frac{3}{\pi}\log(4g+3).$$

In other words, for $g \gg 0$ a Jacobian has a period of unusually short length.

Our main result states that the Seshadri constant of $A$ is bounded below in terms of the mimimal length of a period:

**Theorem.** *One has the inequality*

$$\epsilon(A, \Theta) \geq \frac{\pi}{4} m(A, \Theta).$$

Note that in general (and maybe always) the inequality is strict, as one sees already in the one dimensional case.

This inequality has a number of pleasant consequences. In the first place, combining the Theorem with the bound (BS1) of Buser and Sarnak, we obtain the

**Corollary.** *Let $(A, \Theta)$ be a very general principally polarized abelian variety. Then*

$$\epsilon(A, \Theta) \;\geq\; \frac{2^{\frac{1}{g}}}{4} \sqrt[g]{g!} \;\approx\; \frac{g}{4e}.$$

The hypothesis on $A$ means that the inequality is valid off the union of countably many proper subvarieties of the moduli space $\mathcal{A}_g$. In the approximation, which holds for $g \gg 0$, we are ignoring the factor of $2^{1/g}$. Observe that this lower bound differs from the upper bound $\epsilon(A) \leq \sqrt[g]{g!}$ by a factor of less than 4. It would be interesting to know whether $\epsilon(A_{\text{very general}}) = (g!)^{1/g}$ for large $g$.[2]

Now let $C$ be a compact Riemann surface of genus $g \geq 2$, and as above let $(J(C), \Theta_C)$ be its polarized Jacobian. It is rather easy to obtain upper bounds on the Seshadri constants of $J(C)$:

**Proposition.** (i). *One has*

$$\epsilon(J(C), \Theta_C) \leq \sqrt{g}.$$

(ii). *Suppose that $C$ can be expressed as a $d$-sheeted branched covering $\phi : C \longrightarrow \mathbb{P}^1$. Then*

$$\epsilon(J(C), \Theta_C) \leq \frac{gd}{g + d - 1}.$$

For hyperelliptic curves (when $d = 2$), the inequality (ii) was established by Steffens [S]. Combining statement (i) and the Theorem, one arrives at an elementary new proof that a Jacobian has a period of small length, although the specific inequality that comes out is not as strong as (BS2). On the other hand, we see from (ii) that if $C$ is a $d$-sheeted covering of $\mathbb{P}^1$, then in fact

$$m(J(C), \Theta_C) \leq \frac{4d}{\pi}.$$

[2] Proposition 3 of [S] asserts that equality never holds, but the proof is erroneous (a circumstance for which the present author must share some culpability).

This seems to be new. In the other direction, Buser and Sarnak construct examples of curves to show that the supremum of $m(J(C))$ on the moduli space $\mathcal{M}_g$ is $\geq c \cdot \log(g)$, where $c$ is a small positive constant. Hence the Seshadri constant $\epsilon(J(C_{\text{very general}}))$ of the Jacobian of a very general curve satisfies the same inequality (with a slightly different constant). It would be interesting to know how $\epsilon(J(C_{\text{very general}}))$ actually grows with $g$. It is also tempting to wonder to what extent small Seshadri constants might characterize Jacobians among all irreducible p.p.a.v.'s.

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## §1. Proofs of Theorem and Proposition

The Theorem is a simple consequence of the construction of the symplectic blowing up of a point, as explained for example in the paper [McDP] of McDuff and Polterovich. The basic fact, which is implicit in [McDP], is a relation between Seshadri constants and radii of symplectically embedded holomorphic balls. This connection was exploited in a related but more sophisticated manner in [McDP].

We start by fixing notation. In $\mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$, denote by

$$\omega_{\text{std}} = \sum dx_j \wedge dy_j = \frac{i}{2} \sum dz_j \wedge d\overline{z}_j$$

the standard symplectic form. Write $B(\lambda) \subset \mathbb{C}^n$ for the open ball of radius $\lambda$ centered at the origin:

$$B(\lambda) = \{ z \in \mathbb{C}^n \mid |z|^2 < \lambda^2 \}.$$

We view $B(\lambda)$ as a complex manifold, and also as a symplectic manifold via $\omega_{\text{std}}$.

Now let $X$ be a smooth projective variety of dimension $n$, $L$ an ample line bundle on $X$, and $\omega_L$ a Kähler form[3] on $X$ representing $c_1(L)$. We view $(X, \omega_L)$ as a symplectic manifold. Given $x \in X$ we define a real number $\lambda(x) = \lambda(\omega_L, x) \geq 0$ by looking for the largest radius $\lambda > 0$ for which there exists a holomorphic and symplectic embedding

$$(*) \qquad j = j_\lambda : (B(\lambda), \omega_{\text{std}}) \hookrightarrow (X, \omega_L) \quad \text{with} \quad 0 \mapsto x.$$

More precisely, if there is no $\lambda > 0$ for which an embedding (*) exists, set $\lambda(x) = 0$. Otherwise, put

$$\lambda(\omega_L, x) = \sup \{ \lambda > 0 \mid \ \exists \text{ holomorphic and symplectic } j_\lambda \text{ as in } (*) \ \}.$$

[3] I.e. a closed positive $(1,1)$ form.

**Main Lemma.** *One has the inequality*

$$\epsilon(L, x) \geq \pi\lambda(\omega_L, x)^2.$$

By way of proof, it would probably be almost enough just to refer to [McDP], (5.1)-(5.3). But since the lemma isn't stated there explicitly, and since it involves some ideas that are not standard algebro-geometrically, we will summarize the argument for the benefit of the reader in §2. In the meantime, we grant the lemma.

The rest of the proofs are quite immediate:

*Proof of Theorem.* Let $\pi : V \longrightarrow A$ be the universal covering, and as above let $H$ be the Hermitian form on $V$ determined by $\Theta$. In the natural way, we may view the imaginary part $\omega = \operatorname{im} H$ as a symplectic form on $V$, which is in fact the pull-back $\omega = \pi^*\omega_\Theta$ of a Kähler form $\omega_\Theta$ on $A$ representing $c_1(\mathcal{O}_A(\Theta))$. We fix a basis of $V$ with respect to which $H$ is the standard Hermitian form

$$H(v, w) \;=\; {}^t\overline{v} \cdot\; w$$

on $\mathbb{C}^g$. Then taking $z_j$ to be the corresponding complex coordinates, one has

$$\omega = \pi^*\omega_\Theta = \omega_{\mathrm{std}}, \tag{*}$$

and $H(x, x) = |x|^2$ is just the usual Euclidean length. In particular,

$$m(A) = \min_{x \in \Lambda - \{0\}} \left\{ \; |x|^2 \; \right\}.$$

Now let $\lambda = \sqrt{m(A)}/2$. Then given any two points $x, y \in B(\lambda)$ one has $|x - y| < 2\lambda = \sqrt{m(A)}$. Therefore no two points of $B(\lambda)$ are congruent (modulo $\Lambda$), and consequently the composition

$$j_\lambda : B(\lambda) \hookrightarrow V \xrightarrow{\pi} A$$

is an embedding. But $j_\lambda$ is of course holomorphic, and thanks to (*) it is symplectic as well. Therefore

$$\lambda(\omega_\Theta, 0) \geq \frac{\sqrt{m(A)}}{2},$$

and the Theorem follows from the Main Lemma. $\square$

*Proof of Proposition.* We assume to begin with that $C$ is non-hyperelliptic. Consider the subtraction map

$$s : C \times C \longrightarrow J(C), \quad (x, y) \mapsto \mathcal{O}_C(x - y) \in Pic^0(C) = J(C),$$

and let $\Sigma \subset J(C)$ be its image. It is elementary and well known (cf. [ACGH, pp. 223, 263]) that if $C$ is non-hyperelliptic, then $s$ is an isomorphism off the diagonal $\Delta \subset C \times C$, and blows $\Delta$ down to the origin $0 \in \Sigma$, which is a point of multiplicity $2g-2$. Moreover $\Delta$ is the scheme-theoretic inverse image of the singular point $0 \in \Sigma$. The required inequalities will follow from some computations in the intersection ring of $C \times C$. To this end, let $F_1, F_2 \subset C \times C$ be the preimages of a point of $C$ under the two projections. Then working with numerical equivalence of divisors, one checks that

$$s^*(\Theta) \equiv (g-1)(F_1 + F_2) + \Delta$$

(cf. [R]). It follows with a calculation that the degree of $\Sigma$ with respect to $\Theta$ is

$$\begin{aligned} \deg_\Theta(\Sigma) = \Theta^2 \cdot \Sigma &= \big((g-1)(F_1 + F_2) + \Delta\big)^2 \\ &= 2g(g-1). \end{aligned}$$

Then by [Dem, (6.7)]:

$$\begin{aligned} \epsilon(J(C), \Theta) &\leq \sqrt{\frac{\deg_\Theta(\Sigma)}{\mathrm{mult}_0 \Sigma}} \\ &= \sqrt{\frac{2g(g-1)}{2g-2}} \\ &= \sqrt{g}. \end{aligned}$$

Turning to statement (ii), let $L = \phi^* \mathcal{O}_{\mathbb{P}^1}(1)$. Then there is an effective divisor $\Gamma \subset C \times C$ with

$$\Gamma \in |pr_1^* L \otimes pr_2^* L \otimes \mathcal{O}_{C \times C}(-\Delta)|.$$

Geometrically, for instance, we may realize $\Gamma$ as the closure of $\Gamma_0 = \{(x, y) \mid x \neq y, \phi(x) = \phi(y)\}$. Now if $\epsilon = \epsilon(J(C), \mathcal{O}_J(\Theta))$, then $s^*(\Theta) - \epsilon \cdot \Delta$ is nef on $C \times C$. Hence

$$\begin{aligned} \Gamma \cdot (s^*(\Theta) - \epsilon\Delta) &= \big(d(F_1 + F_2) - \Delta\big) \cdot \big((g-1)(F_1 + F_2) + (1 - \epsilon)\Delta\big) \\ &\geq 0, \end{aligned}$$

and with another calculation this leads to the second assertion of the Proposition. Finally, if $C$ is hyperelliptic the only thing that needs proof is statement (ii) with $d = 2$, and this follows by looking at the image in $J(C)$ of the curve $\Gamma$ just constructed. $\square$

## §2. Sketch of proof of Main Lemma

Finally, for the benefit of readers not versed in symplectic matters, we outline the proof of the Main Lemma. We follow [McDP], §5 (also pp. 414 ff), quite closely. The essential point, which seems to go back at least as far as [GS], is to construct explicitly a Kähler form on the blow-up $\mathrm{Bl}_0(\mathbb{C}^n)$ which agrees with the standard form off a ball of specified radius. The presence of a symplectically embedded holomorphic ball allows one to carry over the local construction to a global setting, and then the inequality of the Main Lemma follows from the positivity of the form so constructed.

Turning to the details, let

$$V \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$$

be the blowing up of $0 \in \mathbb{C}^n$, embedded in the usual way as an incidence correspondence. Write

$$f : V \longrightarrow \mathbb{C}^n, \qquad q : V \longrightarrow \mathbb{P}^{n-1}$$

for the projections, so that $f$ is the blowing-up, and $q$ realizes $V$ as the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Denote by $V(\lambda)$ the inverse image of the ball $B(\lambda) \subset \mathbb{C}^n$:

$$V(\lambda) = f^{-1}B(\lambda) \subset V,$$

so that $V(\lambda)$ is an open neighborhood of the exceptional divisor $E = \mathbb{P}^{n-1} \subset V$. Finally, let $\sigma$ be the usual Fubini-Study Kähler form on $\mathbb{P}^{n-1}$, normalized so that $\int_{\mathbb{P}^1} \sigma = \pi$, the integral being taken over a line in $\mathbb{P}^{n-1}$. This normalization is chosen so that if $S = S^{2n-1} \subset \mathbb{C}^n$ is the unit sphere, and $\kappa : S \longrightarrow \mathbb{P}^{n-1}$ is the Hopf map, then $\kappa^*\sigma = \omega_{\mathrm{std}}|S$.

The crucial ingredient is the following

**Basic Local Construction.** *Fix $\lambda > 0$. Given any small $\eta > 0$, there exists some $0 < \delta \ll 1$, plus a Kähler form $\overline{\tau} = \overline{\tau}(\lambda, \eta)$ on $V$ such that*

(i). $\overline{\tau} = f^*(\omega_{\mathrm{std}})$ *on* $V - \overline{V\left(\lambda(1+\eta)\right)}$;

(ii). $\overline{\tau} = f^*(\omega_{\mathrm{std}}) + \lambda^2 q^*(\sigma)$ *on* $V(\delta)$.

In other words, $\overline{\tau}$ coincides with the standard Kähler form on $\mathbb{C}^n$ off a ball of radius (a tiny bit larger than) $\lambda$, whereas in a neighborhood of the exceptional divisor, we are "twisting" by a form representing $\pi\lambda^2 q^* c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$. ∎ This is an extremely slight variant of [McDP, (5.1)], proved exactly as in

[McDP, (5.2), (5.3)], and we refer the reader to the very clear exposition there.[4] See also [McDS, §6.2].

Given this local construction, the proof of the Main Lemma is rather evident. Let $f : Y = \mathrm{Bl}_x(X) \longrightarrow X$ be the blowing up of $X$, with exceptional divisor $E \subset Y$, and fix any $\lambda < \lambda(\omega_L, x)$. It is enough to show that

(*) the $\mathbb{R}$-divisor class $f^*(c_1(L)) - \pi\lambda^2[E]$ is nef on $Y$.

To this end, fix $0 < \eta \ll 1$ so that $\lambda \cdot (1 + 3\eta) < \lambda(\omega_L, x)$. We have a holomorphic and symplectic embedding

$$B(\lambda \cdot (1 + 3\eta)) \hookrightarrow X, \tag{**}$$

and so for $\nu < \lambda \cdot (1 + 3\eta)$ we can view the local model $V(\nu)$ as being embedded in $Y$ as a neighborhood of the exceptional divisor. Thanks to property (i) and the fact that the embedding (**) is symplectic, the basic local construction guarantees the existence of a Kähler form $\overline{\omega}_L$ on $Y$, agreeing with $\omega_L$ off $V(\lambda(1+2\eta))$, and being given by (ii) in a neighborhood $V(\delta)$ of $E$. Since $\overline{\omega}_L$ is Kähler, and in particular positive, (*) will follow once we know that its cohomology class satisfies

$$[\overline{\omega}_L] = f^*[\omega_L] - \pi\lambda^2[E] = f^*c_1(L) - \pi\lambda^2[E]. \tag{***}$$

But $\overline{\omega}_L - f^*\omega_L$ is supported in a small neigborhood of $E$, and then (***) follows easily using (ii) and the normalization of $\sigma$. $\square$

[4] In brief, choose a monotone increasing smooth function $\phi(r)$ such that $\phi(r) = \sqrt{\lambda^2 + r^2}$ for $0 < r < \delta \ll 1$, and such that $\phi(r) = r$ for $r > \lambda(1 + \eta)$, and then consider the smooth mapping

$$F : \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}^n, \quad F(z) = \frac{\phi(|z|)}{|z|} \cdot z.$$

Then $\overline{\tau} = f^*F^*\omega_{\mathrm{std}}$, extended over $E$ by (ii), has the required properties.

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