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A COUNTEREXAMPLE FOR MAXIMAL OPERATORS OVER A CANTOR SET OF DIRECTIONS

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Abstract. We produce a counterexample to the boundedness of the Cantor set maximal operator. We also produce a sharp counterexample for the restriction of the maximal operator of a truncated Cantor set to a strip.

§0. Introduction

Let $\mathcal{R}_N$ be the set of rectangles in $\mathbb{R}^2$ having length 1 and width $\frac{1}{N}$. Let us define the maximal operator on $\mathbb{R}^2$ given by

$$(\mathcal{M}_N f)(x) = \sup_{x \in R \in \mathcal{R}_N} \frac{1}{|R|} \int_R |f|.$$

Cordoba in [C] proved that for any $f \in L^2(\mathbb{R}^2)$, one has the inequality,

$$||\mathcal{M}_N f||_{L^2} \leq C\sqrt{1 + \log N}||f||_{L^2}. \tag{0.1}$$

In fact, he showed something stronger. He demonstrated that for any $S \in \mathcal{R}_N$ where $\chi_S(x)$ is the characteristic function of S that

$$||\chi_S \mathcal{M}_N^S f||_{L^2} \leq C\frac{\sqrt{1 + \log N}}{\sqrt{N}}||f||_{L^2}. \tag{0.2}$$

Here $\mathcal{M}_N^S$ is defined just as $\mathcal{M}_N$ by

$$(\mathcal{M}_N^S f)(x) = \sup_{x \in R \in \mathcal{R}_N^S} \frac{1}{|R|} \int_R |f|,$$

where $\mathcal{R}_N^S$ is defined as the set of elements of $\mathcal{R}_N$ making an angle of at least $\frac{\pi}{4}$ with S. He proved (0.1) by observing that it is sufficient to work on a square and combining (0.2) on N parallel rectangles S.

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One line of research is to determine whether restrictions on the directions of allowed rectangles will improve the estimate (0.1). That is let $\Sigma \subset [0,1]$ and define $\mathcal{R}_N^{\Sigma}$ to be those elements of $\mathcal{R}_N$ having slope in Σ. Then we define

$$(\mathcal{M}_N^{\Sigma} f)(x) = \sup_{x \in R \in \mathcal{R}_N^{\Sigma}} \frac{1}{|R|} \int_R |f|.$$

It suffices to consider sets Σ whose elements are fractions with denominator N. If Σ is a lacunary set, then in particular it can have at most $C \log N$ elements. By a result of [NSW], one has that

$$||\mathcal{M}_N^{\Sigma} f||_{L^2} \leq C ||f||_{L^2}, \tag{0.3}$$

where the constant C depends only on the order of lacunarity. Now, in light of the ideas in [K], (There, boundedness with constant $\sqrt{\log N}$ is obtained for a certain class of paraproducts whose boundedness implies boundedness of the maximal operator over unit intervals in N directions. Boundedness of the same class of paraproducts multiplied by the scalar $\frac{1}{\sqrt{N}}$ implies boundedness of the maximal operator restricted to a strip) for any set Σ with $\log N$ elements, one gets automatically

$$||\chi_S \mathcal{M}_N^{\Sigma} f||_{L^2} \leq C \frac{\sqrt{1 + \log \log N}}{\sqrt{N}} ||f||_{L^2},$$

but for Σ lacunary, it is quite easy to obtain

$$||\chi_S \mathcal{M}_N^{\Sigma} f||_{L^2} \leq C \frac{1}{\sqrt{N}} ||f||_{L^2}, \tag{0.4}$$

where, in both cases, S is parallel to the y-axis.

Now letting $N = 3^n$ for some integer n, if Σ is the truncated Cantor set (i.e. Σ consists of all numbers of the form 0,1, or $\sum_{j=1}^{n} \frac{a_j}{3^j}$ where a_j is 0 or 2.), then it has been conjectured (e.g. [V]) that one obtains (0.3). Positive results for radial functions may be found in [DV]. However, we point out that (0.4) fails, i.e that

Theorem 1. *Let Σ be the Cantor set above, let S be parallel to the y-axis. The exists a function f and a constant $c > 0$ so that*

$$||\chi_S \mathcal{M}_N^{\Sigma} f||_{L^2} \geq c \frac{\sqrt{\log N}}{\sqrt{N}} ||f||_{L^2}.$$

Thus, in light of [C], the constant $\frac{\sqrt{\log N}}{\sqrt{N}}$ is sharp (as far as the exponent of $\log N$ for $\chi_S \mathcal{M}_N^{\Sigma}$.) We also prove the possibly less sharp

Theorem 2. *Let Σ be as above. There exists a function f and a constant c so that*

$$||\mathcal{M}_N^{\Sigma} f||_{L^2} \geq c\sqrt{\log\log N}||f||_{L^2}.$$

Theorem 2 also implies that for any $p < 2$, there exists c_p and a function g so that

$$||\mathcal{M}_N^{\Sigma} g||_{L^p} \geq c_p(\log\log N)^{\frac{1}{p}}||f||_{L^p}.$$

(In particular, we may choose $g = f^{\frac{2}{p}}$.) For $p > 1 + \frac{\log 2}{\log 3}$, this is a new result. However, for $p > 2$, the problem of boundedness or unboundedness of M_N^{Σ} in L^p remains open.

§1. Proof of Theorem 1

It suffices to let $n > 2$ for otherwise the theorem is trivial. It suffices to work on $S = [0, 3^{-n}] \times [0, \frac{1}{3}]$. We define $Q(l)$ to be the square of sidelength $\frac{1}{N}$ described as $[0, 3^{-n}] \times [(l-1)3^{-n}, l3^{-n})$. For $1 \leq l \leq \frac{N}{3}$, we assign $s(l) \in \Sigma$. We define the linear operator L taking functions on $\mathbb{R}^2$ to functions on S by

$$Lf(x) = \sum_{j=1}^{N} \chi_{Q_j}(x)(\frac{1}{|R_j|}\int_{R_j} f).$$

Here R_j is the rectangle with dimensions $3 \times (\frac{1}{3N})$ centered at the center of Q_j having slope $s(j)$. To prove the theorem, it suffices to show that there exists f, a function on S, a choice of $s(j)$ and a constant $c > 0$ not depending on N so that

$$||LL^* f||_{L^2} \geq \frac{c\log N}{N}||f||_{L^2}.$$

We let $f = \chi_S$. Then $||f||_{L^2} = \frac{1}{\sqrt{N}}$. Thus we must show there exists a choice of $s(j)$ with

$$||LL^* f||_{L^2} \geq \frac{c\log N}{N^{\frac{3}{2}}}.$$

Now by definition

$$||LL^* f||_{L^2} = (\int_S (LL^* f)^2)^{\frac{1}{2}},$$

whilst by Hölder's inequality,

$$\int_S LL^* f \leq (\int_S (LL^* f)^2)^{\frac{1}{2}}(|S|^{\frac{1}{2}}),$$

so that it suffices to show

$$\int_S LL^* f \geq \frac{c \log N}{N^2},$$

which is what we shall do.

Now,

$$L^* f = \sum_{j=1}^{n} (\int_{Q_j} f) \frac{1}{|R_j|} \chi_{R_j} \geq c \frac{1}{N} \sum_{j=1}^{N} \chi_{R_j}.$$

Thus for $x \in Q_k$, one has

$$LL^* f(x) \geq c \sum_{j=1}^{N} |R_j \cap R_k|,$$

so that

$$\int LL^* f \geq \frac{c}{N^2} \sum_{k=1}^{N} \sum_{j=1}^{N} |R_j \cap R_k|.$$

Now whenever $|s(j) - s(k)| \geq \frac{|j-k|}{N}$, we have that

$$|R_j \cap R_k| \geq \frac{c}{|s(j) - s(k)| N^2}.$$

We let $K(j,k) = 1$ when $|s(j) - s(k)| \geq \frac{|j-k|}{N}$ and $K(j,k) = 0$ otherwise. Then it suffices to show there exists a choice of the map s so that

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \frac{K(j,k)}{|s(j) - s(k)| N^2} = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{K(j,k)}{|s(j) - s(k)|} \geq cn. \tag{1.1}$$

We will find such a map s.

We produce a continuous version of s. Let $\mathfrak{C}$ be the middle thirds Cantor set contained in $[0, 1]$. Let d be the triadic distance on $[0, 1]$. For any f measurable taking $[0, \frac{1}{3}]$ to $\mathfrak{C}$, we define $K_f(x, y) = 1$ when $|f(x) - f(y)| \geq |x - y|$ and 0 otherwise. Clearly $K_f(x, y)$ cannot be 1 everywhere since then f^{-1} would be a well defined Lipschitz map from $\mathfrak{C}$ onto $[0, \frac{1}{3}]$ which is impossible.

Proposition 1.1. *There exists f defined a.e. from $[0, \frac{1}{3}]$ to $\mathfrak{C}$ so that for every $n > 0$,*

$$\int_{d(x,y) \geq 3^{-n}} \frac{K_f(x, y) dx dy}{|f(x) - f(y)|} \geq cn.$$

Proof. Almost every $x \in [0, \frac{1}{3}]$ has a unique infinite ternary expansion $x = 0.0x_1x_2\ldots$. Every infinite ternary expansion $0.y_1y_2\ldots$ denotes a unique element of $\mathfrak{C}$ as long as for every j, one has $y_j = 0$ or $y_j = 2$. We define

$$f(0.0x_1x_2\ldots) = .g(x_1)g(x_2)\ldots,$$

where $g : \{0,1,2\} \longrightarrow \{0,2\}$ by $g(0) = g(1) = 0$ and $g(2) = 2$. For any I triadic with $|I| = 3^{-j}$, we observe by rescaling that

$$\begin{aligned} &\int_{\{(x,y)\in I\times I: d(x,y)=3^{-j}\}} \frac{K_f(x,y)dxdy}{f(x)-f(y)} = \\ &3^{1-j}\int_{\{(x,y)\in[0,\frac{1}{3}]\times[0,\frac{1}{3}]: d(x,y)=\frac{1}{3}\}} \frac{K_f(x,y)dxdy}{f(x)-f(y)}. \end{aligned} \tag{1.2}$$

On the other hand

$$\int_{\{(x,y)\in[0,1]\times[0,1]: d(x,y)=1\}} \frac{K_f(x,y)dxdy}{f(x)-f(y)} \geq \frac{4}{81},$$

since $\frac{1}{3} \leq |f(x) - f(y)| \leq 1$ when $x \in [0, \frac{2}{9}]$ and $y \in [\frac{2}{9}, \frac{1}{3}]$. Thus for any l, we have that

$$\int_{d(x,y)=3^{-l}} \frac{K_f(x,y)dxdy}{|f(x)-f(y)|} \geq \frac{4}{81},$$

by summing over all triadic $J \subset [0,1]$ with $|J| = 3^{-l}$. Summing over l going from 1 to $c\log N$, we obtain the proposition. □

Now we are ready to define $s(j)$. If $j = \sum_{l=1}^{n-1} a_l 3^{n-l-1}$, then $s(j) = \sum_{l=1}^{n} g(a_l)3^{-l}$. By the same reasoning as the proposition, we prove (1.1) and Theorem 1.

§2. Proof of Theorem 2

We let $n > 10$ for otherwise the theorem is trivial. We let the rectangles R_j be as before. We define the function

$$f = \sum_{j=1}^{3^{n-1}} \chi_{R_j}.$$

We will prove the following three lemmas:

Lemma 2.1. *With $C > 0$ a universal constant independent of N,*

$$\int f^2 \leq C\log N. \tag{2.1}$$

Lemma 2.2. *There exists a constant $c > 0$ independent of N, so that for all $1 \leq j \leq 3^{n-1}$,*

$$\frac{1}{|R_j|}\int_{R_j} f \geq c \log N. \tag{2.2}$$

Lemma 2.3. *There exists a constant $c > 0$ independent of N so that*

$$|\cup R_j| \geq \frac{c \log\log N}{\log N}. \tag{2.3}$$

We first prove Theorem 2 from these three lemmas.

Proof of Theorem 2. By (2.2), we have that everywhere on $\cup R_j$,

$$\mathcal{M}_N^{\Sigma} f \geq c \log N.$$

Now by (2.3) this yields

$$\int_{\cup R_j} (M_N^{\Sigma} f)^2 \geq c(\log\log N)(\log N). \tag{2.4}$$

The inequalities (2.1) and (2.4) yield Theorem 2 □

Now we need only prove the three lemmas.

Proof of Lemma 2.1. By [C], the operator defined by

$$(Lg)(x) = \sum_{j=1}^{3^{n-1}} \chi_{Q_j}(x)\left(\frac{1}{|R_j|}\int_{R_j} g\right),$$

is bounded on L^2 with norm less than or equal to $\frac{C\sqrt{\log N}}{N}$. Define the strip $S = \cup Q_j$. Then

$$||\chi_S||_{L^2} \leq \frac{C}{\sqrt{N}}.$$

Thus

$$||L^* \chi_S||_{L^2} \leq \frac{C\sqrt{\log N}}{N}.$$

But $L^* \chi S = C\frac{1}{N}\sum_{j=1}^{3^{n-1}} \chi_{R_j}$, by definition. So that

$$||f||_{L^2} \leq C\sqrt{\log N},$$

which was to be shown. □

Proof of Lemma 2.2. Observe that by definition,

$$\frac{1}{|R_j|}\int_{R_j} f = cN\sum_{k=1}^{3^{n-1}} |R_j \cap R_k|.$$

We let $d(s,t)$ denote the triadic distance between s and t for any $s,t \in [0,1]$ (i.e. $d(s,t)$ is the length of the smallest triadic interval containing both s and t.) It suffices to show that for any $1 \le l \le n-1$, we have

$$\sum_{\{k:d(\frac{j}{n},\frac{k}{n})=3^{-l}\}} |R_j \cap R_k| \ge \frac{c}{N}. \tag{2.5}$$

For any j, there are $c3^{n-l-2}$ numbers k with $d(\frac{j}{N},\frac{k}{N}) = 3^{-l}$ and with $3^{-l} \le |s(j)-s(k)| \le 3^{1-l}$. These are all k's so that if the $l+1$st digit of $\frac{j}{N}$ is 0 or 1 then the $l+1$st digit of $\frac{k}{N}$ is 2 or vice versa. For each such k, we have that

$$|R_j \cap R_k| \ge c\frac{3^l}{N^2}.$$

Summing over 3^{n-l-2} values, we obtain (2.5). Summing over l yields the lemma. □

Proof of Lemma 2.3. Let $D = \cup_{j=1}^{3^{n-1}} R_j$ and define the auxiliary functions

$$t_j(x) = \frac{\chi_{R_j}(x)}{\sum_{k=1}^{3^{n-1}} \chi_{R_k}(x)},$$

on D. Clearly, we have that

$$\sum_{j=1}^{3^{n-1}} \int t_j(x)dx = |D|,$$

which is the quantity we wish to estimate. We define the set R_j^λ to be the part of R_j which is at distance approximately $\frac{1}{\lambda}$ from the y-axis. In other words,

$$R_j^\lambda = \{(x_1,x_2) \in R_j : \frac{1}{\lambda} \le |x_1| \le \frac{2}{\lambda}\}.$$

We claim that

$$\sum_{j=1}^{3^{n-1}} \int_{R_j^\lambda} (t_j(x))^{-1} \le C(\frac{1}{\lambda} + \frac{\log N}{\lambda^2}). \tag{2.6}$$

We may readily see that the left hand side of (2.6) is simply the same as

$$\sum_{j=1}^{3^{n-1}} \sum_{k=1}^{3^{n-1}} |R_j^\lambda \cap R_k^\lambda|.$$

One immediately obtains that

$$\sum_{j=1}^{3^{n-1}} |R_j^\lambda| \leq \frac{C}{\lambda},$$

so that it suffices to show

$$\sum_{j=1}^{3^{n-1}} \sum_{k \neq j} |R_j^\lambda \cap R_k^\lambda| \leq C(\frac{\log N}{\lambda^2}). \tag{2.7}$$

Now, we observe that $|s(j) - s(k)| \leq 3d(\frac{j}{N}, \frac{k}{N})$ so that the summand in (2.7) is only nonzero for (j, k) with

$$d(\frac{j}{N}, \frac{k}{N}) \geq \frac{\lambda}{9}(\frac{|j-k|}{N}). \tag{2.8}$$

However for λ sufficiently large (81 will suffice), one also has that (2.8) implies

$$|s(j) - s(k)| \geq \frac{1}{3} d(\frac{j}{N}, \frac{k}{N}). \tag{2.9}$$

The inequality (2.9) is true since s maps the string 02 to 02, while sending 10 to 00,and 12 to 02 while sending 20 to 20. Now we proceed to show (2.7). It suffices to show that for any fixed l,

$$\sum_{j=1}^{3^{n-1}} \sum_{d(\frac{j}{N}, \frac{k}{N})=3^{-l}} |R_j^\lambda \cap R_k^\lambda| \leq C(\frac{1}{\lambda^2}). \tag{2.10}$$

There are fewer than $\frac{CN3^{n-l}}{\lambda^2}$ pairs (j, k) satisfying (2.8) with $d(\frac{j}{N}, \frac{k}{N}) = 3^{-l}$. However by (2.9) each such pair contributes not more than $C\frac{3^l}{N^2}$ to the sum. Multiplying these two numbers gives (2.10) and summing over l gives (2.7) and thus (2.6).

From this point on, we restrict to $\lambda \leq \log N$ so that (2.6) becomes

$$\sum_{j=1}^{3^{n-1}} \int_{R_j^\lambda} (t_j(x))^{-1} \leq C(\frac{\log N}{\lambda^2}).$$

This means by Tchebycheff that for at least $\frac{N}{6}$ values of j one has

$$\int_{R_j^\lambda} (t_j(x))^{-1} \leq 2C(\frac{\log N}{N\lambda^2}).$$

From this we obtain for these j,

$$\frac{1}{|R_j^\lambda|} \int_{R_j^\lambda} (t_j(x))^{-1} \leq 2C(\frac{\log N}{\lambda}),$$

which implies by Jensen's inequality that

$$\frac{1}{|R_j^\lambda|} \int_{R_j^\lambda} (t_j(x)) \geq c(\frac{\lambda}{\log N}),$$

so that

$$\int_{R_j^\lambda} t_j(x) \geq c(\frac{1}{N \log N}). \tag{2.11}$$

Summing over the j's for which (2.11) is valid, we obtain

$$\sum_j \int_{R_j^\lambda} t_j(x) \geq c(\frac{1}{\log N}). \tag{2.12}$$

Now we restrict λ to being a member of the set,

$$\Lambda = \{2^0, 2^1, \ldots, 2^{\log \log N}\}.$$

For fixed j and any $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$ one has $R_j^{\lambda_1} \cap R_j^{\lambda_2} = \emptyset$. Now summing (2.12) over $\lambda \in \Lambda$, we obtain

$$\sum_j \int t_j \geq c(\frac{\log \log N}{\log N}),$$

which was to be shown. □

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