Mathematical Research Letters **3**, 537–540 (1996)

EXAMPLE OF A NON-LOG-CONCAVE DUISTERMAAT-HECKMAN MEASURE

Yael Karshon

Abstract. We construct a compact symplectic manifold with a Hamiltonian circle action for which the Duistermaat-Heckman function is not log-concave. This provides a counter-example to a conjecture of Ginzburg and Knudsen.

1. Introduction

Let T be a torus and $\mathfrak{t}$ its Lie algebra. Let (M, ω) be a symplectic manifold with an action of T and with a moment map

$$\Phi : M \to \mathfrak{t}^*.$$

Recall, this means that for every $\xi \in \mathfrak{t}$, if ξ_M is the corresponding vector field on M, $\iota(\xi_M)\omega = -d\,\langle \Phi, \xi \rangle$.

Liouville measure on M associates to an open set U the measure $\int_U \omega^n$ where n is half the dimension of the manifold and where we integrate with respect to the symplectic orientation. The *Duistermaat-Heckman measure* [DH] on $\mathfrak{t}^*$ is the push-forward of Liouville measure via the moment map Φ. If T acts effectively, the Duistermaat-Heckman measure is absolutely continuous with respect to Lebesgue measure, and the density function on $\mathfrak{t}^*$ is called the *Duistermaat-Heckman function.*

Ginzburg and Knudsen conjectured [Gi, Kn] that for any Hamiltonian torus action on a compact manifold, the Duistermaat-Heckman function is log-concave, i.e., its logarithm is concave. This conjecture was supported by the following evidence.

(1) The image of the moment map is a convex polytope [GS, At]. If the dimension of T is half the dimension of M and T acts effectively, the Duistermaat-Heckman function is equal to one on this polytope and zero outside [De]. This function is log-concave, i.e., its logarithm is concave.

Received June 19, 1996.
The author is partially supported by NSF grant DMS–9404404.
dg-ga/9606007.

(2) If we restrict a Hamiltonian action of a torus T to a subtorus H, the Duistermaat-Heckman function for H evaluated at a point $\alpha \in \mathfrak{h}^*$ is equal to the integral of the Duistermaat-Heckman function for T over the fiber $\pi^{-1}(\alpha)$ of the linear projection $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$. If we start from a log-concave function on $\mathfrak{t}^*$, we get in this way a log-concave function on $\mathfrak{h}^*$ [Pr, Theorem 6].

The log-concavity conjecture was proved for circle actions on four manifolds by the author [Ka, §2.6], for coadjoint orbits in classical groups and for arbitrary projective algebraic varieties by A. Okounkov [Ok1, Ok2], and for arbitrary Kähler manifolds by W. Graham [Gr]. In this note we construct a counterexample to the conjecture; we construct a Hamiltonian circle action on a compact symplectic 6-manifold for which Duistermaat-Heckman function is not log-concave. Our manifold does not admit an equivariant Kähler structure; this follows from Graham's result.

The construction came from investigating an example of Dusa McDuff of a 6-manifold with a circle valued moment map [MD]. I use her notation wherever possible.

Our conventions regarding factors of 2π etc. are irrelevant and will not be made explicit.

2. The construction

Let T^4 be the four dimensional torus with periodic coordinates x_i, $1 \leq i \leq 4$, and let $\sigma_{ij} = dx_i \wedge dx_j$ and $\sigma_{1234} = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. Let L be a complex Hermitian line bundle over T^4 with Chern class $[-\sigma_{14} - \sigma_{32}]$. Let Θ be a connection one-form with curvature $-\sigma_{14} - \sigma_{32}$. By this we mean that Θ is defined on L outside the zero section, that the restriction of Θ to a fiber of L is $d\theta$ in polar coordinates on the fiber, and that $d\Theta$ is the pullback of $-\sigma_{14} - \sigma_{32}$ via the bundle map $L \to T^4$. Denote by the same letters $\sigma_{ij}, \sigma_{1234}$ the pullbacks of these forms to L. Let the function $\Phi : L \to \mathbb{R}$ be the norm squared, with respect to the fiberwise Hermitian metric on L. Consider the two-form

$$\omega = \sigma_{12} + \sigma_{34} + (2 - \Phi)\sigma_{14} + (3 - \Phi)\sigma_{32} + d\Phi \wedge \Theta \tag{1}$$

on L minus its zero section. It is easy to check that ω is closed and that its top power is

$$\omega^3 = 6(1 + (2 - \Phi)(3 - \Phi))\sigma_{1234} \wedge d\Phi \wedge \Theta.$$

Since $\sigma_{1234} \wedge d\Phi \wedge \Theta \neq 0$ and since the function $(1 + (2 - s)(3 - s))$ is always positive, ω is symplectic.

The circle group acts on L by fiberwise rotation. Let ξ be the generating vector field. From (1) it is clear that $\iota(\xi)\omega = -d\Phi$, so Φ is a moment map for the circle action. The Duistermaat-Heckman function is a constant positive multiple of the function

$$\rho(s) = 1 + (2-s)(3-s). \tag{2}$$

This function decreases for $0 < s < 2.5$ and increases for $2.5 < s < \infty$, so it is not log-concave.

To make a compact example out of our noncompact one, we perform Lerman's "symplectic cutting" [Le]: choose any two numbers, $0 < A < 2.5$ and $2.5 < B < \infty$. Cutting produces a compact symplectic manifold (M, ω) with a circle action and a moment map $\Phi : M \to [A, B]$ such that the preimages in M and in L of the open interval (A, B) are equivariantly symplectomorphic. Consequently, the Duistermaat-Heckman functions are the same: for the compact manifold M we get the function (2) restricted to the interval $A \leq s \leq B$, and this function is not log-concave.

Acknowledgement

I wish to thank E. Lerman for suggesting to publish this example, Y. Peres for advising me on convex sets and log-concave functions, and Dror B-N., E. Lerman, and the anonymous referee for commenting on the manuscript.

References

[At] M. Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.

[De] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339.

[DH] J. J. Duistermaat and G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69** (1982), 259–269.

[Gi] V. Ginzburg (University of Chicago), August 1994.

[Gr] W. Graham, *Logarithmic convexity of push-forward measures*, Invent. Math. **123** (1996), 315–322.

[GS] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[Ka] Y. Karshon, *Periodic Hamiltonian flows on four dimensional manifolds*, dg-ga/9510004.

[Kn] A. Knudsen, private communication, May 1994.

[Le] E. Lerman, *Symplectic cuts*, Math. Res. Lett. **2** (1995), 247–258.

[MD] D. McDuff, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. **5** (1988), 149–160.

[Ok1] A. Okounkov, *Log-concavity of multiplicities with an application to characters of $U(\infty)$*, and: *Newton polyhedrons of spherical varieties*, preprints (1994).

[Ok2] _____, *Brunn-Minkowski inequality for multiplicities*, Invent. Math., (1996), to appear.

[Pr] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) **34** (1973), 335–343.

Institute of Mathematics, The Hebrew University of Jerusalem, Giv'at Ram, Jerusalem 91904, Israel

E-mail address: karshon@math.huji.ac.il