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$\underline{SW}$ = MILNOR TORSION

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ABSTRACT. In this paper we announce that, for a connected, compact, oriented and homology oriented 3-manifold with nonzero first Betti number and vanishing Euler characteristic, a version of the Seiberg-Witten invariant is the same as the Milnor torsion invariant.

1. Introduction

Let X be a connected, compact and oriented 3-manifold with nonzero first Betti number b_1 and vanishing Euler characteristic, so either X is closed or ∂X is a disjoint union of tori.

We also assume that X is homology oriented, i.e., an orientation of $\oplus_{i=0}^{i=3} H_i(X;\mathbb{R})$ is given. We would like to remark here that if X is closed, due to Poincare duality, there is a canonical homology orientation: Let $(t_1, \cdots, t_{b_1})$ be a basis for $H_1(X;\mathbb{R})$, for each i, let $\hat{t}_i$ be the Poincare dual of t_i, then the canonical homology orientation can be represented by the basis $([pt], t_1, \cdots, t_{b_1}, \hat{t}_1, \cdots, \hat{t}_{b_1}, [X])$ for $\oplus_{i=0}^{i=3} H_i(X;\mathbb{R})$.

Let $H(X)$ be the quotient group of $H_1(X)$ by its torsion subgroup which is the same as the quotient group of $H^2_{compt}(X)$ by its torsion subgroup. For simplicity, we often write H for $H(X)$. Then H is a free abelian group of rank b_1. Denote by $\mathbb{Z}[H]$ the integral group ring of H, here we view H as a multiplicative group. Note that H acts on $\mathbb{Z}[H]$, and the set of orbits is denoted by $\mathbb{Z}[H]/H$.

In section 2, we define the Seiberg-Witten invariant for X, which we denote by $SW(X)$. We also explain how $SW(X)$ defines an element, $\underline{SW}(X) \in \mathbb{Z}[H]/H$ in the case $b_1 > 1$; while in the $b_1 = 1$ case, let t be a generator of the infinite cyclic group H, then $(1-t)^{\epsilon(X)}\underline{SW}(X)$ defines an element in $\mathbb{Z}[H]/H$. Here $\epsilon(X) = 2$ if ∂X is empty, and otherwise $\epsilon(X) = 1$.

In section 3, following [10], [11], [21] and [22], we review the (refined) Milnor torsion invariant of X, which we denote by $\tau(X)$. The latter is also in $\mathbb{Z}[H]/H$ when $b_1 > 1$. And, in the $b_1 = 1$ case, $(1-t)^{\epsilon(X)}\tau(X)$ is in $\mathbb{Z}[H]/H$. Note for use below, $\mathbb{Z}[H]$ has an involution, called "conjugation" and denoted by $-$, which is induced by sending $h \in H$ to $h^{-1} \in H$.

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We are now ready to state our main theorem.

Theorem 1.1 (Main Theorem). *Let X be an oriented and homology oriented 3-manifold with $b_1 > 0$ and vanishing Euler characteristic. Then $\underline{SW}(X) = \tau(X)$. Furthermore, when $\partial X = \emptyset$, then there is a unique element in $\tau(X)$, determined by $SW(X)$, which is invariant under the involution which comes from the conjugation involution on $\mathbb{Z}[H]$.*

One of the well known application of the Seiberg-Witten invariants in dimension 4 has been to give obstructions to the existence of metrics of positive scalar curvature on closed 4-manifolds. (see, e.g. [24], [9]) In dimension 3, closed manifolds with positive scalar curvature are known by the foundational work of Schoen-Yau[18], [19], and later the work of Gromov-Lawson [4], to be restricted to connect sums of $S^1 \times S^2$'s, and finite quotients of homotopy spheres. A direct calculation shows that $\tau(X)$ is essentially trivial for the manifolds in this list. Thus, the triviality of $\tau(X)$ is a necessary condition for a closed manifold to have a positive scalar curvature metric. Our main theorem now gives an alternative proof of this conclusion.

There is an invariant of X, the Alexander invariant, denoted by $\Delta(X)$, which is related to the Milnor torsion, but easy to compute from a link presentation, or Heegard splitting of X. And, from [18], [19], and [4], or now Theorem 1.1, the triviality of $\Delta(X)$ is a necessary condition for X to admit a metric of positive scalar curvature. (The Alexander invariant of X is related to the Milnor torsion by a theorem of Milnor [10] and Turaev [21] as follows: Let $\bar{\tau}(X) = \tau(X) \cup -\tau(X)$. 1) If $b_1 > 1$, then $\Delta(X) = \bar{\tau}(X)$. 2) If $b_1 = 1$, and t is a generator for the infinite cyclic group H, then $\Delta(X) = (1-t)^{\epsilon(X)}\bar{\tau}(X)$. We say $\Delta(X)$ is trivial when $\Delta(X) = 0$ in the case $b_1 > 1$, or when $\Delta(X) = \pm|Tor(H_1(X))| \cdot H$ in the case $b_1 = 1$. For example, $\Delta(X)$ is nontrivial if X is a circle bundle over a genus greater than zero Riemann surface.)

Y. Lim [8] has independently deduced a part of Theorem 1.1. M. Hutchings and Y.-J. Lee [5] have a very different approach to the Seiberg-Witten equations which should prove the assertion in Theorem 1.1. D. Salamon [17] has a third approach to Theorem 1.1 in the case where X is closed and fibers over S^1. Theorem 1.1 in this same case can also be deduced from the main theorem in [20].

2. The Seiberg-Witten invariant in dimension 3

Let X be as described in Theorem 1.1. If $\partial X \neq \emptyset$, then glue the half cylinder $\partial X \times [0, \infty)$ to X along their common boundaries to get an open manifold X'. The original manifold with boundary X embeds in X' in the obvious way. The compliment of X in X' will be called the end of X'.

With the preceding understood, agree to drop the prime, and assume henceforth that X is a manifold with an end, N, that has a canonical product structure as $Y \times [0, \infty)$, with Y being the disjoint union of a finite set of tori.

2.1. Preliminaries. Endow X with a Riemannian metric which restricts to N as the product of a flat metric on Y with the standard Euclidean metric dt^2 on $(0, \infty) \subset \mathbb{R}$. (Note that metrics of the sort just described are complete.) Of course, the metric defines the principal $SO(3)$ bundle of oriented, orthonormal frames, $Fr \to X$.

Aside from the Riemannian metric, the Seiberg-Witten equations on X require the choice of a 2-form μ on X which has the following additional properties:

(1)
1) μ is closed.
2) μ is non-zero and covariantly constant on N.
3) There is a closed 1-form ν on X which restricts to N as $*\mu$.
4) If $\partial X = \emptyset$ and $b_1 = 1$, then μ is not exact.

(Note that there are, in every case, forms μ which obey these constraints.)

2.2. Equations. A $Spin^{\mathbb{C}}$ structure on X is a lift of Fr to a principal $U(2)$ bundle, $Fu \to X$. Using the determinant representation of $U(2)$ into $U(1)$, a $Spin^{\mathbb{C}}$ structure, s, determines an associated complex line bundle which will be denoted by $det(s)$. Also, the fiducial representation of $U(2)$ on $\mathbb{C}^2$ defines a $\mathbb{C}^2$ vector bundle $S \to X$ which is a Clifford module for T^*X. (There are two possible Clifford module structures up to isomorphism. We choose the non-standard one, where the oriented volume form acts as 1.)

Let $\mathcal{S}$ denote the set of $Spin^{\mathbb{C}}$ structures s with the property that $det(s)$ is trivial on N. (This is no constraint when X is closed, or when ∂X is connected.) Note that $\mathcal{S}$ is a module for the abelian group which is the kernel of the homomorphism from $H^2(X; \mathbb{Z})$ to $H^2(N; \mathbb{Z})$.

Fix $s \in \mathcal{S}$. The Seiberg-Witten equations ([24],[6], [13]) are equations for a pair (A, ψ) where A is a connection on $det(S)$ and where ψ is a section of S. The equation read:

$$\left\{\begin{array}{lcl} F_A & = & q(\psi) - i\mu \\ \\ D_A\psi & = & 0 \end{array}\right. \tag{2}$$

Here, F_A is the curvature tensor of A, while $q(\psi)$ is a certain bilinear expression in ψ and its Hermitian conjugate, and D_A is the Dirac operator. (The term $q(\psi)$ in (2) can be defined explicitly as follows: First, choose an oriented, orthonormal frame $\{e^1, e^2, e^3\}$ for T^*X. Then set $q(\psi) = -\frac{1}{4}\sum_{i,j} e^i \wedge e^j \psi^+ cl(e^i)\cdot cl(e^j)\psi$. As for D_A, remember that a connection A plus the Levi-Civita connection on the tangent bundle of X canonically specify a covariant derivative, ∇_A: $C^\infty(S) \to C^\infty(S \otimes T^*X)$. The Dirac operator is obtained by following ∇_A with the Clifford homomorphism from $S \otimes T^*X$ to S.)

In the case where X is not compact, the equations in (2) are augmented with the "boundary condition":

$$\int_X |F_A|^2 < \infty. \tag{3}$$

2.3. The solution space. If (A, ψ) solves (2) and (3), then so does $(A + 2\cdot u \cdot du^{-1}, u\cdot\psi)$ if u is a smooth map from X to S^1. This action of $C^\infty(X; S^1)$ defines an equivalence relation on the set of triples $(s, (A, \psi))$, where $s \in \mathcal{S}$ and where (A, ψ) consists of a connection on $det(s)$ and a section of S which obey (2) and (3). Let $\mathcal{M}$ denote the set of equivalence classes of solutions to (2) and (3).

Here are some important properties of $\mathcal{M}$:

(4)

1) $\mathcal{M}$ has the natural structure of a real analytic variety. Moreover, $\mathcal{M}$ is a zero dimensional, oriented manifold when μ is chosen from a certain Baire set of solutions to (1).

2) The assignment of $\frac{F_A}{2\pi i}$ to a triple $(s, (A, \psi)) \in \mathcal{M}$ defines, via integration, a locally constant map Φ from $\mathcal{M}$ to $Hom(H_2(X, N; \mathbb{Z}); \mathbb{Z}) \cong H^2_{compt}(X; \mathbb{Z})/Torsion$.

3) If X is compact, then so is $\mathcal{M}$. If X is non-compact, then the Φ inverse images in $\mathcal{M}$ are compact when μ is chosen from a certain Baire set of solutions to (1).

Here are some further comments regarding the preceding points: Concerning the second point, the form $\frac{F_A}{2\pi i}$ decays exponentially fast as $t \to \infty$ along N. Thus, it can be defined in $Hom(H_2(X, N; \mathbb{Z}); \mathbb{Z})$ by integration over non-compact cycles in X which restrict to N as a triangulation of the product of $(0, \infty)$ with a 1-cycle. Also concerning this second point, remark that the association of $(s, (A, \psi))$ to the first Chern class, $c_1(s)$, of $det(s)$ defines a locally constant map from $\mathcal{M}$ into $H^2(X; \mathbb{Z})$. The image, $\bar{c}_1(s)$, of the latter map in $H^2/Torsion$ agrees with the composition of Φ with the tautological homomorphism i: $H^2_{compact}/Torsion \to H^2/Torsion$. Concerning the third point, when N has two or more components, then $\mathcal{M}$ itself is compact when μ is chosen from a certain Baire set of solutions to (1).

2.4. The invariant SW. This subsection defines the invariant SW. The definition requires a preliminary, two part digression. The first part of the digression introduces the symbol $\underline{\mathcal{S}}$ to denote the set of pairs $(s, x) \in \mathcal{S} \times H^2_{compt}(X; \mathbb{Z})/Torsion$ with the property that $\bar{c}_1(s) = i(x)$.

Part 2 of the digression remarks that an orientation of a 0-dimensional manifold is simply a choice of generator for H^0. Since H^0 has a canonical generator, 1, an orientation is, equivalently, a choice of sign ($\pm$) for each component of the manifold.

End the digression. Here is the definition of SW:

Definition 2.1. *In the case where $b_1 > 1$ or when $N \neq \emptyset$, define SW: $\underline{\mathcal{S}} \to \mathbb{Z}$ to be the map that sends a pair (s, x) to the sum of signs which are given by the orientation in (4) to the subset of points $(s, (A, \psi))$ in $\mathcal{M}$ for which $\Phi((A, \psi)) = x$. Here, it is assumed that $\mathcal{M}$ is defined by μ from the Baire sets mentioned in (4). In the case $b_1 = 1$, then SW is defined on the set of triples (s, x, o) where s and x are as before, and where o is a generator of $H^1(X; \mathbb{Z}) \cong \mathbb{Z}$. The definition is the same as above, but with the following added restriction: When X is compact, the cohomology class $[\mu]$ of μ must obey $([\mu] \cup o, [M]) > 2\pi(c_1(s) \cup o, [M])$. Here, $[M]$ denotes the fundamental calss of M, and $(,)$ denotes the natural pairing between H^3 and H_3. When X is not compact, then $\nu = \alpha \cdot o$ where $\alpha > 0$.*

(Note that in the case $b_1 = 1$ where X is noncompact, o determines a homology orientation and vice versa.)

The following theorem asserts, in part, that SW is well defined:

Theorem 2.1. *When $b_1 > 1$, the map SW from Definition 2.1 depends only on the triple (X, s, x) up to natural equivalence of $Spin^{\mathbb{C}}$ structures under change of metric, and up to proper, orientation preserving diffeomorphisms which also preserve the homology orientation. When $b_1 = 1$, the map SW from Definition 2.1 depends only on (X, s, x, o) up to natural equivalence of $Spin^{\mathbb{C}}$ structures under change of metric and up to proper orientation preserving diffeomorphism which also preserve o and the homology orientation.*

Remark that in the case where X is compact, the preceding theorem can be deduced from the properties of the 4-dimensional Seiberg-Witten invariant for the manifold $S^1 \times X$. (See, e.g. [14] for a detailed description of the 4-dimensional invariant. A number of authors have also discussed this invariant in the compact case from a strictly 3-dimensional point of view. See, e.g. [6], [13], [1], [7], [8].)

2.5. The definition of $\underline{SW}$. Here, as in the introduction, let $H = H^2_{\text{compact}}(X;\mathbb{Z})/\operatorname{Torsion}$. Now, let $\mathbb{Z}[H]$ denote the integral group ring of H which is the set of finitely supported $\mathbb{Z}$-valued function on H, and let $\mathbb{Z}[[H]]$ denote the set of $\mathbb{Z}$-valued function on H. (Thus, $\mathbb{Z}[H] \subset \mathbb{Z}[[H]]$. Note that H acts naturally on $\mathbb{Z}[[H]]$ (by pull-back) preserving $\mathbb{Z}[H]$)

Fix a pair $y = (s_0, x_0) \in \underline{\mathcal{S}}$, then if $(s, x) \in \underline{S}$, we have $x - x_0 = 2z$ with $z \in H$, hence such a choice defines a finite to one map Ψ: $\underline{\mathcal{S}} \to H$. With y chosen, define $SW_y \in \mathbb{Z}[[H]]$ by the rule

$$SW_y(z) = \sum_{\{(s,x)\in\underline{\mathcal{S}}:\Psi(s,x)=z\}} SW(s,x)$$

The preceding definition has the following important features:

$$\text{(5)}\qquad \left.\begin{array}{l} \text{1) Change } y \text{ to } y' \text{ and } SW_y \text{ differs from } SW_{y'} \text{ by the} \\ \text{action of } H \text{ on } \mathbb{Z}[[H]]. \\[1ex] \text{2) If } b_1 > 1, \text{ then } SW_y \in \mathbb{Z}[H]. \\[1ex] \text{3) If } b_1 = 1, \text{ then } (1-t)^{\epsilon(X)} \cdot SW_y \in \mathbb{Z}[H], \text{ where } \epsilon(X) = 2 \\ \text{when } X \text{ is compact, and 1 otherwise.} \end{array}\right\}$$

(Remark that $b_1 > 1$ unless N is empty or connected.)

Here is a formal restatement of (5):

Theorem 2.2. *The set $\{SW_y\}$ defines a unique element $\underline{SW}$ which is in $\mathbb{Z}[H]/H$ when $b_1 > 1$, and in $\mathbb{Z}[[H]]/H$ when $b_1 = 1$.*

3. The Milnor torsion invariant

We follow [10], [11], [21] and [22].

3.1. Algebraic preliminaries. Let $C = (C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_0} C_0)$ be a finite dimensional chain complex over a field F. We suppose that for each i we have fixed a basis c_i for C_i and a basis h_i for $H_i(C)$ (by definition, a 0 dimensional vector space has a unique basis).

We write $c = (c_0, c_1, \cdots, c_m)$ and $h = (h_0, h_1, \cdots, h_m)$, the basis for the graded vector space C_* and $H_*(C)$ respectively.

3.1.1. *The torsion* $\hat{\tau}$. We define the torsion $\hat{\tau}(C) \in F \setminus 0$ in the following way. For each i, let $\hat{h}_i$ be a sequence of vectors in $ker(\partial_{i-1})$ which is a lift of h_i. And let b_i be a sequence of vectors in C_i for which $\partial_{i-1}(b_i)$ is a basis in $Im\partial_{i-1}$. We put $N(C) = \sum_{i=0}^{m} \alpha_i(C)\beta_i(C)$, where

$$\alpha_i(C) = \sum_{j=0}^{i} dimC_j \tag{6}$$

and

$$\beta_i(C) = \sum_{j=0}^{i} dimH_j(C) \tag{7}$$

Now for every i, there are two basis for the determinant line $\wedge C_i$: $(\wedge\partial_i(b_{i+1})) \wedge (\wedge\hat{h}_i) \wedge (\wedge b_i)$ and $\wedge c_i$, the ratio of the first to the second is denoted by $[\partial_i(b_{i+1})\hat{h}_i b_i/c_i]$, and we define

Definition 3.1.

$$\hat{\tau}(\{C; c, h\}) = (-1)^{N(C)} \prod_{i=0}^{m} [\partial_i(b_{i+1})\hat{h}_i b_i/c_i]^{(-1)^{i+1}} \in F \setminus 0 \tag{8}$$

It is easy to see that $\hat{\tau}(\{C; c, h\})$ depends only on C, $\wedge c$ and $\wedge h$.

3.1.2. *The torsion* τ. We define the torsion $\tau(C) \in F$ in the following way. $\tau(C)$ is defined to be zero if C is not acyclic. In case C is acyclic, let b_i be a sequence of vectors in C_i for which $\partial_{i-1}(b_i)$ is a basis in $Im\partial_{i-1}$. Now for every i, there are two basis for the determinant line $\wedge C_i$: $(\wedge\partial_i(b_{i+1}))\wedge(\wedge b_i)$ and $\wedge c_i$, the ratio of the first to the second is denoted by $[\partial_i(b_{i+1})b_i/c_i]$, and we define

Definition 3.2.

$$\tau(\{C; c\}) = \prod_{i=0}^{m} [\partial_i(b_{i+1})b_i/c_i]^{(-1)^{i+1}} \in F \setminus 0 \tag{9}$$

It is easy to see that $\tau(\{C; c\})$ depends only on C and $\wedge c$.

3.2. Milnor torsion. Let X be a connected, compact, oriented and also homologically oriented 3-manifold with nonzero first Betti number b_1 and vanishing Euler characteristic. The Milnor torsion of X is a subset of $\mathbb{Q}(H)$ (where $\mathbb{Q}(H)$ is the quotient field of $\mathbb{Z}[H]$) defined as follows. We consider the universal free abelian cover p: $\tilde{X} \to X$ of X. This is a regular cover whose deck transformation group is H. A CW-decomposition of X can be lifted in an obvious way to an equivariant decomposition of $\tilde{X}$. We consider integer-valued cellular complex $C_*(\tilde{X})$ of $\tilde{X}$. The action of H on $\tilde{X}$ gives this complex the structure of a free $\mathbb{Z}[H]$-chain complex, and moreover the number of free generators for the module of the i-dimensional chains is the number of i-dimensional cells in X. Let I: $\mathbb{Z}[H] \to \mathbb{Q}(H)$ be the inclusion, then $\mathbb{Q}(H)$ is a $\mathbb{Z}[H]$-module via inclusion I. Denote the $\mathbb{Q}(H)$-chain complex $\mathbb{Q}(H) \otimes_{\mathbb{Z}[H]} C_*(\tilde{X})$ by $C_*^I(X)$. Let e be a sequence of oriented cells of $\tilde{X}$ with the property that over every cell of X there lies exactly one cell of the sequence e. Then e defines a basis for both $\mathbb{Q}(H)$-chain complex $C_*^I(X)$ and $\mathbb{R}$-chain complex $C_*(X, \mathbb{R})$. Choose a basis h for $H_*(X, \mathbb{R})$ such that h determines the given homology orientation of X. Let $\xi = \hat{\tau}(\{C_*(X, \mathbb{R}); e, h\})$. We put $\tau(X, e, h) = sign(\xi)\tau(\{C_*^I(X); e\})$, it can be seen to be independent of the choice of homology basis h. The totality of $\tau(X, e, h)$ for all possible choice of e is denoted by $\tau(X)$, and it is called the (refined) Milnor torsion. (see [22]. The original Milnor torsion in [10] is only defined up to a sign.) It is not hard to see that $\tau(X) \in \mathbb{Q}(H)/H$ - the set of orbits of the action of H on $\mathbb{Q}(H)$.

3.3. Relation to the Alexander invariant. Let X be as before, $\pi_1(X)$ be the fundmental group of X. Let $\tilde{X}$ be the universal free abelian covering of X, so it is a regular covering with H as the covering transformation group. Since $\mathbb{Z}[H]$ is Noetherian, $H_1(\tilde{X})$ is a finitely generated module over $\mathbb{Z}[H]$, called the Alexander module of X. We represent this module as the cokernel of a $\mathbb{Z}[H]$-homomorphism f from a free module of rank m to a free module of rank n with $\infty > m \geq n$. Now f is reprenseted by a $m \times n$ matrix over $\mathbb{Z}[H]$, called the presentation matrix (for the Alexander module). Since $\mathbb{Z}[H]$ is g.c.d. domain and its units are $\pm$ the elements of H, then the $n \times n$-minors of the presentation matrix have a greatest common divisor which is unique up to multiplication by $\pm h$ with $h \in H$, and the collection of all of these greatest common divisors is, by definition, the Alexander invariant of X, and is denoted by $\Delta(X)$. Clearly $\Delta(X) \in \mathbb{Z}[H]/(\pm H)$, and it does not depends on the orientation or the homology orientation of X.

The relation between the Milnor torsion invariant and the Alexander invariant has been remarked in the introduction.

3.4. Relation to multivariable Alexander polynomial of links in a rational homology 3-sphere. If X is the link compliment of an ordered oriented link of n components in a rational homology 3-sphere, then $H(X)$ is a free abelian group of rank n with the canonical basis $t_1, t_2, \cdots, t_n$, in terms of which, $\Delta(X)$ is just a Laurent polynomial in n variable $t_1, t_2, \cdots, t_n$, and this polynomial is the Alexander polynomial of the link. We would like to remark that the Alexander polynomial of a link with at least two components is a strickly stronger invariant than the Alexander invariant of the link compliment. We would also like to remark that the sign ambiguity can be removed by observing that there is a canonical homology orientation for the link compliment of an ordered oriented link which is independent of the order but sensitive to the orientation of the link. (see [22])

4. Main properties of the invariant

In the following we assume that X is as in Theorem 1.1. The purpose of this section is to remark that many of the properties of τ and $\underline{SW}$ can be proved independently without invoking the Main theorem. For example, let $A(X) = \underline{SW}(X)$ or $\tau(X)$. Then

Proposition 4.1 (unstability). *If $X = X_1 \# X_2$ with $b_1(X_i) \geq 1$ for each i, then $A(X) = 0$.*

In the case where $A = \underline{SW}$, this theorem is the $3-d$ version of a corresponding $4-d$ connect sum theorem. The proof mimics the $4-d$ proof. See, e.g. [23], [15]. In the case $A = \tau$, this theorem follows from (4.6) and (5.1) in [2].

Proposition 4.2.

$$A(X) = \overline{A(X)}$$

Moreover, if X is closed, there is a unique element $a(X) \in A(X)$ such that

$$a(X) = \overline{a(X)}$$

For $A = \underline{SW}$, this theorem express charge conjugation invariance as noted in [24]. For $A = \tau$, the first half of this theorem is called Poincare-Franz-Milnor duality, see [3], [10]; and the second half is shown in [22].

Proposition 4.3. *Let Y be a rational homology 3-sphere, then*

$$A(X \# Y) = |H_1(Y)| A(X)$$

where $|H_1(Y)|$ is the order of the first homology group of Y.

For $A = \underline{SW}$, this result also follows from the $3-d$ version of the connect sum theorem in [23], [15]. For $A = \tau$, this theorem follows from (4.6) and (5.1) in [2].

Proposition 4.4. *If X is closed, with the canonical homology orientation on X and $-X$ assumed, $A(-X) = (-1)^{b_1(X)+1}A(X)$.*

This follows in both cases from the effect of an orientation change on the canonical homology orientation. Here is a simple corollary of this last property:

Corollary 4.1. *A closed oriented 3-manifold with nontrivial Alexander invariant and even first Betti number does not admit an orientation reversing automorphism.*

For example, a nontrivial principal $U(1)$ bundles over a Riemannan surface with genus at least one has even first Betti number and nontrivial Alexander invariant, hence admits no orientation reversing automorphisms.

As remarked in the introduction, the Seiberg-Witten invariants also satisfy a well known vanishing theorem when X is a closed, $b_1 \geq 1$ manifold with a metric of positive scalar curvature [24]:

Proposition 4.5. *If X is closed with $b_1 \geq 1$, and admits a metric with non-negative scalar curvature which is positive somewhere, then $\underline{SW}(X)$ is trivial in the sense that $\underline{SW}(X) = 0$ if $b_1(X) > 1$ and $\underline{SW}(X) = |Tor(H_1(X))|(t-1)^{-2} \cdot H$ if $b_1(X) = 1$.*

(The case $b_1(X) = 1$ is proved using the wall crossing formula in [9], see also [16].)

5. Strategy of the proof

There are three main steps in the complete proof of the main theorem. (The full details will appear elsewhere.)

Step 1. Reduce the main theorem in the general case to the case of link compliments in rational homology 3-spheres as follows: Let n be the number of the boundary components of X, then $b_1 \geq n$. Let $e = b_1 - n$, then e is the rank of $E := image(H_1(X;\mathbb{Q}) \to H_1(X, \partial X;\mathbb{Q}))$. Choose e disjoint solid tori $T_1, T_2, \cdots, T_e$ inside X which represent a basis in E, then $X_1 := X \setminus interior(T_1 \cup \cdots T_e)$ is a compact manifold with b_1 being equal to the number of boundary components, i.e., $n + e$, and can be viewed as the compliment of a link in a rational homology 3-sphere. Clearly $H(X_1) =$

$H(X)$. For each j, we fix a generator $\hat{t}_j$ for $H_1(T_j;\mathbb{Z})$. $\hat{t}_j$ determines a homology orientation for T_j and with such a homology orientation we have

$$\underline{SW}(T_j) = \tau(T_j) = \frac{1}{1-\hat{t}_j}\{\hat{t}_j^n | n \in \mathbb{Z}\}$$

by direct computations.

Let $T := T_1 \cup T_2 \cdots T_e$, then $X = X_1 \cup T$ and T has a homology orientation determined by $\hat{t}_1, \cdots, \hat{t}_e$ which in turn determines a homology orientation on $T/\partial T = X/X_1$ via Poincare Duality. Since the Euler characteristic of T is zero, the homology orientation on X/X_1 thus determined is independent of the orientation of T.

In Theorem 5.1 below, the homology orientations on X, X_1 and T are assigned such that the sign of the torsion of the long exact sequence of real homology groups of the pair (X, X_1) with the homology orientation on X, X_1 and X/X_1 (determined from the homology orientation on T) is $(-1)^{e(1+b_2(X))}$. (This is $+1$ if X is closed, or has one boundary component, or is the link compliment of a link in a rational homology 3-sphere.)

With the above homology orientations on X, X_1 and T understood, here is the gluing theorem which we need to finish step 1:

Theorem 5.1. *Let A be either $\underline{SW}$ or τ, and for each j, let t_j be the image of $\hat{t}_j$ in $H(X)$, then*

$$A(X) = A(X_1)\prod_{i=1}^{e}\frac{1}{1-t_i}$$

This is a special case of the more general gluing formula:

Theorem 5.2 (Gluing formula). *Let X, X_1 and X_2 be connected, oriented 3-manifolds with $b_1 > 0$ and vanishing Euler characteristic number. Let $X = X_1 \cup X_2$, where $X_1 \cap X_2$ is a disjoint union of some boundary components. For $j = 1, 2$, suppose the homomorphism i_j: $H(X_j) \to H(X)$ induced from the inclusion of X_j into X is nontrivial. Then*

1) *For each j, i_j always induces a homomorphism (called i_j also) from $\mathbb{Z}[H(X_j)]$ to $\mathbb{Z}[H(X)]$, and under the given assumption, from $\mathbb{Q}(H(X_j))$ to $\mathbb{Q}(H(X))$ in case $b_1(X_j) = 1$.*

2) *With homology orientation for X, X_1 and X_2 chosen, let $A = \underline{SW}$ or τ. Then*

$$A(X) = \epsilon i_1(A(X_1)) \cdot i_2(A(X_2))$$

(as an equality in $\mathbb{Q}(H(X))/H(X)$) where ϵ is a sign which is determined by the choosen homology orientation for X, X_1 and X_2 and is the same for both $\underline{SW}$ and τ.

(Via a different argument, Y. Lim [8] can also prove a version of this theorem for $A = \underline{SW}$. A related, 4-dimensional gluing theorem for the Seiberg-Witten invariants has also been announced by Morgan and Szabo [12].)

Step 2. Prove the statement of the theorem for link compliments in S^3 by checking that the analytic invariant satisfies the axioms for the topological invariant. In this case these are the axioms for the multivariable Alexander polynomial of links [22]. This step also uses Theorem 5.1 and also a version of Theorem 5.3 which is described below.

Step 3. Prove the statement of the theorem for link compliments in general rational homology 3-spheres based on Step 2 as follows:

Let Y be a rational homology 3-sphere. If $K \subset Y$ is an oriented knot in Y, then Y can be changed to a new manifold, $Y_{q/p}(K)$ (here q and p are relatively prime), by the q/p-Dehn surgery on K as follows: Let V be the compliment of the interior of a solid torus neighborhood $T(K)$ of K in Y. Fix a basis a, b for $H_1(V \cap T(K); \mathbb{Z})$ such that the image of a in $H_1(T(K); \mathbb{Z})$ is trivial and the image of b in $H_1(V; \mathbb{Z})$ is of finite order. Then $Y_{q/p}(K)$ is the 3-manifold obtained by attaching the standard $S^1 \times D^2$ to V via an orientation reversing diffeomorphism f from $\partial(S^1 \times D^2)$ onto $V \cap T(K)$ such that $f_*([pt \times \partial D^2]) = qa + pb$.

Let Y be as above, let $L \subset Y$ be a n-component link and let K be a knot in $X = Y \setminus L$. Note that q/p-Dehn sugery on K naturally identifies L as a n-component link in $Y_{q/p}(K)$. With this understood, we let $X_{q/p}(K) = Y_{q/p}(K) \setminus L$. Now a finite sequence of q/p-Dehn surgeries on knots in $X = Y \setminus L$ (with $q \neq 0$) turn X into the compliment of a link in S^3. To follow the change in $A(X) = \underline{SW}(X)$ or $\tau(X)$ under these surgeries, we need Theorem 5.1 when the knot in question represents a rationally non-trivial homology class in X and Theorem 5.3 below when the knot in question represents a rationally trivial homology class in X. In the statement of the theorem below, we assign a homology orientation for any $X_{q/p}(K)$ as follows: Fix once and for all a homology orientation on $S^1 \times D^2$ and a homology orientation on V, then a homology orientation on each $X_{q/p}(K)$ can be assigned by demanding that the sign of the torsion of the long exact sequence of real homology groups of the pair $(X_{q/p}, V)$ with the homology orientation on $X_{q/p}$, V and $X_{q/p}/V$ (determined from the homology orientation on $S^1 \times D^2$) be $(-1)^{[b_1(X_{q/p}(K))-1]\cdot b_1(X)}$. Note that $X_{q/p}(K)$ and $X_{(-q)/(-p)}(K)$ are the same as an oriented manifold, but have opposite homology orientation. Note also that, when X is closed, the homology orientation on $X_{1/0}$ is the canonical one if and only if the homology orientation on $X_{0/1}$ is the canonical one. With these homology orientations understood below, we have

Theorem 5.3 (Surgery formula). *Let X be a connected, oriented 3-manifold with $b_1 > 0$ and vanishing Euler characteristic number. Let K be an oriented knot in the interior of X which represents a rationaly trivial homology class. Then $H(X) = H(X_{q/p}(K))$ if $q \neq 0$, and $H(X_{0/1}(K)) = H(X) \oplus \mathbb{Z} \cdot t$ where t is the class in $H(X_{0/1}(K))$ represented by K. In addition, if A is either $\underline{SW}$ or τ, then for $q \neq 0$, one has*

$$A(X_{q/p}(K)) = qA(X_{1/0}(K)) + pA(X_{0/1}(K))|_{t=1}$$

(This theorem has also been proved by Y. Lim [8] for $A = \underline{SW}$)

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