

THE CLASSIFICATION OF RULED SYMPLECTIC 4-MANIFOLDS

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ABSTRACT. Let M be an oriented S^2 -bundle over a compact Riemann surface Σ . We show that up to diffeomorphism there is at most one symplectic form on M in each cohomology class. Since the possible cohomology classes of symplectic forms on M are known, this completes the classification of symplectic forms on these manifolds. Our proof relies on a simplification of our previous arguments and on the equivalence between Gromov and Seiberg-Witten invariants that we apply twice.

1. Introduction

A 4-manifold (M, ω) is said to be *ruled* if it is the total space of an S^2 -fibration $\pi : M \rightarrow \Sigma$. A symplectic form ω on a ruled manifold M is *compatible* with the ruling π if it is nondegenerate on the fibers. The ruling π is then said to be symplectic. The paper [6] made a start on classifying ruled manifolds up to symplectomorphism. However, contrary to what was claimed in [6], the arguments there only work under a somewhat restrictive cohomological assumption: see Lalonde [2] or McDuff [6] Erratum. In this note we complete the classification of ruled symplectic submanifolds, proving the following theorem.

Theorem 1.1. *Let ω_0, ω_1 be two cohomologous symplectic forms on the ruled 4-manifold $\pi : M \rightarrow \Sigma$. Then there is a diffeomorphism ϕ of M such that $\phi^*(\omega_0) = \omega_1$. Moreover, if we assume that the forms ω_0, ω_1 are both compatible with π then they are isotopic.*

Remark. Note that the classification up to diffeomorphism is the best that one can achieve at present because it is not yet known whether there is a diffeomorphism f of $S^2 \times S^2$ that acts trivially on homology but which is not isotopic to the identity. If such a diffeomorphism did exist then there would be two cohomologous but nonisotopic symplectic forms

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on $S^2 \times S^2$. To see this, let $\omega = \omega_0 \oplus \omega_0$ be a split form, and consider the symplectic form $\omega' = f^*(\omega)$, which is cohomologous to ω . If ω' were isotopic to ω , there would be an isotopy g_t from the identity to g_1 with $g_1^*(\omega) = \omega'$. Then $h = fg_1^{-1}$ would be ω -symplectic and, since it acts trivially on homology, it would be isotopic to the identity by Gromov's characterization of $\text{Diff}_\omega(S^2 \times S^2)$. But then the family $h_t g_t$ would join the identity to f , a contradiction. In fact, this argument shows that there is a bijection between isotopy classes of diffeomorphisms of $S^2 \times S^2$ that act trivially on homology and isotopy classes of cohomologous symplectic forms on $S^2 \times S^2$.

The first step in proving this theorem is to show that every symplectic form ω on a ruled 4-manifold M is compatible with some ruling. It was shown by McDuff [6] that any symplectic 4-manifold which contains a symplectically embedded 2-sphere S of nonnegative self-intersection is a blow-up either of CP^2 or of a ruled manifold. In particular, if S is a symplectically embedded 2-sphere with $S \cdot S = 0$ and if $M - S$ is minimal (that is, it contains no symplectically embedded 2-spheres of self-intersection -1), then S may be included in a symplectic ruling of M . Therefore, to complete this first step, one just has to produce a suitable sphere S . The existence of such S was proved by Li-Liu [5] and subsequently by Ohta-Ono [12], using Taubes's recent work in [14, 15] relating Gromov invariants to Seiberg-Witten invariants.¹ Because any two rulings of a given 4-manifold are diffeomorphic, this first step shows that it is sufficient to prove the last sentence of the main theorem.

The next step involves considering the structure of symplectically ruled manifolds. The first result in this direction is due to Gromov, who in his pioneering paper [1] established uniqueness when the base is a sphere of the same size as the fiber and the fibration admits a symplectic section of zero self-intersection. McDuff in [6, 7, 9] extended his methods to prove uniqueness for all ruled surfaces with the sphere or torus as base. She also proved the following result which will be the starting point of the current proof. Recall that two symplectic forms are said to be *deformation equivalent* (or *pseudoisotopic*) or if they may be joined by a path of symplectic but not necessarily cohomologous forms (such paths will be called *deformation paths*), while they are *isotopic* if the path ω_t consists of cohomologous symplectic forms (these paths are called *isotopies*). In the latter case, Moser stability implies that there is a path of diffeomorphisms ϕ_t of M starting at the identity such that $\phi_t^*(\omega_t) = \omega_0$.

Proposition 1.2 ([6]). *All symplectic forms on a ruled manifold compat-*

¹In fact these authors show that any symplectic 4-manifold which admits a metric of positive scalar curvature is a blow-up either of CP^2 or of a ruled manifold.

ible with a given ruling are deformation equivalent.

This proposition is proved by enlarging the base to make enough room in which to cut open the ruled surface over a set of loops in the base. This yields a ruled surface over a cell in R^2 with a symplectic form which is standard near the boundary of the cell. One can then complete this to a ruled surface over S^2 and invoke uniqueness for ruled surfaces over S^2 .

The aim of this paper is to show:

Proposition 1.3. *Let ω, ω' be two cohomologous symplectic forms compatible with a given ruling. If they are deformation equivalent, then they are isotopic.*

Clearly, in view of what has been said above, this proposition completes the proof of the Main Theorem.

In our previous papers [8, 2, 9], we proved this proposition only under restrictive hypotheses on the cohomology class of the symplectic forms. In the next section of this paper, we exploit an idea in Lalonde [2] which simplifies the main line of argument and reduces the proof of the above Proposition to the computation of some Gromov invariant. Then, in the last section, we compute that invariant with the help of Taubes' equivalence between Gromov and Seiberg-Witten invariants.

A complete and self-contained article on the classification of rational and ruled symplectic 4-manifolds will appear in [3].

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2. The inflation procedure

In this section we explain how to use a suitable family of symplectic submanifolds Z_t to change a deformation path into an isotopy. The submanifolds Z_t are constructed in the next section.

Recall that for each base Σ , there are exactly two S^2 -bundles up to fiberwise diffeomorphism: the trivial bundle $\pi : \Sigma \times S^2 \rightarrow \Sigma$ and the non-trivial one $\pi : M_\Sigma \rightarrow \Sigma$. (If we think of M as the projectivization of a rank 2 complex vector bundle E over Σ , then these bundles may be distinguished by the Stiefel-Whitney class $w_2(E)$, which is zero in the trivial case and nonzero otherwise.) We will suppose that both the fiber F and base Σ are oriented and, without loss of generality, will only consider symplectic forms compatible with these orientations.

For simplicity we will first explain the proof for the trivial bundle, and will write $\{a_F, a_\Sigma\}$ for the basis of $H^2(\Sigma \times S^2; R)$ which is dual to the homology basis $\{[F] = [pt \times S^2], [\Sigma] = [\Sigma \times pt]\}$. Thus

$$a_F(F) = 1, \quad a_F(\Sigma) = 0, \quad a_\Sigma(F) = 0, \quad a_\Sigma(\Sigma) = 1.$$

Clearly, there is a (compatibly oriented) symplectic form in the class $xa_F + ya_\Sigma$ if and only if $x, y > 0$.

Proposition 2.1. *Let $\pi : M \rightarrow \Sigma$ be a topologically trivial S^2 -bundle. Let τ_t , $t \in [0, 1]$, be any deformation path joining two π -compatible symplectic forms with $[\tau_0] = [\tau_1] = xa_F + ya_\Sigma$, where $x, y > 0$. Assume that there is a smooth 1-parameter family of embedded τ_t -symplectic submanifolds Z_t in a class $m[F] + n[\Sigma]$ which satisfies $n > mx/y$. Then the path τ_t can be transformed into an isotopy through a family of deformation paths which join the same endpoints τ_0, τ_1 .*

Proof. Let S_t be a 1-parameter family of τ_t -symplectic 2-spheres in class $[F]$. (These can be constructed as J_t -holomorphic curves, where J_t is a generic family of τ_t -tame almost complex structures: see [3] for example.) Following an argument of [2], we use the two families S_t and Z_t to transform the deformation path τ_t into a genuine isotopy with same endpoints τ_0, τ_1 .

The first step is to construct smooth families of forms σ_t, ρ_t which represent the Poincaré duals of S_t, Z_t respectively, and are such that the forms $\tau_t + s\sigma_t$ and $\tau_t + r\rho_t$ are symplectic for all $s, r \geq 0$ and all t . Here is a construction for the forms σ_t . Because S_t has trivial normal bundle, the symplectic neighborhood theorem (see [11] for example) implies that there is a smooth family of diffeomorphisms ψ_t of a neighborhood $\mathcal{N}(S_t)$ of S_t into a neighborhood of $S^2 \times \{0\}$ in the product $S^2 \times D^2$ which pushes τ_t forward to the product form $\alpha_0 + \alpha_1$. Choose an open neighborhood V of $\{0\}$ in the disc D^2 such that $S^2 \times V$ is in the image of ψ_t for all t , let f be a bump function with support in V and then set

$$\sigma_t = \psi_t^*(\alpha_0 + f\alpha_1).$$

The forms ρ_t may be constructed using a similar (but slightly more complicated) normal form for the symplectic neighborhood of Z_t : see [8, Lemma 3.7].

Next, take a_F, a_Σ as ordered basis for $H^2(M, R)$ and identify $H^2(M, R)$ with R^2 . For all t , $[\tau_t]$ belongs to the open sector of $H^2(M, R)$ comprised between the positive horizontal axis (generated by a_F) and the positive vertical axis (generated by a_Σ), because the nondegeneracy condition of these forms imply that their cohomology classes cannot cross the axes. The half-line $L = \{C[\tau_0] : C > 0\}$, which contains both endpoints of

the path $[\tau_t]$, divides this quadrant into two open sectors, say Q_1, Q_2 , respectively the sector comprised between the horizontal axis and L , and the one comprised between L and the vertical axis. When $[\tau_t]$ belongs to Q_1 , there is a unique positive real number s_t such that $[\omega_t = \tau_t + s_t \sigma_t]$ belongs to L , and when $[\tau_t]$ belongs to Q_2 , there is a unique positive real number s_t such that $[\omega_t = \tau_t + s_t \rho_t]$ belongs to L because the above inequality $n > m \frac{x}{y}$ means that the slope m/n of $[\rho_t]$ is smaller than the slope y/x of L . Clearly s_t is a piecewise smooth function of t . Finally, let $\kappa_t > 0$ be such that $\kappa_t[\omega_t] = [\tau_0]$. Then the 1-parameter family $\kappa_t \omega_t$ (reparametrized in t to make it smooth) is a genuine isotopy between $\omega_0 = \tau_0$ and $\kappa_1 \omega_1 = \tau_1$. \square

A similar argument works in the case of the nontrivial bundle. Before describing it we must introduce some notation. Let us denote by $[F], [\Sigma_k] \in H_2(M, \mathbb{Z})$ the classes of the fiber and of the section of self-intersection k . (If the bundle is topologically nontrivial, the self-intersections of sections are always odd, and any odd integers can appear in this way.) Thus $[F], [S_1]$ is a basis for $H_2(M, \mathbb{Z})$, and $[F], [\Sigma] = \frac{1}{2}([\Sigma_{-1}] + [\Sigma_1])$ is a basis for $H_2(M, \mathbb{R})$. If the genus g of the base is > 0 , McDuff showed in [6] that a class $\alpha \in H^2(M, \mathbb{R})$ contains a π -compatible symplectic form exactly when $\alpha([F])$ and $\alpha([\Sigma])$ are strictly positive.² Therefore, if we define $\{a_F, a_\Sigma\}$ as before to be the dual basis to $\{[F], [\Sigma]\}$ the previous argument applies word for word. This proves:

Proposition 2.2. *With the above definition of the classes $a_F, a_\Sigma \in H^2(M)$ and $[F], [\Sigma] \in H_2(M)$, Proposition 2.1 also holds for the nontrivial bundle.*

3. Calculating the Gromov invariants of ruled 4-manifolds

The Gromov invariants of the symplectic 4-manifold (M, ω) defined by Taubes in [14] roughly speaking count the number of J -holomorphic curves in a given homology class A . More precisely, given $A \in H_2(M; \mathbb{Z})$ put $k(A) = \frac{1}{2}(c_1(A) + A \cdot A)$ and choose a set Ω of $k(A)$ distinct points in M . Then, for each ω -tame almost complex structure J , let $H_J(A)$ be the set of all pairs (ϕ, C) which satisfy the following conditions:

- ϕ is a J -holomorphic map from the possibly disconnected but closed Riemann surface C to M such that $\phi_*([C]) = A$;
- $\Omega \subset \phi(C)$.

²Interestingly enough, this is not true when the base is S^2 since in this case the class $[\Sigma_{-1}]$ is always represented by a symplectically embedded submanifold (in fact, by an exceptional sphere) and so we must have $\alpha(\Sigma_{-1}) > 0$.

Taubes shows that for generic J there are a finite number of elements of $H_J(A)$. Moreover, if $C_i, i = 1, \dots, k$, are the components of C , the restriction ϕ_i of ϕ to the component C_i is an embedding, except possibly if C_i is a torus. (In this case, ϕ_i could be the multiple covering of an embedding, and extra information is needed to count them: see [16]. However, this case is not relevant to our work here.) Each map ϕ_i inherits a natural sign $\varepsilon_i = \pm 1$ from the moduli space to which it belongs, and we define the sign of ϕ itself to be

$$\varepsilon(\phi) = \prod_i \phi_i.$$

The Gromov invariant is then defined as:

$$Gr(A) = \sum_{(\phi, C) \in H_J(A)} \varepsilon(\phi), \quad \text{for generic } J.$$

The next proposition shows that all we have to do to find the submanifolds Z_t needed in Proposition 2.1 is calculate some Gromov invariant.

Proposition 3.1. *Let (M, τ_0) be a symplectically ruled surface over a base manifold Σ of genus $g > 0$, and let $A = (g-1)[F] + n[\Sigma] \in H_2(M, \mathbb{Z})$ where $n \geq g$. In the case $g = 1$, assume that $n = 1$ if M is the topologically trivial S^2 -bundle and that $n = 2$ otherwise. Suppose that $Gr(A) \neq 0$. Then any deformation path τ_t can be reparametrized so that there is a family of τ_t -symplectic submanifolds Z_t in class A as required by Propositions 2.1 and 2.2.*

Proof. Let \mathcal{J} be the space of all C^∞ almost complex structures on M which are τ_t -tame for some $t \in [0, 1]$, and let J_t be a path in \mathcal{J} which joins a generic τ_0 -tame almost complex structure J_0 to a generic τ_1 -tame element J_1 . We show below that for all t the set $H_{J_t}(A)$ contains only connected curves. This means that $H_{J_t}(A)$ is a subset of the moduli space of J_t -holomorphic A -curves (it consists of all curves through the $k(A)$ points of Ω), and the theory of J -holomorphic curves shows that for generic path J_t (with fixed endpoints), the set

$$H = \bigcup_t H_{J_t}(A)$$

is a compact oriented 1-dimensional manifold which provides a cobordism between $H_{J_0}(A)$ and $H_{J_1}(A)$. (See, for example, [10, Proposition 7.2.1] and [13].) Since $Gr(A) \neq 0$ there must be at least one arc $(\phi_s, C_s), s \in [0, 1]$, in H which starts on $H_{J_0}(A)$ and ends on $H_{J_1}(A)$. Clearly, there is a continuous map $\beta : [0, 1] \rightarrow [0, 1]$ with $\beta(0) = 0, \beta(1) = 1$ such that ϕ_s is J -holomorphic for some J which is $\tau_{\beta(s)}$ -tame. By slightly

perturbing everything if necessary, we may assume that β is smooth. Then reparametrize τ_t to be the family

$$t \mapsto \tau_{\beta(t)},$$

and set Z_t equal to the image of ϕ_t . Note that, by Taubes' definition of the Gromov invariant, each ϕ_t must be an embedding when $g > 1$. When $g = 1$, the choice of n in the statement of the proposition means that the class A is primitive. Hence the elliptic curve ϕ_t cannot be a multiple covering and must therefore be embedded.

It remains to show that $H_J(A)$ consists of connected curves. Suppose that $(\phi, C) \in H_J(A)$ has ℓ connected components in the (nonzero) classes B_1, \dots, B_ℓ . Write $B_i = m_i[F] + n_i[\Sigma]$. (Observe that if $M \rightarrow \Sigma$ is the nontrivial bundle, m_i, n_i need not be integers.) Then for each i we must have

$$c_1(B_i) + B_i \cdot B_i = 2(m_i - n_i(g - 1 - m_i)) = 2(m_i - n_i(g - 1) + n_i m_i) \geq 0,$$

since this is the condition for the (formal) dimension of the moduli space of J -holomorphic B_i -curves to be nonnegative. We saw above that $Gr([F]) = 1$. Therefore $[F]$ always has a J -holomorphic representative and so, by positivity of intersections, $n_1 \geq 0$. Hence, the above inequality implies that $m_i \geq 0$ for all i . When $g = 1$ this means that $m_i = 0$ for all i , and the desired conclusion easily follows from the choice of n . (Recall that in the case of the nontrivial bundle $[\Sigma]$ is not itself an integral class.) Now suppose that $g > 1$, and observe that, because $\sum m_i = g - 1$, $g - 1 - m_i \geq 0$. But $\sum n_i = n \geq g$ so that $n_i > m_i$ for at least one i , say $i = 1$. It then follows that $m_1 = g - 1$, so that $m_i = 0, n_i = 0$ for $i > 1$. In other words, $\ell = 1$, as claimed. \square

Thus it remains to calculate some Gromov invariants. The direct way to do this is to find a generic almost complex structure (if one is lucky it will be integrable) and then count the relevant J -holomorphic curves. This approach is manageable when A is the class of a section, and was used in [8, 9] to classify symplectic ruled surfaces over the torus. When the base has genus > 1 the above proposition shows that we need to look at more general classes A . No doubt, a similar approach via complex analysis would work for these classes. The main difficulty that we see in implementing this would be in finding suitable generic complex structures, or, more generally, in finding a framework in which one could deal with nongeneric complex structures. The calculation of the invariants would then rely on essentially the same computation as that needed to establish the wallcrossing formula that we use below. However, since we have already used Taubes's results (to show that every symplectic form on a ruled manifold is compatible

with some ruling), and since they lead easily to the result we need now, we will go by that route.

From their definition, it is clear that the Gromov invariants of (M, ω) depend only on the deformation class of ω . Taubes's main result is that they coincide with certain Seiberg–Witten invariants and so depend only on the smooth structure of M together with the first Chern class $c_1(J)$ of any ω -tame almost complex structure J . We must state his result somewhat carefully since ruled surfaces have $b_2^+ = 1$, which means that the Seiberg–Witten invariants depend both on the metric and the perturbation used to define them. Normally we will consider metrics of the form g_J defined by

$$g_J(x, y) = \omega(x, Jy),$$

where here we assume that J is ω -compatible, ie that $\omega(Jv, Jw) = \omega(v, w)$ as well as $\omega(v, Jv) > 0$.

To be consistent with usual notation we denote by K the complex line bundle with first Chern class $-c_1(\omega)$ and let E be a complex line bundle whose Chern class $c_1(E)$ is denoted $e \in H^2(M, \mathbb{Z})$. Given such E let W_E denote the Spin^c -structure on M with determinant bundle $L_E = K^{-1} \otimes E^2$. The (perturbed) Seiberg–Witten equations on W_E may be written as

$$(1) \quad D_A(\Phi) = 0, \quad F_A^+ = \sigma(\Phi) - i\eta,$$

where F_A^+ is the self-dual part of the curvature of a connection A on L_E , σ is a quadratic function of the spinor Φ , and η is a real self-dual 2-form. The number of solutions of equations (1) (counted with sign) is independent of the choice of J and η as these vary along a generic path (J_t, η_t) provided that this path does not cross the “wall” where there are reducible solutions. (These are solutions with $\Phi = 0$, which pose problems because the gauge group does not act freely at such points.) Because $[iF_A] = 2\pi c_1(L_E)$ and because $\omega \wedge \alpha \equiv 0$ for all antiselfdual forms α , such reducible solutions occur exactly when

$$2\pi c_1(L_E) \cup [\omega] = [\eta] \cup [\omega].$$

Taubes considers perturbations of the form

$$\eta_r = 4r\omega + iF_{A_0}^+, \quad r \rightarrow \infty,$$

where A_0 is the connection on K^{-1} which is defined by g_J . We define $SW_K(L_E)$ to be the number of solutions of equations (1) for some fixed (large) value of r . It is not hard to see that this invariant is well-defined and independent of ω up to deformation. (In fact, it depends only on K : see [5, 3].) Moreover, Taubes's theorem can be stated in this language as

$$SW_K(L_E) = \pm Gr(PD(e)).$$

Therefore, by Proposition 3.1 it remains to show that $SW_K(L_E)$ is nonzero for $e = (g-1)a_\Sigma + na_F$, where $n \geq g$. As observed by Li–Liu and Ohta–Ono, this can be done by using a wall-crossing formula. The point is that ruled surfaces always have metrics g of positive scalar curvature, and it is well-known that the unperturbed Seiberg–Witten equations have no solutions in this case. Further, the number of times that a path from the pair $(g, 0)$ to a Taubes pair (g_J, η_r) crosses the wall is 1 (when counted with multiplicities). Therefore, provided that the wall-crossing number (that is the jump in the number of Seiberg–Witten solutions) is nonzero for the class e , the Taubes invariant $SW_K(L_E)$ will be nonzero.³ By calculating this wall-crossing number, one shows:

Proposition 3.2 (Li–Liu, Ohta–Ono). *Let (M, ω) be a symplectically ruled surface over Σ where $g = \text{genus}(\Sigma) > 0$, and let $e \in H^2(M, \mathbb{Z})$ be Poincaré dual to $A = m[F] + n[\Sigma]$, where the fiber class $[F]$ and base $[\Sigma]$ are as defined in §3. Then, if $k(A) \geq 0$,*

$$\pm Gr(A) = SW_K(L_E) = (n+1)^g.$$

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³Note that this is possible only when the formal dimension of the Seiberg–Witten solution space is ≥ 0 . When $e \in H^2(M)$ is Poincaré dual to A , this dimension is exactly the number $2k(A) = c_1(A) + A \cdot A$ which occurs in the definition of the Gromov invariant.

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