

ALMOST COMPLEX STRUCTURES AND GEOMETRIC QUANTIZATION

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ABSTRACT. We study two quantization schemes for compact symplectic manifolds with almost complex structures. The first of these is the Spin^c quantization. We prove the analog of Kodaira vanishing for the Spin^c Dirac operator, which shows that the index space of this operator provides an honest (not virtual) vector space semiclassically. We also introduce a new quantization scheme, based on a rescaled Laplacian, for which we are able to prove strong semiclassical properties. The two quantizations are shown to be close semiclassically.

1. Introduction

In the theory of geometric quantization of Kostant and Souriau a key role is played by the concept of polarization. Recall that if X is a symplectic manifold, a polarization of X is a Lagrangian sub-bundle, \mathcal{P} , of $TX \otimes \mathbb{C}$ that satisfies the Frobenius integrability condition. If \mathcal{P} is the $(0, 1)$ bundle associated to an almost complex structure, then by the Newlander-Nirenberg theorem the integrability condition ensures that the almost complex structure comes from a complex structure, i.e. X is a Kähler manifold. The quantization of Kähler manifolds has had numerous applications, from Bargmann's work in quantum mechanics to Kostant and Kirillov's in the orbit method of representation theory. In addition, this theory has excellent semiclassical properties: see [1], [2] and [3].

Compact Kähler manifolds have been, until recently, the only compact symplectic manifolds for which one has had a general method of quantization, where by this term we mean a method for associating to X a Hilbert space \mathcal{H} , and to functions on X operators on \mathcal{H} . (There exist other approaches to quantization; we mention Karasev's [11].) Recently Vergne has generalized the Kostant-Souriau scheme by replacing the polarization with an operator of Dirac-type [17]. Guillemin, [9], has pointed out that a natural choice for this operator is the Spin^c Dirac operator, which is a generalization of the operator $\bar{\partial} + \bar{\partial}^*$ of the Kähler theory and is constructed

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starting with an almost complex structure which may not be integrable (more details below). The resulting “Spin^c quantization” has attracted a great deal of attention recently because of its nice properties with regard to symplectic reduction [8], [13], [14], [16], [18]. The original definition associates a virtual vector space to X , but we will prove a vanishing theorem that shows that semiclassically this is an “honest” Hilbert space.

We will also propose here another quantization method which is well-defined semiclassically, and is based on some old results of Guillemin and one of us [10]. Instead of generalizing the operator $\bar{\partial} + \bar{\partial}^*$, our idea is to generalize the Hodge Laplacian $\Delta^\bullet = \bar{\partial}^* \bar{\partial}$ on sections of a holomorphic pre-quantum line bundle $L \rightarrow X$. As we will see, this can also be done starting with an almost complex structure on X . It turns out that the semiclassical theory of this quantization method, which we’ll refer to as “almost Kähler quantization,” is as good as that of Kähler quantization. This is based on the results of [10] and the calculus of Hermite Fourier integral operators. In fact the methods of proof of the main results of [1], [2] and [3] generalize directly to this setup. We will also prove that semiclassically the almost Kähler quantization is close to the Spin^c quantization.

Before describing these two approaches in more detail, we note that in both schemes the quantization of functions can be defined in the same way by the “Toeplitz recipe.” In either case the space of sections has an L^2 inner product. If Π denotes the orthogonal projection from the L^2 space of sections onto the quantizing Hilbert space, \mathcal{H} , then the quantization of $f \in C^\infty(X)$ is the operator $\Pi M(f)$ restricted to \mathcal{H} , where $M(f)$ denotes the operator of multiplication by f .

The Spin^c Dirac operator arises as follows. Suppose for a moment that X is a compact complex manifold. Given a Hermitian holomorphic line bundle $L \rightarrow X$, consider the twisted Dolbeault complex:

$$0 \longrightarrow C^\infty(X, \mathcal{E}^0 \otimes L) \xrightarrow{\bar{\partial}} C^\infty(\mathcal{E}^1 \otimes L) \longrightarrow \dots \longrightarrow C^\infty(\mathcal{E}^n \otimes L) \longrightarrow 0,$$

where \mathcal{E}^q is the bundle of forms of type $(0, q)$ and n is half the real dimension of X . The Kodaira vanishing theorem says that for L sufficiently positive this complex has non-zero cohomology only in degree zero. Let \mathcal{E} be the direct sum over q of the bundles \mathcal{E}^q . The vanishing theorem can also be stated in terms of the operator $\bar{\partial} + \bar{\partial}^*$, acting on sections of $\mathcal{E} \otimes L$: If L is sufficiently positive, then the kernel of $\bar{\partial} + \bar{\partial}^*$ has only zero-degree components (just holomorphic sections of L). The index of the Dolbeault complex is the Riemann-Roch number $RR(X, L)$, which by the Riemann-Roch-Hirzebruch index theorem equals

$$RR(X, L) = \int_X \text{Td}(X) \text{ch}(L),$$

where $\text{Td}(X)$ is the Todd class of X .

Now suppose that X has merely an almost complex structure and L is a Hermitian line bundle with compatible Hermitian connection. The Dolbeault complex may be replaced with the “rolled up” version

$$(1.1) \quad (\bar{\partial} + \bar{\partial}^*) : C^\infty(X, \mathcal{E}^\pm \otimes L) \rightarrow C^\infty(X, \mathcal{E}^\mp \otimes L),$$

where \mathcal{E}^+ and \mathcal{E}^- are the direct sum of the \mathcal{E}^q for even and odd q , respectively. The Riemann-Roch formula still computes the index of this two-term complex.

The Spin^c Dirac operator D is an alternative to the operator $\bar{\partial} + \bar{\partial}^*$ in this situation. For X symplectic, to define it requires choosing a compatible almost complex structure on X and a Hermitian connection on the dual of the canonical line bundle of X . (For details, see Duistermaat’s excellent book, [7].) D acts on the same rolled-up complex (1.1). Moreover, D has the same principal symbol as $2(\bar{\partial} + \bar{\partial}^*)$, so its index is the again $RR(X, L)$. But D has more natural properties in many respects. For instance, there is an explicit local formula for the integrand in the Riemann-Roch formula in terms of heat kernels [7].

Following the ideas of Vergne and Guillemin, if L is a pre-quantum line bundle of X (i.e. L is Hermitian with a compatible connection and its curvature is the symplectic form) one defines the quantization of X to be the virtual vector space

$$\text{ind } D = \ker D^+ - \ker D^-.$$

More generally, if we consider the tensor power $L^{\otimes k}$ (which has curvature $k\omega$), then $1/k$ plays the role of Planck’s constant. One of the main points of this paper is to prove an analog of Kodaira vanishing for the Spin^c Dirac operator. Namely, we will prove in Section 2 that $\ker D^- = 0$ for k sufficiently large.

In the almost Kähler quantization, which will be introduced fully in Section 3, the quantum Hilbert space will be defined as the span certain eigenfunctions of a rescaled Laplacian operator. The motivation is as follows. In the Kähler setting, the space of holomorphic sections of $L^{\otimes k}$ is just the kernel of the Hodge Laplacian $\bar{\partial}^* \bar{\partial}$. Another natural choice of operator in this setting would be the metric Laplacian on sections, Δ_k , defined using the Hermitian connection on L . The relationship between the two is

$$(1.2) \quad 4\bar{\partial}^* \bar{\partial} = \Delta_k - nk,$$

where $2n$ is the real dimension of X , so the Hodge Laplacian be seen as a rescaling of Δ_k .

In the non-integrable case, we take the right-hand side of (1.2) as the definition of the rescaled Laplacian, Δ_k^\bullet , and we define the Hilbert space \mathcal{H}_k to be the span of the first d_k eigenfunctions of Δ_k^\bullet , where

$$d_k := \text{RR}(X, L^{\otimes k}).$$

This is in fact a more natural generalization of the Kähler situation than it might seem. By the results of [10] and our Theorem 4.2, the first d_k eigenvalues of Δ_k^\bullet can be bounded independently of k , while the rest of the spectrum drifts to the right with a gap of $O(k)$. So as $k \rightarrow \infty$ the first d_k eigenvalues (which are zero in the integrable case) remain within some fixed interval around zero even in the non-integrable case.

In fact, the methods used to prove this fact in [10] imply more than this. Let $\Pi_k : L^2(X, L^{\otimes k}) \rightarrow \mathcal{H}_k$ be the orthogonal projection. Then [10] shows that the Π_k are the components of a projector of Szegő type. That is, they define a generalized Toeplitz structure in the sense of [4]. This means that the semiclassical theorems of [1], [2], and [3], which are based on [4], can immediately be extended to this quantization.

In Section 4 we will apply the vanishing theorem of Section 2 to demonstrate that the spaces $\ker D^+$ and \mathcal{H}_k approach each other semiclassically. This gives a relation between the two methods of quantization (Spin^c and almost Kähler) that is strong enough to extend some, but not all, of the semiclassical theorems to the Spin^c quantization.

2. The Spin^c Dirac vanishing theorem

2.1. Drift of the Laplacian. Our proof of the vanishing theorem is based on a theorem concerning the large k behavior of the spectrum of the Laplacian acting on sections of $\mathcal{E} \otimes L^{\otimes k}$, where \mathcal{E} is any Hermitian vector bundle and L a Hermitian line bundle, both with Hermitian connections. In particular, we will prove in this subsection that as $k \rightarrow \infty$ the spectrum of Δ_k drifts to the right, at a rate governed by the “non-flatness” of the connection on L , provided the curvature of this connection has constant rank. For the Laplacian acting purely on sections of $L^{\otimes k}$ this fact was proven in [10].

Start with X , a compact manifold of real dimension $2n$ with Riemannian structure β . Let ∇^L and $\nabla^\mathcal{E}$ denote the connections on L and \mathcal{E} , respectively, and let ∇_k be the induced connection on $\mathcal{E} \otimes L^{\otimes k}$. The metric Laplacian on sections of $\mathcal{E} \otimes L^{\otimes k}$ is $\Delta_k := \nabla_k^* \nabla_k$.

Let $\omega = \text{curv}(\nabla^L)$, a possibly degenerate 2-form on X . For $x \in X$ define the skew-symmetric linear map $J_x : T_x X \rightarrow T_x X$ by

$$\omega(v, w) = \beta_x(v, J_x w), \quad \text{for } v, w \in T_x X.$$

The eigenvalues of J are purely imaginary. Let $\tau(x) = \text{Tr}^+ J_x := \mu_1 + \dots + \mu_l$, where $i\mu_j, j = 1, \dots, l$, are the eigenvalues of J_x for which $\mu_j > 0$. Let $\tau_0 = \inf \tau(x)$.

Theorem 2.1. *With the definitions above, and provided the rank of ω is constant over X , there exists a constant C such that for all k the spectrum of Δ_k lies to the right of $k\tau_0 - C$. In particular, if ω, J , and β define compatible symplectic, almost complex, and Riemannian structures, respectively, then the spectrum of Δ_k lies to the right of $kn - C$.*

Proof. \mathcal{E} will always be an induced vector bundle for some principal G -bundle \mathcal{F} , where G is a compact Lie group (we could always take the unitary frame bundle of \mathcal{E} , for example). Let E be the complex vector space on which G acts so that $\mathcal{E} = \mathcal{F} \times_G E$. We will also have a \mathfrak{g} -valued connection 1-form ϑ on \mathcal{F} which induces $\nabla^{\mathcal{E}}$.

A Riemannian structure can be defined on \mathcal{F} by first choosing an Ad- G invariant inner product $\beta^{\mathfrak{g}}$ on \mathfrak{g} and then defining

$$\beta^{\mathcal{F}}(X, Y) := \beta(d\pi(X), d\pi(Y)) + \beta^{\mathfrak{g}}(\vartheta(X), \vartheta(Y)),$$

where $\pi : \mathcal{F} \rightarrow X$ is the projection. This $\beta^{\mathfrak{g}}$ can be chosen so that the natural identification of sections of $\mathcal{E} \otimes L^{\otimes k}$ with G -equivariant functions $\mathcal{F} \rightarrow E$ extends to an isomorphism

$$L^2(X, \mathcal{E} \otimes L^{\otimes k}) \cong (L^2(\mathcal{F}, \pi^* L^{\otimes k}) \otimes E)^G.$$

Moreover, the action of Δ_k on the left-hand side corresponds, under this isomorphism, to the restriction of the operator

$$\Delta_k^{\mathcal{F}} \otimes I + I \otimes \text{Cas}_E$$

to G -invariant sections, where $\Delta_k^{\mathcal{F}}$ is the Laplacian on sections of $\pi^* L^{\otimes k}$ and $\text{Cas}_E \in \text{End}(E)$ denotes the Casimir operator determined by $\beta^{\mathfrak{g}}$. The proof thereby reduces to showing that the spectrum of $\Delta_k^{\mathcal{F}}$ lies to the right of $k\tau_0 - C$ for C independent of k .

Now we essentially apply the same trick to $\pi^* L^{\otimes k} \rightarrow \mathcal{F}$. Let P be the principal S^1 -bundle associated with L , and α the connection form on P which induces ∇^L . And let $\mathcal{Z} = \pi^* P$, the pullback of P to \mathcal{F} :

$$\begin{array}{ccc} \mathcal{Z} & \rightarrow & P \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\pi} & X \end{array}$$

(As a bundle over X , \mathcal{Z} is just the associated principal bundle to $\mathcal{E} \otimes L$.) We can identify sections of $\pi^* L^{\otimes k}$ with the k -th isotype of the S^1 action on \mathcal{Z} ,

$$L^2(\mathcal{F}, \pi^* L^{\otimes k}) \cong L^2(\mathcal{Z})_k.$$

Under this identification $\Delta^{\mathcal{F}}$ can be written as the restriction of an operator $\tilde{\Delta}$ on \mathcal{Z} which is independent of k . (This $\tilde{\Delta}$ is in fact the horizontal Laplacian for the bundle $\mathcal{Z} \rightarrow \mathcal{F}$.)

Let $\rho : \mathcal{Z} \rightarrow X$ be the projection. We can split the tangent space $T_z\mathcal{Z}$ into $H \oplus V_{S^1} \oplus V_G$, where H is the horizontal lift of $T_{\rho(z)}X$, and V_{S^1} and V_G are tangent to the S^1 and G actions, respectively. Correspondingly we have $T_z^*\mathcal{Z} = H^* \oplus V_{S^1}^* \oplus V_G^*$. Using the natural isomorphisms $H^* \cong T_{\rho(z)}^*X$ and $V_G^* \cong \mathfrak{g}^*$, the principal symbol of $\tilde{\Delta}$ can be written

$$\sigma(\tilde{\Delta})(z, \xi) = \beta_{\rho(z)}^*(\xi_H, \xi_H) + \beta^{\mathfrak{g}^*}(\xi_{\mathfrak{g}^*}, \xi_{\mathfrak{g}^*}),$$

where ξ_H and $\xi_{\mathfrak{g}^*}$ are the H^* and V_G^* components, respectively, of $\xi \in T_z^*\mathcal{Z}$. The subprincipal symbol of $\tilde{\Delta}$ is identically zero.

On the characteristic set $\mathcal{C} = \{(z, \xi) \in T^*\mathcal{Z} \setminus 0 : \xi_H = \xi_{\mathfrak{g}^*} = 0\}$, $\sigma(\tilde{\Delta})$ vanishes to second order. At a point $(z, \xi) \in \mathcal{C}$, we can therefore define the Hamilton map $F_{z, \xi}$ of $\sigma(\tilde{\Delta})$, a skew-symmetric linear map in $T_{z, \xi}(T^*\mathcal{Z})$. The restriction that ω be of constant rank allows us to apply Theorem 22.3.2 of [12], a Melin-type inequality. In our case, with zero subprincipal symbol, the result implies that if we choose a first order pseudodifferential operator A on \mathcal{Z} such that $\sigma(A)(z, \xi) \leq \text{Tr}^+ F_{z, \xi}$ for all $z, \xi \in \mathcal{C}$, then there exists a constant C so that

$$(2.1) \quad \langle \tilde{\Delta}f, f \rangle \geq \langle Af, f \rangle - C \|f\|^2,$$

for all $f \in C^\infty(\mathcal{Z})$. Here Tr^+ is defined exactly as for J_x above.

The map $F_{z, \xi}$ depends only on the Hessian of $\sigma(\tilde{\Delta})$, and one quickly sees that to compute the non-zero eigenvalues one need only consider the restriction of the Hessian to the symplectic subspace $H \oplus H^* \subset T_{z, \xi}(T^*\mathcal{Z})$. A computation in local coordinates shows that

$$\text{Tr}^+ F_{z, \xi} = \tau(\rho(z)) \cdot \xi_\theta,$$

where ξ_θ is the $V_{S^1}^*$ component of ξ . Let D_θ be the generator of the S^1 action on \mathcal{Z} , whose symbol is ξ_θ . We apply (2.1) with $A = \tau_0 D_\theta$ to give

$$\langle \tilde{\Delta}f, f \rangle \geq \tau_0 \langle D_\theta f, f \rangle - C \|f\|^2.$$

Noting that if $f \in C^\infty(\mathcal{Z})$ comes from a section of $\pi^*L^{\otimes k}$ then $D_\theta f = kf$, the proof is complete. \square

2.2. Vanishing theorem. Now we specialize to the case where β, ω, J are compatible Riemannian, symplectic, and almost complex structures on X . As in the introduction let \mathcal{E}^q be the bundle of $(0, q)$ -forms on X , with \mathcal{E} denoting the direct sum over q of these and \mathcal{E}^\pm the even and odd subbundles.

We also choose a Hermitian connection on the dual of the canonical bundle of X . These data then define the Spin^c Dirac operator D [7], which we decompose into even and odd components:

$$D^\pm : C^\infty(X, \mathcal{E}^\pm \otimes L^{\otimes k}) \rightarrow C^\infty(X, \mathcal{E}^\mp \otimes L^{\otimes k}).$$

The principal symbol of D (which is also the principal of $2(\bar{\partial} + \bar{\partial}^*)$) is given as follows. To a vector $\xi \in T_x^*X$ we can associate an endomorphism of \mathcal{E}_x by

$$c(\xi) : \nu \mapsto (\xi - iJ\xi) \wedge \nu - \text{int}(\beta_x^{-1}(\xi))\nu, \quad \text{for } \nu \in T_x^*X^{(0,*)},$$

where $\text{int}(\cdot)$ denotes interior multiplication by a vector in T_xX . This map c in fact extends to an isomorphism of the complexified Clifford algebra of T_x^*X with $\text{End}(\mathcal{E}_x)$. The principal symbol of D is given by $\sigma(D)(x, \xi) = ic(\xi)$.

Our proof of the vanishing theorem rests on the following calculation of D^2 .

Theorem 2.2. (Theorem 6.1 of [7])

$$D^2 = \Delta_k + k\sigma + R,$$

where Δ_k is defined as in Section 2.1 and $R \in \text{End}(\mathcal{E}) \otimes I$. The operator σ is defined by

$$\sigma = -i \sum_{j>k} \omega(v_j, v_k) c(\xi_j) c(\xi_k),$$

where v_j is a local orthonormal frame for TX and ξ_j is the corresponding dual frame.

Note that (in contrast to $(\bar{\partial} + \bar{\partial}^*)^2$ in the Kähler case) D^2 preserves degree only mod 2 in general. A simple calculation shows that on forms of degree q , σ acts as multiplication by $2q - n$.

We can now prove the main result of this section. Our proof is inspired by the proof in [5] of the Kodaira vanishing theorem based on a Lichnerowicz-type formula.

Theorem 2.3. *There exist constants C, K such that, for $\phi \in C^\infty(X, \mathcal{E} \otimes L^{\otimes k})$, $k > K$,*

$$D\phi = 0 \implies \|\psi\| < Ck^{-1} \|\phi_0\|,$$

where $\phi = \phi_0 + \psi$ is the decomposition of ϕ into zero and higher degree components.

Proof. In the formula of Theorem 2.2, only the R term will mix zero-degree components with those of higher degree. We therefore have

$$\langle \psi, D^2 \phi \rangle = \langle \psi, (\Delta_k + k\sigma + R)\psi + R\phi_0 \rangle.$$

If $D\phi = 0$ then

$$(2.2) \quad |\langle \psi, (\Delta_k + k\sigma + R)\psi \rangle| = |\langle \psi, R\phi_0 \rangle| \leq \|R\| \|\psi\| \|\phi_0\|.$$

By Theorem 2.1, and the fact that σ acts as multiplication by $2q - n$, there exists a constant C' so that for large k

$$|\langle \psi, (\Delta_k + k\sigma + R)\psi \rangle| > C'k \|\psi\|^2.$$

Combining this with (2.2), factoring out a $\|\psi\|$, and absorbing $\|R\|$ into the constant yields

$$\|\psi\| < Ck^{-1} \|\phi_0\|.$$

□

So if k is large, sections in $\ker D$ are dominated by their zero-degree components. If $\phi \in \ker D^-$, then $\phi_0 = 0$, so the vanishing result follows immediately:

Corollary 2.4. *For D acting on $C^\infty(X, \mathcal{E} \otimes L^{\otimes k})$, if $k > K$ then*

$$\ker D^- = 0.$$

Remark. We have stated the vanishing result in the context of geometric quantization. But for the proof we do not really need the fact that $\text{curv}(\nabla^L)$ is symplectic, or even non-degenerate. The more general statement is that $\ker D^- = 0$ for any sufficiently positive Hermitian line bundle L with curvature of constant rank.

3. Almost Kähler quantization

3.1. Definitions. Let (X, ω) be a compact symplectic manifold of real dimension $2n$ and $L \rightarrow X$ a pre-quantum line bundle, i.e. a Hermitian line bundle with a compatible connection, ∇ , whose curvature equals ω . (This means that $[\omega/2\pi]$ must be an integral cohomology class.) Let J be a compatible almost complex structure on X , and define

$$(3.1) \quad \forall k \in \mathbb{Z}^+ \quad \Delta_k^\bullet = \nabla_k^* \nabla_k - nk,$$

where

$$\nabla_k : C^\infty(X, L^{\otimes k}) \rightarrow C^\infty(X, T^*X \otimes L^{\otimes k})$$

is the connection on $L^{\otimes k}$ induced by ∇ . The adjoint of ∇_k is defined using the Riemannian structure on X defined by ω and J , namely

$$\beta(u, v) = \omega(u, J(v)).$$

Let $\{\psi_j^{(k)}\}$ be an orthonormal basis for $L^2(X, L^{\otimes k})$ of eigenfunctions of Δ_k^\bullet :

$$(3.2) \quad \Delta_k^\bullet \psi_j^{(k)} = \lambda_j^{(k)} \psi_j^{(k)},$$

where

$$\lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \dots .$$

Recall that the Riemann-Roch polynomial of (X, ω) is by definition

$$(3.3) \quad d_k = \int_X e^{k\omega/2\pi} \text{Td}(T^{0,1}X),$$

where $\text{Td}(T^{0,1}X)$ is the Todd class of the $(0, 1)$ tangent bundle of X . The Riemann-Roch polynomial is independent of the choice of J , as all such choices are homotopic.

We are now ready to define the Hilbert space of the almost Kähler quantization.

Definition 3.1. For every positive integer k , define

$$\mathcal{H}_k = \text{Span} \{ \psi_j^{(k)} ; 1 \leq j \leq d_k \}.$$

In the holomorphic case, i.e. if J is integrable and L holomorphic, Δ_k^\bullet is a multiple of $\bar{\partial}^* \bar{\partial}$ by the Bochner identity. Therefore, by the Riemann-Roch theorem and the Kodaira vanishing theorem, there exists a K such that for all $k > K$,

$$(3.4) \quad \mathcal{H}_k = \ker \Delta_k^\bullet = H^0(X, L^{\otimes k}).$$

In the non-integrable case there is a vestigial form of the first equality, namely:

Theorem 3.2. [10] *There exist positive constants a, b, K such that for all integers $k > K$*

- (1) $\forall j \in \{1, \dots, d_k\}, \quad \lambda_j^{(k)} \in (-a, a).$
- (2) $\forall j > d_k, \quad \lambda_j^{(k)} \geq bk.$

Thus for large k the first d_k eigenvalues of Δ_k^\bullet (counted with multiplicities) lie in $(-a, a)$, and the rest of the spectrum drifts to the right with a gap of $O(k)$.

Remark 1. The result of [10] is more general: One can start with an arbitrary Riemannian metric on (X, ω) and suitably modify the metric Laplacian, $\nabla_k^* \nabla_k$ (again by a zeroth order perturbation), to obtain an operator for which the previous result remains true. Furthermore the semiclassical results that we will state below remain valid in this setting as well.

Remark 2. Strictly speaking, the result of [10] states that for some integer k_0 and some constants a, b, K , if $k > K$ then $\lambda_j^{(k)} \in (-a, a)$ if $j \leq d_{k+k_0}$ and $\lambda_j^{(k)} > bk$ if $j > d_{k+k_0}$. However, a consequence of Theorem 4.2 is that $k_0 = 0$.

We define the quantization of real-valued functions on X as in the integrable case:

Definition 3.3. For every positive integer k , let

$$\Pi_k : L^2(X, L^{\otimes k}) \rightarrow \mathcal{H}_k$$

denote the orthogonal projection. For $f \in C^\infty(X)$ we define

$$T_k(f) = \Pi_k M(f) \Pi_k,$$

where $M(f)$ denotes the operator of multiplication by f . We usually regard Π_k as an operator on \mathcal{H}_k .

3.2. Semiclassical results. As we will now see, the quantization scheme defined above has excellent semiclassical properties. We will briefly state two theorems; the first is a “deformation quantization” result, generalizing [1], and the second is the so-called trace formula, generalizing [3]. The proofs of these generalizations are the same as those of the cited results, because, as we will describe below, the microlocal structure of the family of projectors $\{\Pi_k\}$ is the same in the integrable and non-integrable cases.

Theorem 3.4. For all $f, g \in C^\infty(X)$,

- (1) $\|T_k(f)\| = \|f\|_\infty + O(1/k)$
- (2) $\|T_k(f)T_k(g) - T_k(fg)\| = O(1/k)$
- (3) $\|k[T_k(f), T_k(g)] - T_k(\{f, g\})\| = O(1/k)$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket on X .

To state the trace formula, we fix a Hamiltonian function $H \in C^\infty(X)$. Let $E_i^{(k)}$ be the eigenvalues of $T_k(H)$. The trace formula is an asymptotic expansion for a weighted trace of $T_k(H)$ taken near some fixed energy E ,

which we assume to be a regular value of H . The weighting is given by a test function φ with compactly supported Fourier transform. Let ϕ denote the Hamiltonian flow of H .

Theorem 3.5. *If the flow ϕ is clean on $H^{-1}(E)$, then we have an asymptotic expansion:*

$$\sum_{i=0}^{\infty} \varphi(k(E_i^{(k)} - E)) \sim \sum_{j \in J} \sum_{l=0}^{\infty} C_{j,l} e^{ik\theta_j} k^{(d_j-1)/2-l},$$

where J indexes the connected components of the set of pairs $(x, \tau) \in H^{-1}(E) \times \mathbb{R}$ with $\phi_\tau(x) = x$, d_j denotes the dimension of the j -th component, and the angles θ_j are the holonomy angles for the closed trajectories.

The coefficients $C_{j,0}$ can be expressed as integrals over the corresponding fixed point sets. For example, the $\tau = 0$ component contributes

$$C_{0,0} = (2\pi)^{-n} \hat{\varphi}(0) \text{vol}(H^{-1}(E)).$$

For the details see [3].

There are other results whose proof is based on the microlocal structure. In [1] several additional semiclassical theorems that follow directly from [4] are pointed out. For instance, a relation between quantum and classical time evolution:

$$\left\| e^{-iktT_k(H)} T_k(f) e^{iktT_k(H)} - T_k(f \circ \phi_t) \right\| = O(1/k).$$

And in addition to the trace formula, [3] also presents an asymptotic expansion for a localized variant of the weighted trace,

$$\sum_{i=0}^{\infty} \varphi(k(E_i^{(k)} - E)) \Psi_i^{(k)}(x_1) \overline{\Psi_i^{(k)}(x_2)},$$

where $\Psi_i^{(k)}$ is the i -th eigenfunction of $T_k(H)$, in terms of classical trajectories joining x_1 to x_2 .

Finally, the results of [2] can be also extended to the non-integrable case. Here families of states in \mathcal{H}_k , $k \in \mathbb{Z}_+$, are associated to certain Lagrangian submanifolds of X . These states concentrate on their associated submanifolds as $k \rightarrow \infty$, resulting in asymptotic formulas for norms and inner products in terms of intersections of submanifolds.

3.3. Microlocal structure. We will now briefly describe the microlocal structure that gives the immediate extension of the proofs in [1], [2], and [3] to our case

Each of the Π_k 's has a smooth Schwarz kernel. Singularities arise, and hence microlocal analysis becomes relevant, when the \mathcal{H}_k 's are "rolled up" together, as follows. Let P be the principal S^1 -bundle associated to L as in §2.1, and identify

$$L^2(X, L^{\otimes k}) \cong L^2(P)_k.$$

By considering \mathcal{H}_k now as a subset of $L^2(P)$, and Π_k as a projector on $L^2(P)$, we may define

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \quad \Pi = \bigoplus_{k=0}^{\infty} \Pi_k.$$

In the Kähler case, P can be regarded as the boundary of the unit disc bundle in L^* , \mathcal{H} is the Hardy space of P , and Π is the Szegő projector for this Hardy space. Boutet de Monvel, Sjöstrand, and Guillemin [4], [6] have shown that (in the Kähler case) the Szegő projector is a Hermite Fourier integral operator associated to the symplectic cone

$$\Sigma = \{(p, r\alpha_p) : p \in P, r > 0\} \subset T^*P,$$

where α is the connection form induced on P , and have worked out its symbol. In [4] the authors in fact develop a theory of "generalized Toeplitz structures," which are projectors expressible as Hermite FIO's associated to a particular symplectic cone and modeled microlocally on the Szegő projector. The semiclassical results of [1], [2], and [3] are all proven using this general theory. The point is that the singularities of Π come from large k (i.e. semiclassical) behavior (since the Π_k were smooth individually).

The basis for the proof of Theorem 3.2 was the following.

Theorem 3.6. [10] *The full projector Π on $L^2(P)$ occurring in the almost Kähler quantization defines a generalized Toeplitz structure associated to the symplectic cone Σ .*

Note that in addition to the semiclassical results already stated, this allows the immediate application of many theorems of [4] on the structure of the full projector Π .

4. Semiclassical properties of Spin^c quantization

Let X, ω, J, L as in §3, and let \mathcal{E} be the bundle of type $(0, *)$ forms as in §2. Choose a Hermitian connection on the dual of the canonical bundle of X so that the Spin^c Dirac operator D is defined. With K the constant occurring in Theorem 2.3, we make the following definitions for the Spin^c quantization.

Definition 4.1. For $k > K$ define

$$\mathcal{Q}_k = \ker D^+ \subset L^2(X, \mathcal{E} \otimes L^{\otimes k}).$$

Let

$$\Theta_k : L^2(X, \mathcal{E} \otimes L^{\otimes k}) \rightarrow \mathcal{Q}_k$$

denote the orthogonal projection, and for $f \in C^\infty(X)$ define

$$S_k(f) := \Theta_k M(F) \Theta_k.$$

Since $\dim \mathcal{Q}_k = \text{ind } D$ for $k > K$ by the vanishing theorem, $\dim \mathcal{Q}_k$ is given again by the Riemann-Roch polynomial d_k . The point of this section is to compare the almost Kähler quantization and the Spin^c quantization semiclassically.

In order to make such comparisons, in this section we'll regard the almost Kähler Hilbert space \mathcal{H}_k as a subspace of $L^2(X, \mathcal{E} \otimes L^{\otimes k})$ containing sections with only zero-degree components. Similarly, we'll extend the domain of Π_k to $L^2(X, \mathcal{E} \otimes L^{\otimes k})$.

Theorem 4.2. $\|\Pi_k - \Theta_k\| = O(1/k)$.

Proof. We begin by showing that $\|(1 - \Pi_k)\Theta_k\| < Ck^{-1}$. Given any $\phi \in L^2(X, \mathcal{E} \otimes L^{\otimes k})$, we can decompose $\phi = \eta_0 + \eta_1 + \psi$, where $\eta_0 \in \mathcal{H}_k$, η_1 is in the orthogonal complement of \mathcal{H}_k in $L^2(X, \mathcal{E}^0 \otimes L^{\otimes k})$, and ψ is the sum of higher-degree components. For $D\phi = 0$, $(1 - \Pi_k)\Theta_k\phi = \eta_1 + \psi$. So we are trying to show that $D\phi = 0$ implies a bound on η_1 and ψ relative to ϕ .

Theorem 2.3 already implies that for large k , $\|\psi\| < Ck^{-1}\|\phi\|$ for any $\phi \in \ker D$. To show that η_1 is small, we appeal again to Theorem 2.2, the formula for D^2 . If $D\phi = 0$ then

$$(4.1) \quad 0 = \langle \eta_1, D^2\phi \rangle = \langle \eta_1, (\Delta_k - nk + R)(\eta_0 + \eta_1) \rangle + \langle \eta_1, R\psi \rangle$$

Since η_1 is in the complement of \mathcal{H}_k , for $\Delta_k - nk$ acting on η_1 we have a lower bound which increases with k :

$$\langle \eta_1, (\Delta_k - nk + R)\eta_1 \rangle > Ck \|\eta_1\|^2.$$

On the other hand, the action of $\Delta_k - nk$ on \mathcal{H}_k is bounded uniformly in k , so

$$\langle \eta_1, (\Delta_k - nk + R)\eta_0 \rangle < C \|\eta_0\| \|\eta_1\|.$$

Returning to (4.1) and using the bound on ψ , we obtain

$$k \|\eta_1\|^2 < C \|\eta_0\| \|\eta_1\| + Ck^{-1} \|\eta_1\| \|\phi\|,$$

which leads directly to the desired $\|\eta\| < Ck^{-1} \|\phi\|$.

This completes the proof that $\|(1 - \Pi_k)\Theta_k\| < Ck^{-1}$. Noting that $\Pi_k(1 - \Theta_k) = ((1 - \Theta_k)\Pi_k)^*$, we can use the same argument to show that $\|\Pi_k(1 - \Theta_k)\| < Ck^{-1}$. Since $\Pi_k(1 - \Theta_k) - (1 - \Pi_k)\Theta_k = \Pi_k - \Theta_k$, this completes the proof. \square

We return now to clarify the remark made after Theorem 3.2. The version of this theorem proven in [10] involves d_{k+k_0} eigenvalues, where k_0 is some fixed but unknown integer, instead of d_k . So we should really have defined $\dim \mathcal{H}_k$ as the span of the first d_{k+k_0} eigenfunctions. But a simple consequence of Theorem 4.2 is that with \mathcal{H}_k so defined, $\dim \mathcal{H}_k = \dim \mathcal{Q}_k$ for k sufficiently large (large enough so that $\|\Pi_k - \Theta_k\| < 1$). So in fact $k_0 = 0$.

Theorem 4.2 has some direct applications to semiclassical analysis of “Spin^c Toeplitz operators.”

Theorem 4.3. *For $f \in C^\infty(X)$,*

$$\|S_k(f) - T_k(f)\| = O(1/k).$$

Corollary 4.4. *For all $f, g \in C^\infty(X)$,*

- (1) $\|S_k(f)\| = \|f\|_\infty + O(1/k)$
- (2) $\|S_k(f)S_k(g) - S_k(fg)\| = O(1/k)$,

Theorem 4.3 follows immediately from Theorem 4.2, and the corollary follows because of Theorem 3.4.

Some additional results may be derived for a more general notion of Toeplitz operator. We may regard the space \mathcal{Q}_k as a subspace of $L^2(\mathcal{Z})$, where \mathcal{Z} is the principal bundle introduced in Section 2. So given a pseudodifferential operator Q on \mathcal{Z} , we can define

$$S_k(Q) := \Theta_k Q \Theta_k.$$

Theorem 4.5. *Let Q and R be pseudodifferential operators on \mathcal{Z} of order q and r , respectively. Then*

$$(4.2) \quad \|S_k(Q)\| = O(k^q),$$

and

$$(4.3) \quad \|[S_k(Q), S_k(R)]\| = O(k^{q+r-1}).$$

Proof. Let D_θ be the generator of the S^1 action on \mathcal{Z} , as in Section 2. To prove (4.2) for $q \leq 0$, note that $(D_\theta)^{-q}Q$ is an operator of order zero, and thus bounded because \mathcal{Z} is compact. Let $\Theta = \oplus_k \Theta_k$, as a projector on $L^2(\mathcal{Z})$. Since $\|\Theta\| = 1$, we have $\|\Theta(D_\theta)^{-q}Q\Theta\| < C$. So $\|S_k((D_\theta)^{-q}Q)\| < C$ for each k , where C is independent of k . The proof is concluded by noting that $\Theta_k D_\theta = k\Theta_k$. For $q > 0$ we must choose a parametrix B for D_θ , and consider $S_k(B^q Q)$. Then $\Theta_k B = k^{-1}\Theta_k$ plus k^{-1} times a smoothing operator, and we may use the result for $q < 0$ to control this correction.

To prove (4.3) we need to appeal to one of the basic facts about generalized Toeplitz structures proven in [4], Proposition 2.13. Namely given a pseudodifferential operator Q , there exists another pseudodifferential operator \tilde{Q} such that $\Pi Q \Pi = \Pi \tilde{Q} \Pi$ and $[\Pi, \tilde{Q}] = 0$. \tilde{Q} necessarily has the same principal symbol as Q .

It turns out that \tilde{Q} also commutes with Θ_k to leading order in k . To see this, note that $[\Theta_k, \tilde{Q}] = [\Theta_k - \Pi_k, \tilde{Q}]$. Inserting D_θ^{-q} as above, and using Theorem 4.2, we obtain

$$(4.4) \quad \|[\Theta_k, \tilde{Q}]\| < Ck^{q-1}.$$

Returning to the second statement of the theorem,

$$[S_k(Q), S_k(R)] = [S_k(\tilde{Q}), S_k(\tilde{R})] + [S_k(Q - \tilde{Q}), S_k(\tilde{R})] + [S_k(Q), S_k(R - \tilde{R})]$$

The norms of the last two commutators on the right can be bounded by Ck^{q+r-1} , by (4.2), so it only remains to bound $[S_k(\tilde{Q}), S_k(\tilde{R})]$. Observe that

$$[S_k(\tilde{Q}), S_k(\tilde{R})] = S_k([\tilde{Q}, \tilde{R}]) + \Theta_k \tilde{Q} [\Theta_k, \tilde{R}] \Theta_k - \Theta_k \tilde{R} [\Theta_k, \tilde{Q}] \Theta_k.$$

All terms on the right have norm bounded by Ck^{q+r-1} ; the first by (4.2) directly, the second and third by (4.2) and (4.4). \square

Of course, by the same argument (4.2) holds for $T_k(Q)$. And in fact there is a refined version of (4.4) for almost Kähler Toeplitz operators. The theory of [4] allows one to pick off the leading term in the commutator:

$$\|[T_k(Q), T_k(R)] - k^{-1}T_k(P)\| = O(k^{q+r-2}),$$

where $\sigma(P) = \{\sigma(Q), \sigma(R)\}$. Part 3 of Theorem 3.4 follows immediately from this fact [1]. Our comparison between Π_k and Θ_k is not strong enough to extend this result to the Spin^c case.

We note also that Theorem 4.3 implies that $\text{dist}(\text{Spec}(S_k(f)), \text{Spec}(T_k(f)))$ is $O(1/k)$. This is enough to extend the Szegő theorem of §13 of [4] to Spin^c Toeplitz operators, but not enough to extend the trace formula (Theorem 3.5).

5. Concluding Remarks

We have seen that there are two methods (at least) of quantizing triples (X, ω, J) consisting of a compact symplectic manifold with a compatible almost complex structure. Although we have demonstrated that the almost Kähler quantization has very good semiclassical properties, we have yet to prove the same for the Spin^c quantization. This seems to require dealing with Fourier integral operators of Hermite type acting on sections of vector bundles. Although there do not seem to be any serious obstacles, as always when dealing with systems matters are more involved. We hope to provide the details of this approach in the future.

From a purely geometrical point of view, the quantizations discussed in this paper raise a number of natural questions. Recall a classical construction in algebraic geometry: if $L \rightarrow X$ is a holomorphic line bundle such that the corresponding linear system is base-point free, then L defines a map of X to a projective space,

$$F : X \longrightarrow \mathbb{P}(H^0(X, L)^*).$$

The definition of F is the following: $\forall x \in X$, $F(x)$ is the hyperplane in $H^0(X, L)$ consisting of all holomorphic sections vanishing at x . (The condition of being base-point free means precisely that $\forall x \in X$ such a set is indeed a hyperplane.) In the setting considered in this article, the Hilbert spaces of both quantization schemes are subspaces of the space of sections of a vector bundle, so one can attempt to define a map F precisely as above, with \mathcal{H}_k (or \mathcal{Q}_k) replacing $H^0(X, L)$. We claim that if k is sufficiently large then this definition is possible, at least for the almost Kähler case. That is, for a given X for all k sufficiently large and for all $x \in X$ there is a $\psi \in \mathcal{H}_k$ such that $\psi(x) \neq 0$, and therefore the space of elements in \mathcal{H}_k vanishing at x has codimension one. Questions on the geometry of these maps will be investigated in future work.

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