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HOMOLOGY SPHERES WITH THE SAME FINITE TYPE INVARIANTS OF BOUNDED ORDERS

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Abstract. For every $n \in \mathbb{N}$, we give a direct geometric construction of integral homology spheres that cannot be distinguished by finite type invariants of orders $\leq n$. In particular we obtain $\mathbb{Z}$-homology spheres that are not homeomorphic to S^3 but cannot be distinguished from S^3 by finite type invariants of orders $\leq n$.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with unit. The purpose of this note is to prove the following theorem:

Theorem. *For every $n \in \mathbb{N}$, there exist $\mathbb{Z}$-homology spheres M and $M^{'}$ such that all the $\mathcal{R}$-valued finite type invariants of orders $\leq n$, of M and $M^{'}$ are equal.*

T. Le proved a similar result (see Proposition 4.6 in [L]) for invariants with rational values only. His proof uses the construction of the universal finite type 3-manifold invariant which is based on the machinery developed in [LMO], as well as results developed in earlier work of Le and Murakami. On the other hand, our proof of the theorem above is a direct geometric argument and it only uses the definition of finite type $\mathbb{Z}$-homology sphere invariants. Moreover it works for all coefficient rings.

This note is organized as follows: In §2 we define the notion of *surgery n-equivalent* knots and we present two ways for constructing examples of such knots. One construction is based on iterating Whitehead doubling. In the second construction the knots are obtained as boundary curves of appropriate Seifert surfaces. In §3 we prove the theorem stated above by showing that the integral homology spheres obtained by ± 1 surgery on two *surgery n-equivalent* knots are not distinguished by finite type invariants of orders $\leq n$.

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2. Surgery modification of knots

2a. Preliminaries. A surgery disk for a knot $K \subset S^3$, is a standard disk $D \subset S^3$ such that K intersects D only in its interior with zero algebraic number. Performing ± 1 surgery ([Ro]) on $K_1 = \partial D$, changes K to another knot $K^{'} \subset S^3$. We will say that $K^{'}$ is obtained from K by *a surgery modification* along D.

To continue let $D_1, \dots, D_n$ be n mutually disjoint surgery disks for a knot K in S^3, such that the boundary components of $D_1, \dots, D_n$ are ± 1 framed. We will call $\bar{D} = \{D_1, \dots, D_n\}$ an *n-collection* in the embedding $K \subset S^3$. For $j = 1, \dots, n$, let $i_j \in \{0,\ 1\}$ and let $\bar{i} := (i_1, \dots, i_n)$. For every $\bar{i}$, we will denote by $K(\bar{i},\ \bar{C})$ the knot obtained from K by surgery modification along each D_j for which $i_j = 1$.

Definition 1. The knots K and $K^{'}$ are called *surgery n-equivalent*, if there exists an *n-collection* $\bar{D}$, for K such that: the knot $K(\bar{i},\ \bar{C})$ is isotopic to $K^{'}$ for every $\bar{i} \neq (0, \dots, 0)$. In particular, if $K^{'}$ is the trivial knot K is called *surgery n-trivial*.

Notice that if $K^{'}$ is *surgery n-trivial* then $K \# K^{'}$ and K are *surgery n-equivalent*, for every knot K. Next, we give two constructions that lead to various examples of knots that are *surgery n-trivial*.

2b. Whitehead doubles. Let K be a knot and let $D^1(K) = D(K)$ be the *untwisted* (positive or negative) Whitehead double of K (see [Ro]). Inductively, define the nth iterated, *untwisted* (positive or negative) Whitehead double of K, by $D^n(K) = D(D^{n-1}(K))$.

Let P be a projection of K and let $w = w(P)$ be the algebraic crossing number of P. Then a projection, say P^*, of $D(K)$ is obtained as follows: Add $-w$ kinks in the projection P, and draw a second copy of K, which is parallel to the first one. Then connect the two copies by an appropriate clasp.

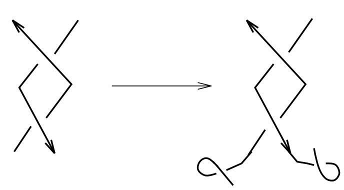

FIGURE 1

Here we have a negative clasp and we have added two full negative twists.

To continue, let us adjust P^* near the clasp by adding two kinks with sign same to that of the clasp. For a picture see Figure 1. Call the resulting projection P_1. A projection P_2 of $D^2(K)$ is obtained by doubling P_1 and connecting the two copies by an appropriate clasp. By iterating this procedure n times we obtain a projection P_n, of $D^n(K) = D(D^{n-1}(K))$. Then, we have:

Lemma 2. *For every knot K and every $n \in \mathbb{N}$, the knot $D^n(K)$ is surgery n-trivial.*

Proof. We obtain an *n-collection*, $\bar{D}$, for $D^n(K)$ as follows: For each of the n collections of parallel clasps of $D^n(K)$ consider a surgery disk that intersects the strings of $D^n(K)$ in its interior, as shown in Figure 2.

For each of these surgery disks we frame the boundary knot by $+1$ or -1 according to whether the corresponding (possibly multiple) clasp is positive or negative. One can see that $\bar{D}$ satisfies the requirements of Definition 1. To see this, let us for simplicity focus on a negative double clasp, say C^2, of $D^n(K)$. Let D be the surgery disk attached to C^2 as above. Finally let $K_1 = \partial D$.

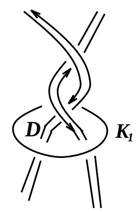

FIGURE 2

By performing -1 surgery on K_1 we untwist the clasp and we insert two positive full twists. See Figure 3. Now to see that this operation unknots the knot $D^n(K)$, observe that C^2 is obtained by doubling the simple clasp, say C^1, of $D^{n-1}(K)$.

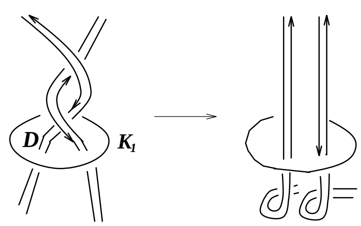

FIGURE 3
Doing -1 surgery on K_1 unties the clasp and introduces
a positive full twist on each of its two branches.

But by construction, two negative kinks have been added to $D^{n-1}(K)$ before constructing $D^n(K)$. These will create two negative full twists on $D^n(K)$ which will cancel with the twists induced after the surgery.

In general, modifying by surgery along a disk which intersects $D^n(K)$ in a k-fold clasp C^k (with $2 \leq k \leq n$), untwists the clasp and induces one full twist, of sign opposite of that of the clasp, on each of the two sets of parallel strings of C^k. However, these twists will cancel with those introduced in $D^{n-k+1}(K)$ before the construction of $D^{n-k+2}(K)$. □

Remark. Iterated *untwisted* Whitehead doubles have been used to construct examples of knots with the same Vassiliev invariants ([V]) of bounded orders. See [Li1] or [P].

2c. A construction with Seifert surfaces. A link L, is called *almost trivial* if every proper sublink of L is trivial. Thus, in particular, each component is unknotted. Let $L = L_1 \cup \ldots \cup L_n$ be an *almost trivial link* with n components. Fix a projection, P, of L that contains no kinks. Pick a point p on the projection plane and connect it to a point on the projection of each L_i, by an arc α_i. This way we obtain a projection a bouquet of n-circles, say W_n. For every circle in W_n we add an unlinked loop $L_{i'}$, which contains a kink.

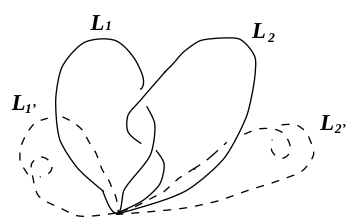

FIGURE 4

This is done in such a way so that the four arcs of L_i and $L_{i'}$ in a disc neighborhood of p, appear in alternating order. See Figure 4. Call the resulting bouquet W_{2n}. Now we construct a Seifert surface, in disc-band form, having W_{2n} as its 1-skeleton, as follows: Let D be a disc neighborhood of p, which contains no singular points of the projection. Then, D intersects the projection of W_{2n} in a bouquet of $4n$ arcs and there are $2n$ arcs outside D. Now, replace each of the arcs outside D by a band, with possible twists if the arc contains kinks.

Lemma 3. *Let S be a Seifert surface obtained with the procedure described above. The boundary knot $K = \partial S$ is surgery n-trivial, where n is the number of components of the almost trivial link used in the construction of S.*

Proof. Simply observe that the knot K can be unknotted in $2^n - 1$ ways by untwisting the bands whose cores correspond to the loops $L_{i'}$ of the discussion before the statement of the lemma. By our construction each band is untwisted by changing a single crossing. See Figure 5 for an example.

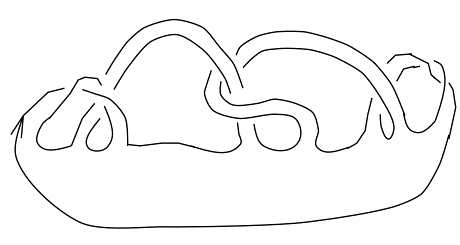

FIGURE 5
Each of the knots obtained by changing a crossing in one or in both of the twisted bands is trivial.

Then the result follows from the fact that a single crossing change can be achieved by surgery modification along an appropriate surgery disc. □

Remark. The reader might have noticed that the following is true: Suppose that a knot K has a projection that contains n single crossings, say $c_1, \dots, c_n$, such that each K_C is the trivial knot. Here, C ranges over all nom-empty subsets of $\{c_1, \dots, c_n\}$ and K_C is the knot obtained from K by changing all crossings contained in C. Then, K is *surgery n-trivial*. Notice, that knots with the property described above are $(n-1)$-*trivial* in the sense of [Gu] (see also [Oh]). Thus, in particular, all of their Vassiliev invariants of orders $\leq n-1$ vanish.

3. The main result

Let us first recall from [O] (see also [Li] and references therein) the definition of *finite type* invariants for $\mathbb{Z}$-homology spheres.

An *algebraically split link* in a $\mathbb{Z}$-homology sphere M is an unoriented link L such that the linking number of any two components of L is zero. In this paragraph all of our links are going to be ± 1-framed algebraically split links. For a link $L \subset M$, let M_L be the $\mathbb{Z}$-homology 3-sphere obtained from M by surgery on L.

Now let $\mathcal{R}$ be a commutative ring with unit, and let $\mathcal{M}$ be the $\mathcal{R}$-module spanned by the set of homeomorphism classes of oriented $\mathbb{Z}$-homology spheres. For $M \in \mathcal{M}$ and $L \subset M$ define $(M,\ L) \in \mathcal{M}$ by

$$(M,\ L) := \sum_{L' \subset L} (-1)^{\# L'} M_{L'},$$

where L' ranges over all sublinks of L (including the emplty one), and $\# L'$ denotes the number of components of L'. Let $\mathcal{M}_n$ be the subspace of $\mathcal{M}$ spanned by $(M,\ L)$ for all M and all links L, whose number of components is greater or equal to n.

Definition 4. [O] A functional $f : \mathcal{M}/\mathcal{M}_{n+1} \longrightarrow \mathcal{R}$ is called an $\mathcal{R}$-valued finite type $\mathbb{Z}$-homology sphere invariant of order $\leq n$.

Now we are ready to state and prove our main result:

Theorem 5. *Let K and K' be two knots in S^3 which are surgery n-equivalent. Let M_K (resp. $M_{K'}$) denote the $\mathbb{Z}$-homology sphere obtained by ± 1 surgery on K (resp. K'). Then all the ($\mathcal{R}$-valued) finite type invariants of M_K and $M_{K'}$, of orders $\leq n$ are the same. In particular, if K is surgery n-trivial then M_K cannot be distinguished from S^3 by any finite type invariant of order $\leq n$.*

Proof. Let $\bar{D} = \{D_1, \dots, D_n\}$ be an $(n+1)$-*collection* for K, as in Definition 1. For each i, let K_i denote the boundary of D_i. By definition each K_i is an unknot whose linking number with K is zero. The manifold obtained by ± 1 surgery on K_i is clearly S^3.

Let L be the $(n+1)$-component, ± 1-framed link $K \cup K_1 \cup \ldots \cup K_n$, where the frames of $K_1 \cup \ldots \cup K_n$ are determined by Definition 1. Finally, let f be a finite type $\mathbb{Z}$-homology sphere invariant of order $\leq n$.

Then, by definition we have

$$f((M,\ L)) = \sum_{L^{'} \subset L} (-1)^{\# L^{'}} f(M_{L^{'}}) = 0, \tag{1}$$

where $L^{'}$ ranges over all sublinks of L (including the emplty one). Clearly, $M_{L^{'}} = S^3$ if $L^{'}$ doesn't contain K. If $L^{'}$ contains K then, by Definition 1, $M_{L^{'}} = M_K$ if $L^{'} = K$ and $M_{L^{'}} = M_{K^{'}}$ otherwise. Then (1) implies that $f(M_K) = f(M_{K^{'}})$. □

By a theorem of Gordon and Luecke ([GL]), M_K is never homeomorphic to S^3 if K is a non trivial knot. Combining this with Theorem 5 and Lemmas 2 and 3, we obtain the following:

Corollary 6. *For every $n \in N$, there exist $\mathbb{Z}$-homology spheres which are not homeomorphic to S^3 and which cannot be distinguished from S^3 by finite type invariants of orders $\leq n$.*

Finally, the theorem stated in the introduction is an immediate consequence of Theorem 5.

Remark. Let L be an *almost trivial link* with n components and let K be a knot constructed from L, as described in 2c. Moreover, suppose that L can be unlinked by changing a single crossing. Arguing as in the proof of Theorem 5, one can see that then all the finite type invariants of M_K , of orders $\leq (n+1)$ will vanish. Thus, for example the Casson invariant (which is the only invariant of order three) of the manifold obtained by ± 1 surgery on the knot of Figure 5, is zero.

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