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A FAKE SMOOTH $\mathbb{CP}^2 \# \mathbb{RP}^4$

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Abstract. We show that the manifold $*\mathbb{CP}^2 \# * \mathbb{RP}^4$, which is homotopy equivalent but not homeomorphic to $\mathbb{CP}^2 \# \mathbb{RP}^4$, is in fact smoothable.

1. Introduction

In Kirby's problem list [Kir97, Problem 4.82] and in a recent lecture at MSRI, P. Teichner raised the question of the smoothability of a certain non-orientable 4-manifold. In this note we show that the manifold in question, denoted $*\mathbb{CP}^2 \# * \mathbb{RP}^4$, which is homotopy equivalent but not homeomorphic to $\mathbb{CP}^2 \# \mathbb{RP}^4$, is in fact smoothable. The smooth model we construct will have the additional property that its universal cover is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. To describe the manifold in question, we remind the reader that one of the first consequences of Freedman's simply-connected surgery theory was a construction of a manifold $*\mathbb{CP}^2$, sometimes called CH in honor of Chern, which is homotopy equivalent but not homeomorphic to $\mathbb{CP}^2$. The manifold $*\mathbb{CP}^2$ is not smoothable for classical reasons: it has non-trivial Kirby-Siebenmann invariant $\mathrm{KS} \in \mathbb{Z}_2$. Given any simply-connected *non-spin* manifold M, a similar construction produces a homotopy equivalent '$*$–partner' $*M$ with opposite Kirby-Siebenmann invariant [Fre82]. In 1983, the first author [Rub84] constructed what is in effect the $*$-partner of $\mathbb{RP}^4$. The connected sum $*\mathbb{CP}^2 \# * \mathbb{RP}^4$ has trivial KS-invariant and so might expected to be smoothable; on the other hand [HKT94] it is not homeomorphic to $\mathbb{CP}^2 \# \mathbb{RP}^4$.

Theorem 1. *The manifold $*\mathbb{CP}^2 \# * \mathbb{RP}^4$ has a smooth structure. Moreover, it has smooth structure such that its universal cover is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.*

The topological classification of non–orientable manifolds with $\pi_1 = \mathbb{Z}_2$ is presented in [HKT94]. There a complete list, up to homeomorphism, of such manifolds is given. The only manifold in this list with vanishing Kirby-Siebenmann invariant which was unknown to be smoothable is the manifold $*\mathbb{CP}^2 \# * \mathbb{RP}^4$. Together with theorem 1 this yields:

Corollary 2. *Let X be a closed non–orientable 4-manifold with $\pi_1(X) = \mathbb{Z}_2$. Then X has a smooth structure if and only if $\mathrm{KS}(X) = 0$.*

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2. Construction of the manifold

The proof of Theorem 1 is constructive; we will find a smooth manifold homeomorphic to $*\mathbb{CP}^2 \# * \mathbb{RP}^4$. The construction uses a homology sphere satisfying the conclusion of the following lemma, whose proof will be given in the next section.

Lemma 2.1. *There is a homology* 3*–sphere* Σ^3 *with the following properties.*

(i) Σ *is obtained by* ± 1 *surgery on a knot* K *in* S^3.
(ii) *The Rohlin invariant* $\mu(\Sigma) = 1 \pmod 2$.
(iii) Σ *admits a free, orientation preserving involution* τ*, which is isotopic to the identity.*

Different Σ's could in principle give rise to different smooth structures on $*\mathbb{CP}^2 \# * \mathbb{RP}^4$, but we know of no way to tell them apart. The situation is quite analogous to that for the fake $\mathbb{RP}^4$'s constructed in [FS81].

Proof of Theorem 1. Let Σ be a homology 3–sphere as described in the lemma; choose an orientation on Σ so that it becomes surgery on a knot with coefficient $= +1$. Items *(i)* and *(ii)* are the ingredients in Freedman's construction [Fre82] of $*\mathbb{CP}^2$. That is, let Y be the result of adding a 2–handle to B^4 along K, with framing 1, then $\partial Y = \Sigma$ and

$$*\mathbb{CP}^2 = Y \cup_\Sigma \Delta^4,$$

where Δ^4 is a contractible 4-manifold with boundary $-\Sigma$. (The sign of the framing is not really important, for the difference between $*\mathbb{CP}^2$ and $*\overline{\mathbb{CP}}^2$ will disappear when we connect sum with $*\mathbb{RP}^4$.) The non-trivial μ-invariant is readily identified with the Kirby-Siebenmann invariant of $*\mathbb{CP}^2$.

Now items (ii) and (iii) are exactly the ingredients for the construction of $*\mathbb{RP}^4$ given in [Rub84], i.e.

$$*\mathbb{RP}^4 = \Delta^4/(x \in \Sigma \sim \tau(x)) = (\Sigma/\tau \mathbin{\widetilde{\times}} I) \cup_\Sigma \Delta^4.$$

(The authors of [HKT94] seem to have been unaware of this earlier construction of $*\mathbb{RP}^4$; compare the discussion in [Kir97, Problem 4.74].)

Let X be the smooth manifold obtained as the union of Y and the mapping cylinder of the orbit map of the free involution τ on Σ, i.e.

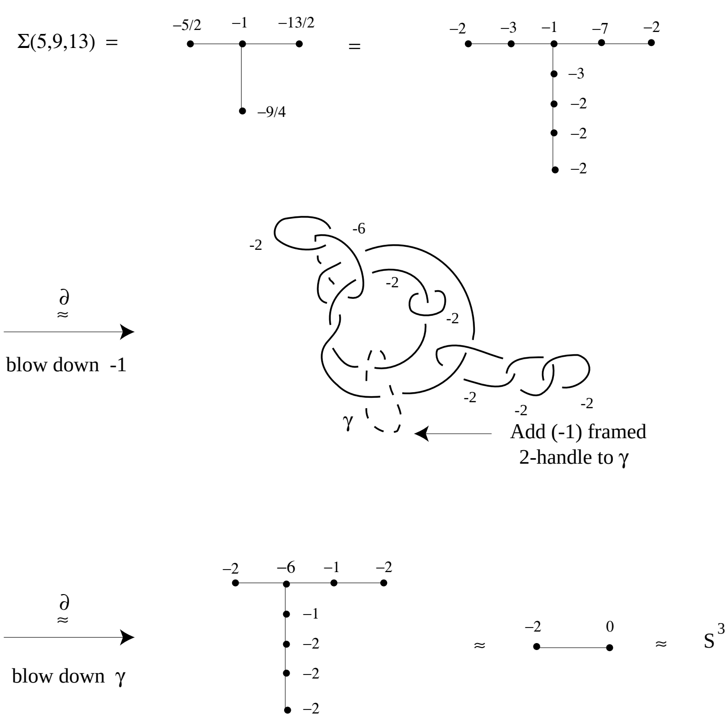
$$X = Y \cup_\Sigma (\Sigma/\tau \mathbin{\widetilde{\times}} I) = Y/(x \in \Sigma \sim \tau(x)).$$

Then X is manifestly smooth, and we claim that it is homeomorphic to $*\mathbb{CP}^2 \# * \mathbb{RP}^4$. This seems quite plausible, for the construction amounts to performing a sort of connected sum, where instead of removing disks and gluing, we remove the 'pseudo-disc' Δ^4 and glue up. More precisely, X is homeomorphic to $*\mathbb{CP}^2 \# * \mathbb{RP}^4$ since, by [Fre82], the simply-connected topological manifolds Y and $(Y \cup_\Sigma \Delta^4) \# \Delta^4$ are homemorphic.

The additional remark about the universal cover of X being standard may be seen as follows(cf. [FS81]). By the construction of X, its cover $\widetilde{X} \cong Y \cup_\tau \overline{Y} \cong Y \cup \overline{Y}$ since τ is isotopic to the identity. On the other hand, $Y \cup \overline{Y}$ is obtained by adding two 2-handles to B^4, together with a 4-handle. The first is added along K, with framing 1, and the second is added along a meridian of K, with framing 0. (This is a standard argument in handle theory, see for example [Kir89].) It is then easy to unknot K, by repeatedly sliding over the 0-framed handle, resulting in a standard picture of $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. □

3. Proof of Lemma 2.1

In this section, we give two examples of homology spheres satisfying the conclusions of Lemma 2.1. Both examples are Brieskorn spheres, i.e. Seifert-fibered homology spheres of the form $\Sigma(p, q, r)$, where p, q, and r are relatively prime odd numbers. The involution τ is nothing more than multiplication by $-1 \in S^1$ in the natural circle action on $\Sigma(p, q, r)$. The condition that the numbers p, q, and r be odd guarantees that τ is free; since -1 is contained in a circle, the involution is isotopic to the identity.



There are many Brieskorn spheres which are integral surgery on a knot–for some examples see [KT90, MM97] or adapt the technique of [CH81]. For most of these constructions one of the indices turns out to be even. One construction is given above, where it is shown that adding a handle (along the curve denoted γ) to the Brieskorn sphere $\Sigma(5, 9, 13)$ yields S^3. The second picture is just the canonical resolution for the singularity of the algebraic surface $z_1^5 + z_2^9 + z_3^{13} = 0$ in $\mathbb{CP}^3$, cf. [NR78]. Turning the picture upside down shows that $\Sigma(5, 9, 13)$ is integral surgery on a knot in S^3. As remarked in the proof of Theorem 1, it doesn't matter whether the coefficient is positive or negative. Again, the μ-invariant is 1 since from the picture just after blowing down the first -1 curve we see that $\Sigma(5, 9, 13)$ bounds a spin manifold with definite intersection form and with signature -8. Thus, this example proves the lemma.

Another construction from the literature which provides Seifert fibered spaces is $rs(p + q)^2 + pq$ surgery on the knot denoted $K_{p,q}(r, s)$ in the recent paper [MM97, §9]. Choosing $p = -13$, $q = 23$, $r = 3$, and $s = 1$ gives the homology sphere $\Sigma(3, 13, 23)$ as $+1$ surgery on a hyperbolic knot. Since $\mu(\Sigma(3, 13, 23)) = 1$, this manifold gives an example which yields the proof of Lemma 2.1. This is the only example of a μ-invariant 1 homology sphere constructible by this method found by a moderately long computer search. It is possible to give a Kirby-calculus proof that $\Sigma(3, 13, 23)$ is surgery on a knot similar to the one for $\Sigma(5, 9, 13)$; aficionados of the subject may wish to check if the knot is the same as the one in the knot from the paper [MM97].

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