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MINIMAL GENUS EMBEDDINGS IN S^2–BUNDLES OVER SURFACES

Bang-He Li and Tian-Jun Li

Abstract. In this paper we completely solve the problem of minimal genus smooth embeddings for the S^2-bundles over surfaces.

1. Introduction

Given a closed oriented smooth 4−manifold N and an integral second homology class ξ, can ξ be represented by a smoothly embedded sphere? The answer is no in general. Lawson [La] gives an excellent survey of the extensive literature on this problem. A further question one can ask is whether ξ is representable by an immersed sphere with a given number of intersection points. Fintushel and Stern [FS] have very general results on this problem.

A more ambitious problem is the minimal genus smooth embedding problem: what is the minimal genus g_ξ of all smoothly embedded orientable surfaces representing ξ? The most general results on this problem are the generalized adjunction inequality for manifolds with $b_2^+ > 1$ and nontrivial Seiberg-Witten invariants or Donaldson invariants, by Kronheimer and Mrowka [KM], and the generalized Thom conjecture for all symplectic 4-manifolds by Morgan, Szabo and Taubes [MST]. Kronheimer and Mrowka [KM] solve the problem completely for $\mathbb{C}P^2$ using Seiberg-Witten invariants. The problem for $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, which are exactly the trivial and nontrivial S^2-bundles over S^2, was first completely solved by Ruberman [R], and then independently by the authors [LLi]. For classes with positive square in $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$, the authors [LLi] solved the problem completely for $2 \le n \le 6$, and obtained partial results for $n = 7, 8$.

In the present paper, the authors completely solve the minimal genus embedding problem for all S^2-bundles over orientable surfaces. These 4−manifolds are symplectic and have $b_2^+ = 1$, hence the generalized adjunction inequality proved in [LLiu] can be applied to give lower bounds for classes with nonnegative self-intersections and positive pairings with some symplectic form. Using various constructions and the Whitney trick, we can construct, for these classes, smoothly embedded surfaces realizing the lower bounds, therefore solving the problem for these classes. By studying the induced action of the self-diffeomorphisms (including orientation-reversing ones), we solve the problem for the remaining classes as well. Interestingly, the minimal genus problem appears in dimension

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4, because the Whitney trick fails in general for dimension 4, while the hardest part of this paper is the use of the Whitney trick to eliminate some intersection points.

An interesitng question is to determine, given a symplectic 4-manifold, which class can be represented by a pseudo-holomorphic curve. The result in this paper shows that there are no differential topological obstructions for S^2-bundles over surfaces.

2. Statements of the results

If $N = S^2 \times M$, where M is a connected closed orientable surfaces with genus g, we denote by x_1 and x_2 the homology classes in N represented by $S \times \{pt\}$ and $\{pt\} \times M$ respectively, where we assume orientations on S^2 and M are given, and the orientation of N is the product orientation. Then $x_1^2 = x_2^2 = 0$ and $x_1 x_2 = 1$. We have

Theorem 1. *Let $N = S^2 \times M$ be as above, $\xi = a_1 x_1 + a_2 x_2 \in H_2(N, \mathbb{Z})$, and g_ξ be the minimal genus of ξ, then*

$$g_\xi = \begin{cases} |a_2|g + (|a_1| - 1)(|a_2| - 1), & if\ (|a_1| + g_2)|a_2| \neq 0 \\ 0, & otherwise. \end{cases}$$

For any nontrivial S^2-bundle over a connected closed orientable surface, we may regard it as a Kahler surface, which is a geometric ruled surface with a holomorphic section whose homology class is denoted by x with $x^2 = 1$. Denote by y the homology class of a fiber. It is easy to see that $y^2 = 0$ and $xy = 1$. Then we have

Theorem 2. *For $\xi = ax + by \in H_2(N, \mathbb{Z})$, where N is the nontrivial S^2-bundle over a surface M with genus $g > 0$, let g_ξ be the minimal genus of ξ, and when $a \neq 0$, let $a' = |a|, b' = \frac{a}{|a|} b$, then*

$$g_\xi = \begin{cases} a'g + \frac{1}{2}a'(a' - 1) + (a' - 1)(b' - 1), & if\ a \neq 0, \\ & \quad a' + 2b' \geq 0 \\ a'g + \frac{1}{2}a'(a' - 1) - (a' - 1)(a' + b' + 1), & if\ a \neq 0, \\ & \quad a' + 2b' \leq 0 \\ 0, & if\ a = 0. \end{cases}$$

Remark 1. Our result in [LLi] for $S^2 \times S^2$ is a special case of Theorem 1 with $g = 0$, while the result therein for $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is not a special case of Theorem 2 with $g = 0$. Actually, our proof for Theorem 2 does not work in this case. However, the problem has been settled in [R] and [LLi], and we exclude it in Theorem 2.

The following results are useful in proving Theorem 2:

Theorem 3. *Let N be a nontrivial S^2-bundle over a surface M with genus > 0, then*

1) *There are orientation-reversing self-diffeomorphisms f and g of N such that*
$$f_*(x) = x - y, \qquad f_*(y) = -y$$
$$g_*(x) = -x + y, \qquad g_*(y) = y$$
2) *id and $-id$ are the only elements in the orthogonal group of $H_2(N, \mathbb{Z})$ induced by orientation-preserving self-diffeomorphisms, where id stands for the identity.*

Remark 2. Denote by $A(N)$ the orthogonal group of the quadratic form on $H_2(N; \mathbb{Z})$, and $D(N)$ the subgroup of $A(N)$ whose elements come from orientation-preserving self-diffeomorphisms. The determination of $D(N)$ is itself an interesting problem. Wall was the pioneer in considering this problem and proved ([W1], [W2]) that $D(N) = A(N)$ if $N = \mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ with $n \leq 9$. Friedman and Morgan [FM] proved the following interesting results: if N is a Dolgachev surface or simply connected with type $(1, n)$ and $n > 9$, then $D(N) \neq A(N)$.

Remark 3. When N is a S^2-bundle over a surface M, $A(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. When M is a sphere, according to [W2], $D(N)$ is also isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. But when M has positive genus, Theorem 3 and an easy argument in the case of a trivial bundle show that $D(N)$ is isomorphic to $\mathbb{Z}_2$, and hence a proper subgroup of $A(N)$.

3. Proof of Theorem 3

In this section, N is a nontrivial S^2-bundle. Since N may be regarded as the sphere bundle of a real orientable $3-$dimensional vector bundle η over M with its second Stiefel-Whitney class $w_2(\eta)$ being the generator of $H^2(M, \mathbb{Z}_2)$, we may take f to be the map which sends every point to its antipodal in the S^2-bundle. Clearly, $f_*(y) = -y$. Since f is orientation-reversing and $x^2 = 1$ we have that $f_*(x)^2 = -1$. f being fiber-preserving and x being represented by a section imply that $f_*(x)$ is also represented by a section, so $f_*(x) = x + ay$ for some $a \in Z$. Now $(x + ay)^2 = x^2 + 2axy = 1 - 2a = f_*(x)^2 = -1$, so $a = -1$ and $f_*(x) = x - y$.

To define g, we take any orientation-reversing self-diffeomorphism of M, say h, then $h^*(\eta)$ is also a real orientable $3-$dimensional vector bundle. Since $h^* : H^2(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2)$ is an isomorphim, $h^* w_2(\eta) = w_2(h^*(\eta)) = w_2(\eta)$. Therefore, $h^*(\eta)$ and η are isomorphic as vector bundles, and there is a fiber-preserving diffeomorphism h' from the sphere bundle of $h^*(\eta)$ to that of η. Let h'' be the natural map from the sphere bundle of η to that of $h^*(\eta)$, and then let $g = h' \circ h''$, we see immediately from the definition of g that $g_*(y) = y$ and $g_*(x) = -x + ay$.

Now g being orientation-reversing implies $-1 = g_*(x)^2 = (-x + ay) = 1 - 2a$. So, $a = 1$ and $g_*(x) = -x + y$, and we have proved 1).

Since $f_*(g_*(x)) = -x, \qquad f_*(g_*(y)) = -y$, we see that id and $-id$ belong to the orthogonal group of $H_2(N, \mathbb{Z})$.

The only other elements of the orthogonal group are h_1 and h_2, where

$$\begin{array}{ll} h_1(x) = -x, & h_1(x-y) = x-y \\ h_2(x) = -x, & h_2(x-y) = -(x-y) \end{array}$$

(cf. [W1], where Wall's basis is $\{x, x-y\}$).

Now $h_1(y) = -2x + y$. If $h_1 = k_*$ for some self-diffeomorphism k of N, then y being represented by an embedded 2−sphere implies that there is a map $l : S^2 \to N$ such that $p_* l_*$ sends a generator of $H_2(S^2, \mathbb{Z})$ to twice a generator of $H_2(M, \mathbb{Z})$, where $p : N \to M$ is the bundle projection. Since the genus of $M > 0$, $\pi_2(M)$ is trivial. Therefore $p_* l_* = 0$. This contradiction shows that $h_1 \neq k_*$. And $h_1 \circ h_2 = -id$ implies $h_2 \neq k_*$ either. We have proved 2), and hence the theorem.

4. Proof of Theorem 1

In this section, N is a trivial S^2−bundle. Since both S^2 and M have orientation-reversing self-diffeomorphisms, there is a self-diffeomorphism of N sending $\xi = a_1 x_1 + a_2 x_2$ to $|a_1| x_1 + |a_2| x_2$. Thus we may assume $a_1 \geq 0$ and $a_2 \geq 0$ with $a_1 + a_2 > 0$, since the case $\xi = 0$ is trivial.

Now, given a Kahler structure on S^2 and M, we have a Kahler surface structure on N which induces a symplectic structure (N, ω) with its symplectic canonical class being $K = (2g-2)\hat{x}_1 - 2\hat{x}_2$, where for any $\xi \in H_2(N; \mathbb{Z})$, $\hat{\xi}$ is the Poincare dual of ξ. Since the x_i are represented by algebraic curves in N, we have $x_i \cdot \omega > 0$, hence for $\xi = a_1 x_1 + a_2 x_2$ satisfying our assumption, we have $\xi \cdot \omega > 0$ and $\xi^2 = 2a_1 a_2 \geq 0$. Therefore, by the generalized adjunction inequality for symplectic 4-manifolds with $b_2^+ = 1$ (Theorem E in [LLiu]), if ξ is represented by a smooth embedding of surface with genus g', then

$$2g' - 2 \geq K(\xi) + \xi^2 = -2a_1 + (2g-2)a_2 + 2a_1 a_2$$

So, $g' \geq a_2 g + (a_1 - 1)(a_2 - 1)$.

Case 1. $\quad a_1 > 0, a_2 > 0$.

Take a_1 disjoint parallel S^2 and a_2 disjoint parallel M, then do surgeries on the $a_1 a_2$ intersection points, we get an embedding of a surface with genus $a_2 g_2 + a_1 a_2 - (a_1 + a_2 - 1) = g_\xi$ which represents $\xi = a_1 x_1 + a_2 x_2$, where $a_1 + a_2 - 1$ intersection points are used to make the surface connected.

Case 2. $\quad a_1 > 0, a_2 = 0$ or $a_1 = 0, a_2 > 0$. Obviously, we may consider only the case $a_1 = 0, a_2 > 0$. Let $a = a_2$.

If $g = 0$, then $\xi = a x_2$ is represented by a disjoint copies of S^2, and hence has minimal genus zero.

Now assume $g > 0$. We need a connected covering surface $\widetilde{M}$ of a sheets with projection $p : \widetilde{M} \to M$ and a smooth map $f : \widetilde{M} \to S^1$ such that for any $m \in M$, f takes a values on $p^{-1}\{m\}$.

Let $T_1, T_2, \cdots, T_g$ be tori in $\mathbb{C}\times\mathbb{R}$ with cores $\{(z,0)/|z| = 3j\}, j = 1, 2, \cdots, g$, and radii 1. Make a connected sum of T_j and T_{j+1} by a tube $T_{j,k}$ in $\mathbb{C}\times\mathbb{R}$ having as core a line from $((3j+1)e^{\sqrt{-1}\frac{2\pi}{a}k}, 0)$ to $((3j+2)e^{\sqrt{-1}\frac{2\pi}{a}k}, 0)$, where $k = 0, 1, \cdots, g-1$, such that the rotation $e^{\sqrt{-1}\frac{2\pi}{a}} \times id$ takes $T_{j,k}$ to $T_{j,k+1}$, and the connected sum of $T_j, j = 1, 2, \cdots, g$, by all $T_{j,k}$ is a smooth surface $\widetilde{M}$ in $\mathbb{C}\times\mathbb{R}$. Then it is obvious that via $e^{\sqrt{-1}\frac{2\pi}{a}} \times id, \mathbb{Z}_a$ acts on $\widetilde{M}$ freely, and the orbit space of the $\mathbb{Z}_a$ action is a surface with genus g which may be regarded as M. The natural map $p : \widetilde{M} \to M$ is a covering map with a sheets, and $\widetilde{M}$ has genus $a(g-1)+1$. For $\widetilde{m} \in \widetilde{M}$, if $\widetilde{m} = (re^{\sqrt{-1}\theta}, t)$, let $f(\widetilde{m}) = e^{\sqrt{-1}\theta}$, then f is a smooth function: $\widetilde{M} \to S^1$ which has a values on $p^{-1}\{m\}$ for any $m \in M$.

Let $\widetilde{f} = p \times f : \widetilde{M} \to M \times \mathbb{C}$, and since $M \times \mathbb{C}$ is naturally embedded in N, we may regard $\widetilde{f}$ as a map into N. It is obvious that $\widetilde{f}$ is a smooth embedding representing $\xi = ax_2$ and $g_\xi = a(g-1)+1$ is the required minimal genus. The proof for Theorem 1 is complete.

Another construction of the covering was shown to us by Prof. Ruberman.

5. Proof of Theorem 2

In this section, N is a nontrivial S^2-bundle. Let $\xi = ax + by$, then $\xi^2 = a(a+2b)$. We first claim that, under the action of $D(N)$, any class $\xi = ax + by$ can be transformed into a class with $a \geq 0$ and $a + 2b \geq 0$. If $a < 0$, using $-id$, we get a class with $a > 0$. Now, if $a + 2b < 0$, using the orientation-reversing map f in Theorem 3, we have $f_*(\xi) = ax - (b+a)y$, and $f_*(\xi)$ is the required class.

If $a = 0$, then obviously $\xi = by$ can be represented by an embedded sphere. So, from now on, we only consider the classes with $a > 0$ and $a + 2b \geq 0$. Consider a symplectic form with cohomology class $\hat{x} + \hat{y}$. Then the symplectic canonical class is $K = -2\hat{x} + (2g-1)\hat{y}$(see Proposition 3.3 in [LLiu]). The pairings between the symplectic form and these classes are clearly positive, and furthermore, these classes have nonegative squares, so Theorem E in [LLiu] can be applied to these classes to give the following inequality: if $\xi = ax + by$ is represented by a smoothly embedded surface with genus g', then

$$g' \geq ag + \frac{1}{2}a(a-1) + (a-1)(b-1).$$

We further separate into two cases,

Case I: $a > 0$ and $b \geq 0$.
Case II: $a > 0, b < 0$ and $a + 2b \geq 0$.

Case I is easy, while Case II is quite difficult. We will deal with them in 5.1 and 5.2 respectively.

5.1. Case I. $a > 0$ and $b \geq 0$.

Take sections $S_1, S_2, \cdots, S_a$ such that

1) each S_i represents x,

2) if $i < j$, then S_i and S_j have unique transversal intersection point S_{ij},
3) if $\{i, j\} \neq \{i', j'\}$, then $S_{ij} \neq S_{i'j'}$.

Then take b disjoint fiber $P_1, \cdots, P_b$ such that each P_k does not meet any S_{ij}.

First, suppose $b > 0$. Then, doing the usual surgeries on the intersection points of P_1 with $S_1, S_2, \cdots, S_a$ and the intersection points of S_1 with $P_2, \cdots, P_b$ yields an immersion of a connected surface with genus ag. All of its intersection points are

$$\begin{array}{ll} S_{ij}, & 1 \leq i < j \leq a \\ P_i \cap S_j, & i = 2, \cdots, b, j = 2, \cdots, a. \end{array}$$

Doing surgeries on these points yields an embedding of a surface with genus

$$ag + \frac{a(a-1)}{2} + (a-1)(b-1)$$

which represents $\xi = ax + by$.

Next, suppose $b = 0$. Then, the surgeries on S_{ij} produce an embedding of a surface with genus

$$ag + \frac{a(a-1)}{2} - (a-1)$$

which represents $\xi = ax$, where $(a-1)$ intersection points are used to make the surface connected. The proof of Case I is complete.

5.2. Case II. $a > 0, b < 0$, and $a + 2b \geq 0$.

For convenience, we change the sign of b and rewrite ξ as $ax - by$ with $a > 0$, $b > 0$, and $a - 2b \geq 0$. First, we give an outline of the proof.

Outline of the Proof. Since the proof is a bit long, we divide it into 3 steps.

In Step 1, we construct, as in section 4, an immersion of the $a-$sheet covering surface $\widetilde{M}$ of M in the complex line bundle E over M with Chern number 1 such that outside a disk D (regarded as the unit disk in $\mathbb{C}$) of M, the immersion is an embedding and its composition with the projection is an $a-$sheeted covering map, while on the disk D, for a trivialization of E, it is given by the graphs of a linear functions $f_k, k = 0, 1, \cdots, a-1$.

In Step 2, we change the f_k in the interior of D to the $\widetilde{f_k}$ which have the following properties:

1) For $k = 1, 3, \cdots, 2b-1, 2b, 2b+1, \cdots, a-1$, $\widetilde{f_k}$ takes values in the unit disk, while for $k = 0, 2, \cdots, 2b-2$, $\widetilde{f_k}$ takes values outside the unit disk (including ∞).
2) The graphs of the $\widetilde{f_k}$ restricted to $[-1, 1] \subset D$ for the first group of the k are in a smoothly embedded surface in $D \times D$, and the graphs of the $\widetilde{f_k}$ restricted to $[-1, 1]$ for the second group of the k are in a smoothly embedded surface in $D \times (\{z/|z| \geq 1\} \cup \infty)$.
3) There are no intersection points for the graphs of $\widetilde{f_k}$ and $\widetilde{f_{k'}}$ over $D \setminus [-1, 1]$, if $k \neq k'$, and their unique intersection point is simple.

The graphs of the $\widetilde{f}_k$ are to be constructed so that the Whitney disks for eliminating $\frac{1}{2}b(b-1)$ pairs of intersection points with opposite signs are easy to see. Actually all Whitney disks will be in a smoothly embedded surface in E.

We do our construction for the $\widetilde{f}_k$ first on $[-1,1]$ to make the Whitney disks clearly visible. Then we extend the $\widetilde{f}_k$ to the region in the disk D between the two line segments $I_+ = \{t+\sqrt{-1}\xi / t \in [-\sqrt{1-\xi^2}, \sqrt{1-\xi^2}]\}$ and $I_- = \{t-\sqrt{-1}\xi / t \in [-\sqrt{1-\xi^2}, \sqrt{1-\xi^2}]\}$, where $\xi > 0$ is a fixed small number. On the line segment I_+, the graphs of the first group of the $\widetilde{f}_k$ and the graphs of the first group of the f_k give us a pair of braids. Similarly, considering the graphs of the second group of the $\widetilde{f}_k$ and the f_k and the line segment I_-, we get three other pairs of braids. We have to prove that each pair of braids are the same. Therefore the $\widetilde{f}_k$ can be extended to the disk D such that $\widetilde{f}_k = f_k$ outside a neighborhood of $[-1,1]$.

Step 2 is the most difficult part of the proof.

In Step 3, from the careful construction of Step 2, we immediately find a Whitney disk and we eliminate a pair of intersection points first. Furthermore, the virtue of the construction in Step 2 is that after a pair of intersection points are removed, another Whitney disk naturally appears. After eliminating $\frac{1}{2}b(b-1)$ pairs of intersections, we do surgeries on the remaining intersections and get the required embedding.

Step 1. Let D be a closed disk in M and E a complex line bundle over M with Chern number 1 with respect to an orientation of M. According to the proof of Theorem 1, there is an $a-$sheeted covering $\widetilde{M}$ of M embedded in $M\times\mathbb{C}$ by the map $p\times f$, where p is the covering map. Since $E|_{M\setminus \mathring{D}}$ is trivial, there is an isomorphism $g : E|_{M\setminus \mathring{D}} \to (M\setminus \mathring{D})\times\mathbb{C}$, where $\mathring{D} = D\setminus\partial D$. So $\widetilde{M}\setminus p^{-1}(\mathring{D})$ is embedded in $E|_{M\setminus \mathring{D}}$ via g^{-1}. These can be summarized in the following commutative diagram:

$$\begin{array}{ccccccccccc} E & \supset & E|_{M\setminus \mathring{D}} & \xrightarrow{g} & (M\setminus \mathring{D})\times\mathbb{C} & \subset & M\times\mathbb{C} & \xleftarrow{p\times f} & \widetilde{M} & \supset & \widetilde{M}\setminus P^{-1}(\mathring{D}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & p\downarrow & & \downarrow \\ M & \supset & M\setminus \mathring{D} & = & M\setminus \mathring{D} & \subset & M & = & M & \supset & M\setminus \mathring{D}. \end{array}$$

Furthermore, we may assume that the point $(3,1)\in\widetilde{M}\subset\mathbb{C}\times\mathbb{R}$ has a closed neighborhood $\widetilde{D'}\subset\widetilde{M}$ disjoint with any tube $T_{j,k}$ such that $\widetilde{D'}$ is homeomorphic to $\{z\,/\,|z-3|\leq\varepsilon\ \}$ for some $\varepsilon>0$ via the projection $\mathbb{C}\times\mathbb{R}\longrightarrow\mathbb{C}$, and for $k=1,\cdots,a-1$

$$\widetilde{D'}\bigcap((e^{\sqrt{-1}\frac{2\pi k}{a}}\times id)(\widetilde{D'}))=\emptyset.$$

Let $\widetilde{D}\subset\widetilde{D'}$ be homeomorphic to $\{z\,/\,|z-3|\leq\frac{\varepsilon}{2}\ \}$, and

$$f_k^{''} : (e^{\sqrt{-1}\frac{2\pi k}{a}}\times id)(\partial\widetilde{D})\longrightarrow S^1$$

be the constant map with value $e^{\sqrt{-1}\frac{2\pi k}{a}}$. We may define a map

$$f'_k : (e^{\sqrt{-1}\frac{2\pi k}{a}} \times id)(\widetilde{D'} \setminus \text{ the interior of } \widetilde{D}) \longrightarrow S^1$$

such that $f'_k|_{\partial \widetilde{D'}} = f,\ f'_k|_{\partial \widetilde{D}} = f''_k$, and image $f'_k \subset$ image $f|_{\widetilde{D'}}$. Then we define f' on

$$\widetilde{M} \setminus \bigcup_{k=0}^{a-1} (e^{\sqrt{-1}\frac{2\pi k}{a}} \times id)(\text{ the interior of } \widetilde{D})$$

by

$$f' = \begin{cases} f'_k, & \text{on } (e^{\sqrt{-1}\frac{2\pi k}{a}} \times id)(\widetilde{D'} \setminus \text{ the interior of } \widetilde{D}) \\ f, & \text{otherwise.} \end{cases}$$

We may assume f' is a smooth map into S^1.

Now, we take $p(\widetilde{D})$ as our previous D. Choose an orientation-preserving diffeomorphism of $\{z \,/\, |z-3| \leq \frac{\varepsilon}{2}\}$ to the unit disk such that $\{z \,/\, |z| = 3\} \bigcap \{z \,/\, |z-3| \leq \frac{\varepsilon}{2}\}$ is sent to the real interval $[-1, 1]$ and $3 + \frac{\varepsilon}{2}$ is sent to $\sqrt{-1}$. Then we use the unit disk as the coordinate chart of D.

Denote $g|_{\partial D} : E|_{\partial D} \longrightarrow \partial D \times \mathbb{C}$ still by g, and let $h : E|_D \longrightarrow D \times \mathbb{C}$ be a trivialization. We may assume that for $(e^{\sqrt{-1}\theta}, z) \in \partial D \times \mathbb{C}$,

$$(h \circ g^{-1})((e^{\sqrt{-1}\theta}, z)) = (e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\theta} z)$$

by a suitable choice of h, since the Chern number of E is 1.

It is easy to see that the composition

$$C : \quad \widetilde{M} \setminus p^{-1}(\overset{\circ}{D}) \xrightarrow{p \times f} (M \setminus \overset{\circ}{D}) \times \mathbb{C} \xrightarrow{g^{-1}} E|_{M \setminus \overset{\circ}{D}}$$

is still an embedding. And $h \circ C$ restricted to $(e^{\sqrt{-1}\frac{2\pi}{a}} \times id)(\partial \widetilde{D})$ gives a functions

$$f_k : S^1 \longrightarrow S^1 \text{ with } f_k(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}(\theta + \frac{2\pi k}{a})}.$$

For $k = 0, 2, \cdots, 2b-2$, extend the f_k to functions

$$f_k : \quad D \longrightarrow \{z \in \mathbb{C} \,/\, |z| \geq 1\} \bigcup \infty.$$

with $f_k(z) = \bar{z}^{-1} e^{\sqrt{-1}\frac{2\pi k}{a}}$, and for other k with $0 \leq k \leq a-1$, extend the f_k to functions

$$f_k : \quad D \longrightarrow D$$

with $f_k(z) = z e^{\sqrt{-1}\frac{2\pi k}{a}}$.

Via h^{-1}, we may regard the graph of f_k as a disk in $E|_D \bigcup E_\infty$, where E_∞ is the infinite section. We identify it with $(e^{\sqrt{-1}\frac{2\pi k}{a}} \times id)(\widetilde{D})$. Then we get an immersion of $\widetilde{M}$ in $E \bigcup\ E_\infty$, which is the nontrivial S^2-bundle (by suitable smoothing process, we may assume the immersion is smooth, and is identified with

$$h^{-1}(\text{ the graph of } f_k) \quad \text{on} \quad (e^{\sqrt{-1}\frac{2\pi k}{a}} \times id)(\widetilde{D}),$$

having only $h^{-1}(0,0)$ and $h^{-1}(0,\infty)$ as multiple points).

It is easy to see that the above immersion represents the homology class $ax - by$.

Step 2. Now we are going to change f_k so that we will get an immersion representing $ax - by$ and having only double points. Look first at those $f_k : D \longrightarrow D$. Consider the restriction of f_k to the real interval $[-1, 1]$. Then, their graphs are in the cylinder $[-1, 1] \times D$.

First we construct a topological embedding $\widetilde{u}$ of the set $[-1, 1] \times [-\varepsilon, 2\pi - 2\varepsilon]$ in $[-1, 1] \times D$ where $0 < \varepsilon < \frac{\pi}{a}$. The idea of the construction is

1) $\widetilde{u}(t, \cdot) = (t, \cdot)$
2) for $t \geq \frac{2}{3}$, the image of $\widetilde{u}$ is the corresponding part of the cone with vertex $(0, 0) \in [-1, 1] \times D$ and base $(1, e^{\sqrt{-1}\theta}), \theta \in [-\varepsilon, 2\pi - 2\varepsilon]$.
3) for $t \leq -\frac{2}{3}$, the image of $\widetilde{u}$ is the corresponding part of the cone with vertex $(0, 0) \in [-1, 1] \times D$ and base $(-1, -e^{\sqrt{-1}\theta}), \theta \in [-\varepsilon, 2\pi - 2\varepsilon]$.
4) for $|t| \leq \frac{1}{3}$, $\widetilde{u}(t, \cdot)$ is the line segment from $(t, e^{\sqrt{-1}(-\frac{3}{2}\varepsilon - \frac{\pi}{2})})$ to $(t, e^{\sqrt{-1}(-\frac{3}{2}\varepsilon + \frac{\pi}{2})})$.
5) for $\frac{1}{3} \leq |t| \leq \frac{2}{3}$, the image of $\widetilde{u}$ is gotten by using some line segments.

More precisely we have

1) $\widetilde{u}(t, \theta) = (t, \widetilde{v}(t, \theta))$
2) for $t \geq \frac{2}{3}, \widetilde{v}(t, \theta) = te^{\sqrt{-1}\theta}$
3) for $t \leq -\frac{2}{3}, \widetilde{v}(t, \theta) = te^{\sqrt{-1}(2\pi - 3\varepsilon - \theta})$
4) for $|t| \leq \frac{1}{3}, \widetilde{v}(t, \theta) = e^{\sqrt{-1}\frac{3}{2}(\pi - \varepsilon)}(\frac{\theta + \varepsilon}{2\pi - \varepsilon} - \frac{1}{2})$
5) for $\frac{1}{3} \leq t \leq \frac{2}{3}, \widetilde{v}(t, \theta) = (2 - 3t)\widetilde{v}(\frac{1}{3}, \theta) + (3t - 1)\widetilde{v}(\frac{2}{3}, \theta)$
6) for $-\frac{1}{3} \leq t \leq -\frac{2}{3}, \widetilde{v}(t, \theta) = (-1 - 3t)\widetilde{v}(\frac{-2}{3}, \theta) + (3t + 2)\widetilde{v}(-\frac{1}{3}, \theta)$.

To see if $\widetilde{u}$ is a topological embedding, we need only to prove that if $|t| \in (\frac{1}{3}, \frac{2}{3})$ and $\widetilde{v}(t, \theta) = \widetilde{v}(t, \theta')$, then $\theta = \theta'$. From the definition of $\widetilde{v}$, we have

$$\begin{aligned} e^{\sqrt{-1}\theta} - e^{\sqrt{-1}\theta'} &= re^{\sqrt{-1}\frac{3}{2}(\pi - \varepsilon)}, &\quad \text{if } t \in (\tfrac{1}{3}, \tfrac{2}{3}) \\ e^{\sqrt{-1}(2\pi - 3\varepsilon - \theta)} - e^{\sqrt{-1}(2\pi - 3\varepsilon - \theta')} &= r'e^{\sqrt{-1}\frac{3}{2}(\pi - \varepsilon)}, &\quad \text{if } -t \in (\tfrac{1}{3}, \tfrac{2}{3}) \end{aligned}$$

where

$$r = \frac{3}{2}\frac{2 - 3t}{3t - 1}\frac{\theta - \theta'}{2\pi - \varepsilon} \qquad -r' = \frac{3}{2}\frac{2 + 3t}{3t + 1}\frac{\theta - \theta'}{2\pi - \varepsilon}.$$

Since $e^{\sqrt{-1}\frac{3}{2}(\pi - \varepsilon)}$ is a tangential direction of the unit circle at $e^{\sqrt{-1}(\pi - \frac{3}{2}\varepsilon)}$, we see that for $t \in (\frac{1}{3}, \frac{2}{3})$

$$\theta - (\pi - \frac{3}{2}\varepsilon) = -(\theta' - (\pi - \frac{3}{2}\varepsilon))$$

and hence

$$\begin{aligned} e^{\sqrt{-1}(\theta - (\pi - \frac{3}{2}\varepsilon))} - e^{\sqrt{-1}(\theta' - (\pi - \frac{3}{2}\varepsilon))} &= r\sqrt{-1} = \\ e^{\sqrt{-1}(\theta - (\pi - \frac{3}{2}\varepsilon))} - e^{-\sqrt{-1}(\theta' - (\pi - \frac{3}{2}\varepsilon))} &= \text{ a real number.} \end{aligned}$$

Therefore $r = 0$, i.e. $\theta = \theta'$. The same argument works for the case of $-t \in (\frac{1}{3}, \frac{2}{3})$. Thus $\widetilde{u}$ is an embedding with "corners" on $|t| = \frac{1}{3}, \frac{2}{3}$. We may use a smooth perturbation to get a smooth embedding u with $u(t,\theta)=(t,v(t,\theta))$ such that

$$v(t,\theta)=\widetilde{v}(t,\theta)),\ \text{ if } \quad |t|\le\frac{1}{3}\ \text{or}\ |t|\ge\frac{2}{3}.$$

It is easy to see that the image of the set

$$[-1,-\frac{2}{3}]\times\{2\pi-3\varepsilon-\frac{2\pi k}{a}\}\bigcup[\frac{2}{3},1]\times\{\frac{2\pi k}{a}\}$$

under u is in the graph of f_k. Now, for $k=1,3,\cdots,2b-1,2b,2b+1,\cdots,a-1$, let

$$a_k=\frac{a-k}{3a},\qquad b_k=2\pi-\frac{2\pi}{a}+(\frac{2\pi}{a}-2\varepsilon)\frac{a-k}{a},$$

and let L_k' be a curve in $[-1,1]\times[-\varepsilon,2\pi-2\varepsilon]$ consisting of the line segments connecting the six points:

$$(1,\frac{2\pi k}{a}),(a_k,\frac{2\pi k}{a}),(a_k-\frac{1}{6a},b_k),(0,b_k),(-\frac{1}{3},2\pi-3\varepsilon-\frac{2\pi k}{a}),$$
$$\text{and } (-1,2\pi-3\varepsilon-\frac{2\pi k}{a}).$$

Smoothing L_k' on some small neighborhood of the corners, we get a smooth curve L_k parametrized by $(t,l_k(t)),\quad t\in[-1,1]$.

We will construct a map $\widetilde{f}_k:D\longrightarrow D$ instead of f_k. First, let

$$\widetilde{f}_k'(z)=f_k(z),\quad \text{if}\quad |z|>\frac{2}{3}.$$

Let the intersection point of L_k and $L_{k'}(k\neq k')$ be $(t_{k,k'},l_k(t_{k,k'}))=(t_{k,k'},l_{k'}(t_{k,k'}))$ for some $t_{k,k'}$. Then there is a neighborhood of $t_{k,k'}$ in $(0,\frac{1}{3})$, where

$$l_k(t)=A_{k,k'}t+B_{k,k'}$$

for some real constants $A_{k,k'}$ and $B_{k,k'}$, and hence $v(t,l_k(t))=\widetilde{v}(t,l_k(t))$ is a real analytic function around $t_{k,k'}$. And in a neighborhood $U_{k,k'}$ of $t_{k,k'}$ in D, we let

$$\widetilde{f}_k'(z)=v(z,l_k(z))=e^{\sqrt{-1}\frac{3}{2}(\pi-\varepsilon)}(\frac{A_{k,k'}z+B_{k,k'}+\varepsilon}{2\pi-\varepsilon}-\frac{1}{2}).$$

Suppose $\varepsilon_{k,k'}>0$ is small so that $\{z\in D\,/\,|z-t_{k,k'}|<2\varepsilon_{k,k'}\}\subset U_{k,k'}$. For small $\eta>0$, let

$$U_k=\{z=t+\sqrt{-1}s\in D\,/\,|s|<2\eta,|t|<\frac{2}{3}+\eta,|z-t_{k,k'}|>\varepsilon_{k,k'}\}$$

where k' runs over all possible values. We define $\widetilde{f}_k'$ on U_k by

$$\widetilde{f}_k'(t+\sqrt{-1}s)=v(t,l_k(t)).$$

Then on the set

$$U_k\bigcup_{k'}U_{k,k'}\bigcup\{z\,/\,\frac{2}{3}\le|z|\le 1\}$$

with its natural open covering, we can use a partition of unity to combine those $\widetilde{f}'_k$ to get a function $\widetilde{f}_k$ with values in D. We may assume that $\widetilde{f}_k$ is well defined on the set

$$U = \{z = t + \sqrt{-1}s \in D \,/\, |s| \leq \eta\}$$

and $\widetilde{f}_k(z) = f_k(z)$ if $1 - \eta \leq |z| \leq 1$. It is obvious that

$$\widetilde{f}_k(t) = v(t, l_k(t)) \quad \text{for} \quad t \in [-1, 1]$$

and for $k' \neq k$, the only intersection of the graphs of $\widetilde{f}_k$ and $\widetilde{f}'_k$ over U is $(t_{k,k'}, l_k(t_{k,k'}))$.

Furthermore, it can be seen that for small $\xi \in (0, \eta]$,

$$\widetilde{f}_k(-\frac{1}{3} + \sqrt{-1}\xi) = v(-\frac{1}{3}, 2\pi - 3\varepsilon - \frac{2\pi k}{a})$$
$$\widetilde{f}_k(\frac{1}{3} + \sqrt{-1}\xi) = v(\frac{1}{3}, \frac{2\pi k}{a})$$

and for $k \neq k'$, the orthogonal projections of the curves $(t, \widetilde{f}_k(t + \sqrt{-1}\xi))$ and $(t, \widetilde{f}_{k'}(t + \sqrt{-1}\xi)), t \in [-\frac{1}{3}, \frac{1}{3}]$ on the plane

$$u([-\frac{1}{3}, \frac{1}{3}] \times [-\varepsilon, 2\pi - 2\varepsilon])$$

can have intersections only in a small neighborhood of $(t_{k,k'}, l_k(t_{k,k'}))$. Now we are going to construct a braid by using the $\widetilde{f}_k(t + \sqrt{-1}\xi)$.

Let L be the diameter of D from $-\sqrt{-1}e^{\sqrt{-1}(2\pi - \frac{3\varepsilon}{2})}$ to $\sqrt{-1}e^{\sqrt{-1}(2\pi - \frac{3\varepsilon}{2})}$ parametrized by

$$L(\theta) = e^{\sqrt{-1}\frac{3}{2}(\pi - \varepsilon)}(\frac{2\theta + 2\varepsilon}{2\pi - \varepsilon} - 1), \qquad \theta \in [-\varepsilon, 2\pi - 2\varepsilon].$$

Then there is an embedding u_+ of $[\sqrt{1 - \xi^2}, 2] \times [-\varepsilon, 2\pi - 2\varepsilon]$ to $[\sqrt{1 - \xi^2}, 2] \times D$ given by

$$u_+(t, \theta) = (t, \quad \frac{2 - t}{2 - \sqrt{1 - \xi^2}} e^{\sqrt{-1}\theta} + \frac{t - \sqrt{1 - \xi^2}}{2 - \sqrt{1 - \xi^2}} L(\theta))$$

and an embedding u_- of $[\sqrt{1 - \xi^2}, 2] \times [-\varepsilon, 2\pi - 2\varepsilon]$ to $[-2, -\sqrt{1 - \xi^2}] \times D$ given by

$$u_-(t, \theta) = (t, \quad \frac{-t - \sqrt{1 - \xi^2}}{2 - \sqrt{1 - \xi^2}} L(\theta) - \frac{t + 2}{2 - \sqrt{1 - \xi^2}} e^{\sqrt{-1}(2\pi - 3\varepsilon - \theta)}).$$

We may assume that ξ is so small that if $\sqrt{1-\xi^2}+\sqrt{-1}\xi = e^{\sqrt{-1}\theta'}$ and $\theta' \in (0, 2\pi)$, then $\theta' < \min\{\varepsilon, \frac{2\pi}{a} - 2\varepsilon\}$. Thus

$$\widetilde{f}_k(\sqrt{1-\xi^2}+\sqrt{-1}\xi) = e^{\sqrt{-1}(\frac{2\pi k}{a}+\theta')} = u_+(\sqrt{1-\xi^2}, \frac{2\pi k}{a}+\theta'),$$
$$\widetilde{f}_k(-\sqrt{1-\xi^2}+\sqrt{-1}\xi) = -e^{\sqrt{-1}(\frac{2\pi k}{a}+\theta')} = u_-(-\sqrt{1-\xi^2}, 2\pi - 3\varepsilon - \frac{2\pi k}{a}+\theta'),$$

and the map $s_k : [-2, 2] \longrightarrow [-2, 2] \times D$ given by

$$s_k(t) = \begin{cases} u_-(t, 2\pi - 3\varepsilon - \frac{2\pi k}{a} + \theta'), & t \in [-2, -\sqrt{1-\xi^2}] \\ (t, \widetilde{f}_k(t+\sqrt{-1}\xi)), & |t| \le \sqrt{1-\xi^2} \\ u_+(t, \frac{2\pi k}{a} + \theta'), & t \in [\sqrt{1-\xi^2}, 2] \end{cases}$$

is continuous.

Regard the image of s_k as a string in $[-2, 2] \times D$, then taking $k = 1, 2, \cdots, 2b-1, 2b, \cdots, a-1$, we get $a - b$ strings which form a braid denoted by S.

Let $\widetilde{s_k} \longrightarrow [-2, 2] \times D$ be the map given by

$$\widetilde{s_k} = \begin{cases} u_-(t, 2\pi - 3\varepsilon - \frac{2\pi k}{a} + \theta'), & t \in [-2, -\sqrt{1-\xi^2}] \\ (t+\sqrt{-1}\xi)e^{\sqrt{-1}\frac{2\pi k}{a}}, & |t| \le \sqrt{1-\xi^2} \\ u_+(t, \frac{2\pi k}{a} + \theta'), & t \in [\sqrt{1-\xi^2}, 2]. \end{cases}$$

Then $\widetilde{s_k}$ is continuous, and the string coming from $\widetilde{s_k}$ has the same endpoints as that coming from s_k. Denote by $\widetilde{S}$ the braid coming from the $\widetilde{s_k}$. We are going to prove that S and $\widetilde{S}$ are the same as braids.

Letting $L_+(\theta) = (2, L(\theta)), L_-(\theta) = (-2, L(\theta))$, $\theta \in [-2, 2\pi - 2\varepsilon]$, we have line segments L_+ and L_- in $[-2, 2] \times D$. Regard the normal direction of L_+ and L_- given by $e^{-\sqrt{-1}\frac{3}{2}\varepsilon}$ as the forward direction in the set $[-2, 2] \times D$. Then it can be seen that $\widetilde{S}$ can be represented by strings $\widetilde{s'_k}$ so that if $k < k'$, then $\widetilde{s'_{k'}}$ is in front of $\widetilde{s'_k}$ (Notice: it takes time to see this important fact!).

For S, we observe first that for the part of S corresponding to $t \in [\frac{1}{3}, 2]$, it is isotopic to the lines connecting

$$(\frac{1}{3}, \widetilde{f}_k(\frac{1}{3}+\sqrt{-1}\xi)) \quad \text{and} \quad (2, L(\frac{2\pi k}{a}+\theta'))$$

with the endpoints at $t = \frac{1}{3}$ and 2 fixed. Notice that $(\frac{1}{3}, \widetilde{f}_k(\frac{1}{3}+\sqrt{-1}\xi))$ is on the line

$$(\frac{1}{3}, v(t, \theta)), \quad \theta \in [-\varepsilon, 2\pi - 2\varepsilon]$$

which is parallel to the lines L_+ and L_-. And for the part of S corresponding to $t \in [-2, -\frac{1}{3}]$, it is isotopic to the lines connecting

$$(-2, L(2\pi - 3\varepsilon - \frac{2\pi k}{a}+\theta')) \quad \text{and} \quad (-\frac{1}{3}, \widetilde{f}_k(-\frac{1}{3}+\sqrt{-1}\xi))$$

with the endpoints at $t = -2$ and $-\frac{1}{3}$ fixed. Denoting by s'_k the strings of the braid S' obtained by S and the isotopies for $t \in [\frac{1}{3}, 2]$ and $t \in [-2, -\frac{1}{3}]$, we have that

$$s'_k(t) = (t, \widetilde{f}_k(t + \sqrt{-1}\xi)) \quad \text{for} \quad t \in [-\frac{1}{3}, \frac{1}{3}]$$

and for $\frac{1}{3} \leq |t| \leq 2, s'_k$ lies in the plane spanned by L_+ and L_- such that if $k \neq k'$, then s'_k and $s'_{k'}$ do not intersect. In $U_{k,k'}$, we have

$$\widetilde{f}_k(t + \sqrt{-1}\xi) = e^{\sqrt{-1}\frac{3}{2}(\pi-\varepsilon)}(\frac{A_{k,k'}(t + \sqrt{-1}\xi) + B_{k,k'} + \varepsilon}{2\pi - \varepsilon} - \frac{1}{2}),$$
$$\widetilde{f}_{k'}(t + \sqrt{-1}\xi) = e^{\sqrt{-1}\frac{3}{2}(\pi-\varepsilon)}(\frac{A_{k',k}(t + \sqrt{-1}\xi) + B_{k',k} + \varepsilon}{2\pi - \varepsilon} - \frac{1}{2}).$$

If $k < k'$, then by our construction of $l_k(t)$,

$$A_{k,k'} < 0 \quad \text{and} \quad A_{k',k} = 0$$

and therefore $s'_{k'}$ lies in the plane spanned by L_+ and L_-. Also,

$$\widetilde{f}_k(t + \sqrt{-1}\xi)) = \alpha(t)e^{\sqrt{-1}\frac{3}{2}(\pi-\varepsilon)} + \sqrt{-1}e^{\sqrt{-1}\frac{3}{2}(\pi-\varepsilon)}\frac{A_{k,k'}\xi}{2\pi - \varepsilon}$$

where $\alpha(t)$ is a real number, and so s'_k lies in back of $s'_{k'}$. This proves that S and $\widetilde{S}$ are the same braid.

Since

$$s_k(t) = \widetilde{s}_k(t) \quad \text{for} \quad \sqrt{1 - \xi^2} \leq |t| \leq 2,$$

an easy argument in homotopy theory shows that as braids for $|t| \leq \sqrt{1 - \xi^2}$, S and $\widetilde{S}$ are also the same. Let

$$\approx{s}_k\ (t) = (t, f_k(t + \sqrt{-1}\eta)), \;\; |t| \leq \sqrt{1 - \eta^2}$$

where $f_k(z) = ze^{\sqrt{-1}\frac{2\pi k}{a}}$. Then the $\approx{s}_k$ define a braid $\approx{S}$ which is isotopic to $\widetilde{S}|_{|t| \leq \sqrt{1-\xi^2}}$ by the homotopy with the endpoints moved:

$$H(t, s) = (t, f_k(t + \sqrt{-1}s)), \quad s \in [\xi, \eta], |t| \leq \sqrt{1 - s^2}.$$

Therefore $S|_{|t| \leq \sqrt{1-\xi^2}}$ is isotopic to $\approx{S}\ |_{|t| \leq \sqrt{1-\eta^2}}$. This shows that $\widetilde{f}_k(z)$ extends to $\{\, t + \sqrt{-1}s \in D \,/\, s \geq 0 \,\}$.

A similar argument proves that $\widetilde{f}_k(z)$ extends to $\{\, t + \sqrt{-1}s \in D \,/\, s \leq 0 \,\}$ and we obtain maps

$$\widetilde{f}_k(z) : D \longrightarrow D$$

such that

$$\widetilde{f}_k(z) = f_k(z), \text{ if } 1 - \eta \leq |z| \leq 1$$

and the unique intersection point of the graphs of $\widetilde{f}_k$ and $\widetilde{f}_{k'}$ for $k \neq k'$ is

$$(t_{k,k'}, l_k(t_{k,k'})).$$

For $k = 0, 2, \cdots, 2b-2$, to construct

$$\widetilde{f}_k : D \longrightarrow \{z \,/\, |z| \leq 1\} \bigcup \{\infty\}$$

we notice that $f_k = H \circ g_k$, where

$$H(z) = \bar{z}^{-1}, \;\; g_k(z) = z e^{\sqrt{-1}\frac{2\pi k}{a}}.$$

We may do a similar thing for $g_k, k = 0, 2, \cdots, 2b-2$, as for $f_k, k = 1, 3, \cdots, 2b-1, 2b, \cdots, a-1$. The main difference is the construction of L'_k.

For the present case, choose a_k and b_k such that

$$a_k = -\frac{1}{3} + \frac{k+1}{3a}, \;\; b_k = -\varepsilon + \frac{k+1}{a}\varepsilon,$$

and let L'_k be a curve in $[-1,1] \times [-\varepsilon, 2\pi - 2\varepsilon]$ consisting of the segments of lines:

$$\text{from } (-1, 2\pi - 3\varepsilon - \frac{2\pi k}{a}) \text{ to } (a_k, 2\pi - 3\varepsilon - \frac{2\pi k}{a}), \text{ and then}$$
$$\text{to } (a_k + \frac{1}{6a}, b_k), \;(0, b_k), \;(\frac{1}{3}, \frac{2\pi k}{a}), \;\text{ and }\; (1, \frac{2\pi k}{a}).$$

Then we do the same thing for g_k to get $\widetilde{g}_k$ as we did for f_k to get $\widetilde{f}_k$. For $k \neq k'$, the graphs of $\widetilde{g}_k$ and $\widetilde{g}_{k'}$ have a unique intersection point of multiplicity 1 coming from the intersection point of L'_k and $L'_{k'}$. Now, let

$$\widetilde{f}_k = H \circ \widetilde{g}_k, \; k = 0, 2, \cdots, 2b-2.$$

Then the intersection of the graphs of $\widetilde{f}_k$ and $\widetilde{f}_{k'}$ has multiplicity -1.

Replacing f_k by $\widetilde{f}_k$ for $k = 0, 1, \cdots, a-1$, we obtain an immersion F of $\widetilde{M}$ having $E \bigcup E_\infty$ with $\frac{1}{2}(a-b)(a-b-1)$ intersection points of positive sign, and $\frac{1}{2}b(b-1)$ intersection points of negative sign.

Step 3. We are now in a position to eliminate $\frac{1}{2}b(b-1)$ pairs of intersection points with opposite signs, by using the Whitney trick.

First notice that the intersection points of the immersion F of $\widetilde{M}$ are all in the circle

$$S = \{(z, 1) \,/\, |z| = 3\} \subset \mathbb{C} \times \mathbb{R}$$

and that

$$F(S) \subset T \bigcup u([-1,1] \times [-\varepsilon, 2\pi - 2\varepsilon]) \bigcup (H \circ u)([-1,1] \times [-\varepsilon, 2\pi - 2\varepsilon])$$

where $T \subset E|_{p(S) \setminus \overset{\circ}{D}}$ is a smooth S^1−bundle over $p(S) \setminus \overset{\circ}{D}$. Hence, we may assume both $T \bigcup u([-1,1] \times [-\varepsilon, 2\pi - 2\varepsilon])$ and $T \bigcup (H \circ u)([-1,1] \times [-\varepsilon, 2\pi - 2\varepsilon])$ are smooth connected surfaces with corners on the boundaries (topologically each of them is a Klein bottle with an open disk deleted).

Notice that for $k = 0, 2, \cdots, 2b-2$, there is a unique curve $\widetilde{c}_k$ in $F(S) \cap T$ from

$$(H \circ u)(-1, 2\pi - 3\varepsilon - \frac{2\pi k}{a}) \text{ to } u(1, \frac{2\pi}{a}(k+1))$$

such that

$$\widetilde{c}_k \cap \widetilde{c_{k'}} = \emptyset, \text{ if } k \neq k'.$$

Denote by $\widetilde{L}_k$ the piece of L_k corresponding to either $t \in [-1, 0]$ or $t \in [0, 1]$ according to k being even or odd, and let

$$\begin{aligned} c''_k &= (H \circ u)(\widetilde{L}_k), && \text{if } k \text{ is even} \\ c'_k &= u(\widetilde{L}_k), && \text{if } k \text{ is odd.} \end{aligned}$$

Let

$$c_k = c'_{k+1} \wedge \widetilde{c}_k \wedge c''_k, \quad k = 0, 2, \cdots, 2b-2$$

where $\wedge$ means the product of curves.

If

$$W = T \bigcup u([0,1] \times [-\varepsilon, 2\pi - 2\varepsilon]) \bigcup (H \circ u)([-1, 0] \times [-\varepsilon, 2\pi - 2\varepsilon])$$

then W is a surface with corners on the boundary (topologically, it is an annulus for any choice of u) such that

$$F(\widetilde{M}) \cap W \supset \cup_{k=0}^{b-1} c_{2k}.$$

Now c_0 and c_2 bound a Whitney disk in W, so we may eliminate the intersection point of c''_0 and c''_2, and that of c'_1 and c'_3 by the Whitney trick. Then the immersion F is regularly homotopic to an immersion $F^{(1)}$ such that $F^{(1)}(\widetilde{M}) \cap W$ is obtained from $F(\widetilde{M}) \cap W$ by deleting c_0 and adding $c_0^{(1)}$, where $c_0^{(1)} \cap c_0 =$ the pieces of c_0 corresponding to $t \in [t_{0,2} + \delta_{0,2}, 0]$ and $t \in [0, t_{1,3} + \delta_{1,3}]$ for some $\delta_{0,2}, \delta_{1,3} > 0$, and with $c_0^{(1)} \setminus c_0$ a curve between c_2 and c_4.

Now $c_0^{(1)}$ and c_4 bound a Whitney disk, and we may get an immersion $F^{(2)}$ regularly homotopic to $F^{(1)}$ by the Whitney trick such that $F^{(2)} \cap W$ is obtained from $F(\widetilde{M}) \cap W$ by deleting $c_0^{(1)}$ and adding $c_0^{(2)}$, where $c_0^{(1)} \cap c_0^{(2)} =$ the pieces of $c_0^{(1)}$ corresponding to $t \in [t_{2,4} + \delta_{2,4}, 0]$ and $t \in [0, t_{3,5} + \delta_{3,5}]$ for some $\delta_{2,4}, \delta_{3,5} > 0$, and with $c_0^{(2)} \setminus c_0^{(1)}$ a curve between c_4 and c_6. Continuing in this way, we may get an immersion $F_1 = F^{(b-1)}$ such that the $b-1$ pairs of intersection points of F contributed by $c_0 \cap c_{2k}, k = 1, 2, \cdots, b-1$, are all eliminated.

Doing the same thing for F_1 to eliminate the $b-2$ pairs of intersection points of F_1 contributed by $c_2 \cap c_{2k}, k = 2, 3, \cdots, b-1$, we will get an immersion F_2. Continuing in this way, we get at last an immersion F_{b-1} with all pairs of intersection points contributed by

$$c_{2k} \cap c_{2k'}, \quad 0 \leq k < k' \leq b-1$$

eliminated. The number of the pairs is $\frac{1}{2}b(b-1)$, and so F_{b-1} has

$$\frac{1}{2}(a-b)(a-b-1)-\frac{1}{2}b(b-1)$$

intersection points of positive sign. Eliminating these intersection points by surgeries, we get an embedding of a new surface $\widetilde{M}'$ representing $ax-by$ with genus

$$a(g-1)+1+\frac{1}{2}(a-b)(a-b-1)-\frac{1}{2}b(b-1)$$

which is the minimal genus by the generalized adjunction inequality. This completes the proof of Theorem 2.

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Institute of System Science, Academia Sinica, Beijing 100080, P. R. CHINA
E-mail address: libh@iss06.iss.ac.cn

Department of Mathematics, Yale University, New Haven, CT 06520, USA
E-mail address: tli@math.yale.edu