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RADIAL VARIATION OF BLOCH FUNCTIONS

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0. The result

In 1971 J. M. Anderson [A] conjectured that for any conformal map φ in the unit disc there exists $\beta, 0 \le \beta \le 2\pi$ such that

$$\int_0^1 |\varphi''(re^{i\beta})|dr < \infty.$$

More recently this problem has been posed in the works of N. Makarov [M], Ch. Pommerenke [P] and D. Gnuschke - Ch. Pommerenke [G-P]. The purpose of the present note is to prove Anderson's conjecture. This will be done by showing the following theorem about the associated Bloch function $b = \log|\varphi'|$.

Theorem 1. *There exists $\beta, 0 \le \beta \le 2\pi$ such that*

$$b(re^{i\beta}) \le -\delta \int_0^r |\nabla b(\rho e^{i\beta})|d\rho + \frac{1}{\delta}, \text{ for } 0 < r < 1,$$

where $\delta > 0$ is independent of $r < 1$.

The proof of Theorem 1 is given in section 3 where we also discuss how Anderson's conjecture follows.

1. Preliminary inequalities

In this section we recall three estimates due to J. Bourgain, Ch. Pommerenke and A. Beurling respectively. In section 2 the construction of stopping time Lipschitz domains is based on Pommerenke's inequality. In section 3 the selection of good directions $e^{i\beta}$ is based on the result of J. Bourgain and estimates for harmonic measure due to A. Beurling.

We first discuss Bourgain's inequality from [B]. For $e^{i\alpha} \in \mathbb{T}$ we let Γ_α be the collection of curves γ which admit the following parametrization. For $0 < r < 1$, $\gamma(r) = re^{i\alpha}e^{i\theta(r)}$ where $|\theta(r)| < C(1-r)$ and $|\theta'(r)| < C$. We fix a non-negative, harmonic function h in $\mathbb{D}$ and we let K be an interval in $\mathbb{T}$. Then the following result was proven in [B].

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Theorem 2. *There exists $e^{i\alpha} \in K$ so that for each curve $\gamma \in \Gamma_\alpha$,*

$$\delta_0 \int_\gamma |\nabla h|(\zeta)|d\zeta| \le h(0),$$

where $\delta_0 > 0$ depends only on C and $|K|$.

The Bloch space B consists of those harmonic functions $b : \mathbb{D} \to \mathbb{R}$ for which $||b||_B = \sup_{z\in\mathbb{D}} |\nabla b(z)|(1 - |z|)$ is finite. Pommerenke's theorem (see [P, p. 78]) is the following.

Theorem 3. *Let $b : \mathbb{D} \to \mathbb{R}$ be in the Bloch space B and $||b||_B \le 1$. Let $I \subseteq \mathbb{T}$ be an interval. Then there exists $e^{i\alpha} \in I$ so that for each $z \in \{re^{i\alpha} : 0 < r < 1\}$, the estimate $|b(z) - b(0)| \le 22|I|^{-1}$ holds.*

For a conformal map $\varphi : \mathbb{D} \to \Omega$ the function $b = \log|\varphi'|$ belongs to the Bloch space B with Bloch norm ≤ 6. This is a consequence of classical distortion theorems. (See [P].) Bloch functions are related to $|\varphi''|$ as follows. Let $g = \log \varphi'$, $g' = \varphi''/\varphi'$, hence $|\varphi''| = |g'||\varphi'|$. Let $b = \log|\varphi'|$, then b is the real part of g. Hence by the Cauchy-Riemann equations $2|g'| = |\nabla b|$, and

$$2|\varphi''| = |\nabla b||\varphi'| = |\nabla b|e^b.$$

This identity provides a link between estimates for the variation of Bloch functions and estimates for the L^1 norm of φ''. This has been exploited in [G-P].

Next we recall a minorization for harmonic measure due to A. Beurling. Fix $e \in \mathbb{D}$ and $0 < \delta < 1$. Then we define the Stolz angle $C(e, \delta)$ to be the convex hull of $\{z \in \mathbb{D} : |z| < (1 - \delta)|e|\}$ and e. If $1/4 < \delta < 1$, then we write simply $C(e)$ for $C(e, \delta)$. Let $W \subseteq \mathbb{D}$ be a Lipschitz domain, $e \in \partial W$, and suppose that the cone $C(e, \delta/2)$ is contained in W. Let D be a connected subset of ∂W, so that $\mathrm{dist}(D, e) < rC_1$ and $\mathrm{diam}(D) > rC_2$. Then the following estimate holds.

Theorem 4. *For each $z \in C(e, \delta)$, with $|e - z| < r$, the harmonic measure satisfies $\omega(z, D, W) > \eta_0$, where η_0 depends only on C_1, C_2, and δ.*

Below we will be concerned with the question whether a given curve in the unit disc remains in a fixed Stolz angle or not. To decide this the following criterion which is a folk theorem uses lower estimates for harmonic measure. We fix $e^{i\alpha} \in \mathbb{T}$, and let $I_\alpha = \{e^{i\theta} : \alpha < \theta < \alpha + \pi/2\}$ and $J_\alpha = \{e^{i\theta} : \alpha - \pi/2 < \theta < \alpha\}$. For $z \in \mathbb{D}$ we denote by $\omega(z, I)$ the harmonic measure of I with respect to $\mathbb{D}$ evaluated at $z \in \mathbb{D}$.

Theorem 5. *For a path Γ in $\mathbb{D}$ the following conditions are equivalent.*

(i) *There exists δ such that $\Gamma \subset C(e^{i\alpha}, \delta)$.*
(ii) *There exists $\eta > 0$ such that for each $z \in \Gamma$, there hold the lower estimates for harmonic measure $\omega(z, I_\alpha) > \eta$ and $\omega(z, J_\alpha) > \eta$.*

We combine this criterion and Beurling's estimates. Let $f : \mathbb{D} \to W$ be the conformal map from the unit disc to the Lipschitz domain W. Fix $e \in \partial W$, and let $e^{i\alpha} = f^{-1}(e)$. Suppose that $C(e, \delta/2) \subset W$, and let Γ be a path in $C(e, \delta)$. Then the following holds.

Theorem 6. *There exists $\delta_0 > 0$, depending only on δ, such that $f^{-1}(\Gamma)$ is contained in $C(e^{i\alpha}, \delta_0)$.*

2. Stopping time Lipschitz domains

In this section we define the stopping time Lipschitz domain $W(z_0)$, and we collect some of its basic properties. We let $b : \mathbb{D} \to \mathbb{R}$ be a Bloch function, and we fix it throughout this section. We also fix $z \in \mathbb{D}$ with $|z| > 15/16$. First we construct an auxiliary domain $V(z)$.

Let $I = \{\zeta \in \mathbb{T} : |z - \zeta| \leq 8(1 - |z|)\}$. The intervals I_1, I_2 have length $1 - |z|$, they are attached to the left respectively right endpoint of I. Let $r_1 = 2|z| - 1$, and let $S_0 = \{w \in \mathbb{D} : |w| = r_1 \text{ and } |w - z| \leq 1 - |z|\}$. The left respectively right endpoint of S_0 are s_1 resp. s_0. By Theorem 3 there are line segments S_i connecting s_i to I_i such that

$$|b(w) - b(s_i)| \leq 25||b||/\omega(s_i, I_i),$$

whenever $w \in S_i$, $i \in \{1, 2\}$. We let $V(z)$ be the domain in $\mathbb{D}$ which is bounded by $S_0 \cup S_1 \cup S_2$.

The domain $V(z)$ satisfies

$$\sup |b(w) - b(z)| \leq ||b||_B/\omega_0,$$

where the supremum is taken over $S_0 \cup S_1 \cup S_2$, and where $\omega_0 = \min\{\omega(s_1, I_1), \omega(s_2, I_2)\}/30$. The boundary of $V(z)$ intersects $\mathbb{T}$ in an interval J. For the harmonic measure of J we have the lower estimate $w(z, J, V(z)) \geq 1/3$. Moreover we observe the following.

(2.1) The angle formed by S_1 and J, resp. S_2 and J is less than $\pi/5$.

Next fix $z_0 \in \mathbb{D}$ with $|z_0| \geq 15/16$, and $M \in \mathbb{N}$ large enough. We now turn to the construction of $W(z_0)$. Using a stopping time $W(z_0)$ is obtained as a subdomain of $V(z_0)$. For an interval $I \subseteq \mathbb{T}$ we let $T(I) = \{w \in \mathbb{D} : w/|w| \in I \text{ and } |I|/2 < 1 - |w| < |I|\}$. Now we let $\{I_i : i \in \mathbb{N}\}$ be the collection of maximal dyadic intervals with the property that there exists $z_i \in T(I_i)$ with

$$b(z_i) - b(z_0) \leq -M. \tag{2.2}$$

The stopping time Lipschitz domain is defined as

$$W(z_0) = V(z_0) \setminus \bigcup_{i=1}^{\infty} V(z_i).$$

For $z_0 = 0$ we define the points z_i using the stopping time condition (2.2). By (2.4), each point z_i satisfying (2.2) must have modulus $\geq 15/16$. Hence $V(z_i)$ is well defined and we put $W(0) = \mathbb{D} \setminus \bigcup_{i=1}^{\infty} V(z_i)$.

The following list of remarks collects the basic properties of $W(z_0)$.

Remarks.

1. It follows from (2.1) that $W(z_0)$ is a Lipschitz domain with starcenter z_0. The Lipschitz constant is independent of z_0.

2. Suppose that $w \in \partial V(z_i)$ for some $i > 0$, and suppose also that $|w| < 1$. Then for M large enough we have that

$$-2M < b(w) - b(z_0) < -M/2. \tag{2.3}$$

3. For M large enough we have $V(z_i) \subset V(z_0)$. Moreover the estimate $b(z_i) - b(z_0) < -M$ implies that

$$1 - |z_i| \leq (1 - |z_0|)/16. \tag{2.4}$$

4. Let $I(z_0) = \{\zeta \in \mathbb{T} : |\zeta - z_0/|z_0|| \leq (1 - |z_0|)/4\}$. Pick $\zeta_0 \in I(z_0)$. Let R be the ray connecting 0 to ζ_0, and let $L = R \setminus V(z_0)$. Then L is contained in the Stolz angle $C(z_0)$.

5. Let $K(z_0)$ be the convex hull of z_0 and $I(z_0)$, and let $D = K(z_0) \cap \partial W(z_0)$, then by Theorem 4, $\omega(z_0, D, W(z_0)) > \omega_0$, where $\omega_0 > 0$ is a universal constant.

3. The selection of a good ray

In this section we fix a conformal map φ in the unit disc, and we will select $e^{i\beta} \in \mathbb{T}$ such that

$$\int_0^1 |\varphi''|(re^{i\beta})dr < \infty. \tag{3.1}$$

Let $b = \log |\varphi'|$ be the Bloch function associated to the Riemann map φ. The ray $L = \{re^{i\beta} : 0 < r < 1\}$ will be chosen so that on L there are points Q_k satisfying

$$b(Q_k) - b(Q_{k-1}) < -M/3, \tag{3.2}$$

and

$$\int_{l_k} |\nabla b(\zeta)| d|\zeta| \leq C_0 M, \tag{3.3}$$

where l_k is the line segment connecting Q_k and Q_{k-1}. Note that (3.2) and (3.3) imply Theorem 1 and (3.1). Indeed, summing (3.2) gives

$$b(Q_k) < -kM/3. \tag{3.4}$$

Using (3.3) we obtain from (3.4),

$$b(\zeta) < -kM/3 + C_0 M, \text{ for } \zeta \in l_k. \tag{3.5}$$

Clearly (3.3) and (3.5) imply the conclusion in Theorem 1. Now recall that $2|\varphi''| = |\nabla b|e^b$, together with (3.3) and (3.5) this identity gives the following estimate.

$$\int_0^1 |\varphi''(re^{i\beta})|dr = \int_0^1 |\nabla b(re^{i\beta})|e^{b(re^{i\beta})}dr \ \le \sum_{k=0}^{\infty}\int_{l_k} |\nabla b|(\zeta)e^{b(\zeta)}|d\zeta|$$
$$\le e^{C_0 M}\sum_{k=0}^{\infty} e^{-kM/3}\int_{l_k}|\nabla b(\zeta)||d\zeta| \ \le C_0 e^{C_0 M}M.$$

Now we begin the proof of Theorem 1. First we give an inductive definition of an auxiliary sequence of points $e_k \in \mathbb{D}$. Their limit will be the point $e^{i\beta}$ satisfying the required properties (3.2) and (3.3).

Fix $M \in \mathbb{N}$ large enough and assume $b(0) = 0$. Let $W(0)$ be the stopping time Lipschitz domain constructed in section 2. Clearly we have $0 \in W(0)$. Let $f : \mathbb{D} \to W(0)$ be the Riemann map normalized such that $f(0) = 0$.

We use the conformal map f to pull back b from $W(0)$ to the unit disc $\mathbb{D}$. The composition $h = b \circ f$ is harmonic and satisfies $h > -2M$ in $\mathbb{D}$, and $h(0) = 0$. By Bourgain's theorem there exists $e^{i\alpha} \in \mathbb{T}$ such that

$$\delta_0 \int_\gamma |\nabla h(\zeta)||d\zeta| \le M, \text{ for } \gamma \in \Gamma_\alpha.$$

Now we let $e_1 = f(e^{i\alpha})$. We distinguish between the cases $|e_1| = 1$ and $|e_1| < 1$. If we have $|e_1| < 1$, then by (2.3) we have $b(e_1) < -M/2$. If $|e_1| = 1$, then we stop the construction of the points $\{e_k\}$.

Next we give the induction step in the construction of the points $\{e_k\}$. We are given $e_1, \ldots, e_l$, points in $\mathbb{D}$, so that $I(e_{k+1}) \subset I(e_k)$ and $|I(e_{k+1})| < |I(e_k)|/4$, for $1 \le k \le l-1$. Let $D = \partial W(e_l) \cap K(e_l)$. D is connected, and by Remark 5 in Section 2, for the harmonic measure we have the estimate $w(e_l, D, W(e_l)) \ge \omega_0$. Let $f : \mathbb{D} \to W(e_l)$ with $f(0) = e_l$ be the Riemann map for the domain $W(e_l)$, then $K = f^{-1}(D)$ is an interval and $|K| \ge \omega_0$.

Again, the conformal map f is used to pull back b from $W(e_l)$ to the unit disc. The composition $h = b \circ f - b \circ f(0)$ is harmonic and satisfies $h > -2M$ in $\mathbb{D}$ and $h(0) = 0$. By Bourgain's theorem there exists $e^{i\alpha} \in K$ such that

$$\delta_0 \int_\gamma |\nabla h(\zeta)||d\zeta| \le M, \text{ for } \gamma \in \Gamma_\alpha. \tag{3.6}$$

Let $e_{l+1} = f(e^{i\alpha})$. As e_{l+1} is a point in $D = \partial W(e_l) \cap K(e_l)$ it follows from (2.4) that

$$I(e_{l+1}) \subset I(e_l) \text{ and } |I(e_{l+1})| < |I(e_l)|/4. \tag{3.7}$$

If $|e_{l+1}| < 1$, then by (2.3),

$$b(e_{l+1}) - b(e_l) < -M/2. \tag{3.8}$$

Otherwise, i.e., when $|e_{l+1}| = 1$ we stop the construction of the points $\{e_k\}$.

Having completed the construction of the points $\{e_k\}$ we let $e^{i\beta} = \lim e_k$. By (3.7) the limit exists and lies in $\mathbb{T}$. Moreover it follows that $e^{i\beta} = \bigcap I(e_k)$. We let L be the ray connecting 0 and $e^{i\beta}$. Notice that L intersects the boundary $\partial W(e_k)$ at least once and at most twice. We let Q_k be the point in $L \cap \partial W(e_k)$ that has the smaller modulus. If we let l_k be the line segment connecting Q_k and Q_{k+1} then clearly l_k coincides with $L \cap \{W(e_k) \setminus V(e_{k+1})\}$. Moreover by Remark 4,

$$l_k \text{ is a subset of } C(e_{k+1}). \tag{3.9}$$

Now we wish to verify (3.2) and (3.3). We fix $k \in \mathbb{N}$ and let $f : \mathbb{D} \to W(e_k)$ be the conformal map that was used in the definition of $e_{k+1} \in \partial W(e_k)$. Then we had determined α by $f(e^{i\alpha}) = e_{k+1}$. Now (3.9) and Theorem 6 imply that the curve

$$f^{-1}(l_k) \text{ is contained in } C(e^{i\alpha}).$$

Moreover by the distortion theorem, the curve $f^{-1}(l_k)$ can be decomposed into say $F_1 \cup \cdots \cup F_{m_0}$, with a universal m_0, so that each of the F_i is contained in a curve $\gamma \in \Gamma_\alpha$. By (3.6) this implies that

$$\int_{f^{-1}(l_k)} |\nabla h(\zeta)||d\zeta| \le C_0 M,$$

where $h = b \circ f - b \circ f(e_k)$. A change of variables gives the desired

$$\int_{l_k} |\nabla b(\zeta)||d\zeta| \le C_0 M.$$

To obtain (3.2) from (3.8) we observe that L hits the boundary $\partial W(e_k)$ near e_k. More precisely, the hyperbolic distance between Q_k and e_k is bounded independent of k. Hence from (3.8) we obtain

$$b(Q_{k+1}) - b(Q_k) < -M/3,$$

provided that M is large enough.

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