Mathematical Research Letters **4**, 445–458 (1997)

ON MELNIKOV'S PERSISTENCY PROBLEM

J. Bourgain

1. Introduction

We consider Melnikov's persistency problem for b-dimensional tori in $\mathbb{R}^{2b} \times \mathbb{R}^{2r}$-phase space for a real analytic Hamiltonian H of the form

$$\begin{aligned} H = H(I,\theta,y) &= H(I_1,\dots,I_b,\theta_1,\dots,\theta_b,y_1,\dots,y_r) \\ &= \langle \lambda_0, I\rangle + \sum_{s=1}^{r} \mu_s |y_s|^2 + |I|^2 + \varepsilon H_1(I,\theta,y) \end{aligned} \tag{1.1}$$

(the last term in perturbative), as considered in [E], [Kuk], [Pos]. Here $I = (I_1,\dots,I_b)$, $\theta = (\theta_1,\dots,\theta_b)$ are action-angle variables for the "tangential" part of phase space and $y = (y_1,\dots,y_r)$ are the "normal" coordinates.

Assume λ_0 a diophantine vector and the normal frequencies satisfying a non-resonance condition

$$\langle \lambda_0, k\rangle - \mu_s \neq 0 \qquad (k \in \mathbb{Z}^b,\ s = 1,\dots,r). \tag{1.2}$$

This last condition is weaker than considered in [E], since we do not require the second assumption

$$\langle \lambda_0, k\rangle + \mu_s - \mu_{s'} \neq 0 \qquad (s \neq s') \tag{1.3}$$

excluding multiplicities in the normal frequencies.

Our aim is to show the persistency of the invariant torus $\mathbb{T}^b \times \{0\} \times \{0\}$ with perturbed frequency vector λ that can be taken of the form

$$\lambda = t\lambda_0,\ t \in \mathbb{R}, t \approx 1 \tag{1.4}$$

thus a multiple of the given λ_0. Thus

Received Janurary 30, 1997.

Theorem. *Let $H(I, \theta, y)$ be the Hamiltonian given by* (1.1)*, where we assume condition* (1.2) *and the perturbation H_1 real analytic in a neighborhood of 0 in phase space. Then, for a fixed small $\varepsilon > 0$ and t taken in a set of positive measure, there is a perturbed torus with frequency vector λ satisfying* (1.4)*, parametrized as*

$$I = I(t) \qquad \theta = \lambda t + \theta'(t) \qquad y = y(t) \tag{1.5}$$

with I, θ', y quasi-periodic with frequency vector λ and size $0(\varepsilon^{1/2})$ - say in appropriate real analytic function space norm.

This result was obtained in [E] under the stronger nonresonance hypothesis (1.2)-(1.3). Assuming (1.2), the result without requiring λ to be of the specific form (1.4), follows from [Bo1] (containing also certain infinite dimensional phase space settings applicable in the PDE-context. Achieving the extra property (1.4) is a main point in this Note. In order to obtain this additional property, we will combine the KAM-method (as explained for instance in [Kuk]) with the Nash-Moser type methods from [C-W], [Bo1, 2, 3]. The main point in this approach is that due to a weaker nonresonance hypothesis, we do not eliminate the y-quadratic perturbative part of the Hamiltonian and therefore, in the KAM scheme, at each step a non-diagonal linear operator needs to be inverted. This operator is the same as encountered in the direct Liapounov-Schmidt approach used in [C-W], [Bo], after expressing the problem as a lattice problem (passing to Fourier transform) and linearization. In this context, there is the extra difficulty that we do not allow $\lambda = (\lambda_1, \dots, \lambda_b)$ to vary as a full b-dimensional parameter since there is the restriction (1.4) $\lambda = t\lambda_0$ with λ_0 fixed. It turns out however that, at least with finitely many normal modes, the analysis of the inversion of the linearized operator may be performed as well in this situation. In fact, establishing the separation properties of "singular" submatrices of given size in the multi-scale analysis will only use the diophantine property of λ_0 and the role of the t-parameter appears only at the final stage.

It seems unclear how this result with the restriction (1.4) may be generalized to an infinite dimensional phase space setting (appearing in the PDE applications, cf. [Kuk], [Bo1, 2, 3]) where the frequency vector λ is allowed to vary as a full parameter.

Finally, as observed by J. Moser, in contrast to the "standard" KAM problem where $r = 0$, one may not expect in the Melnikov problem to specify the perturbed frequency vector independently of the perturbation, unless there are additional parameters (hence $> b$) in the problem (the dilation factor t plays here the role of substitute). A simple example illustrating this fact will be described in the last section of the paper.

What follows is a very schematic, sometimes only formal, overview of the proof of the result stated in the beginning of this section.

2. Melnikov problem (scheme)

Write the Hamiltonian as

$$H = \sum_{j=1}^{b} \lambda_j I_j \tag{2.1}$$

$$+\sum_{s=1}^{r} \mu_s |y_s|^2 \tag{2.2}$$

$$+H_1(I,\theta) \tag{2.3}$$

$$+Re\Big[\sum_{k\in\mathbb{Z}^b,\, s=1,\dots,r} \hat{a}_s(k)\; e^{ik.\theta} y_s\Big] \tag{2.4}$$

$$+Re\Big[\sum_{k\in\mathbb{Z}^b,\, s,\, s'=1,\dots,r} \widehat{b}_{s,s'}(k) e^{ik.\theta} y_s.y_{s'} + \widehat{c}_{s,s'}(k)\, e^{ik.\theta} y_s \overline{y}_{s'}\Big] \tag{2.5}$$

$$+0(|y|.|I|) \tag{2.6}$$

$$+0(|y|^3), \tag{2.7}$$

where $I = (I_1, \dots, I_b)$, $\theta = (\theta_1, \dots, \theta_b)$ are action-angle variables in the tangential part of phase space, $y_s = p_s + iq_s (s = 1, \dots, r)$ with (p_s, q_s) canonical coordinates in the normal part of phase space and with I and y restricted to a neighborhood of 0. The vector $\lambda = (\lambda_1, \dots, \lambda_b)$ is considered as parameter (which may be extracted by amplitude-frequency modulation from the "twist" term $|I|^2 = \sum_{j=1}^{b} I_j^2$ in (1.1)). The terms (2.3)-(2.7) are perturbative. Assuming in (2.5)

$$\widehat{b}_{s,s'}(k) = \widehat{b}_{s',s}(k) \tag{2.8}$$

and

$$\widehat{c}_{s,s'}(k) = \overline{\widehat{c}_{s',s}(-k)}. \tag{2.9}$$

The unperturbed torus corresponds to $\mathbb{T}^b \times \{0\} \times \{0\} \subset \mathbb{R}^{2b} \times \mathbb{R}^{2r}$. Our purpose in order to solve the Melnikov problem is to replace

$$\sum \lambda_j I_j + H_1(I,\theta) \tag{2.10}$$

by an integrable Hamiltonian

$$\sum \lambda_j I_j + H_1'(I) \tag{2.11}$$

and make the y-linear term (2.4) disappear. The first step is achieved by the standard KAM procedure. The second will be achieved from canonical transformations involving y-linear Hamiltonians only. Thus we do not eliminate the

y-quadratic part (2.5) of the Hamiltonian as in [Kuk]. The advantage of this is that the second nonresonance condition (1.3)

$$\mu_s - \mu_{s'} + \langle k, \lambda \rangle \neq 0 \tag{2.12}$$

is unnecessary. The disadvantage is that now solving the homological equations requires to invert a linear operator that is non-diagonal, due to the presence of (2.5). The small divisor problems are in this situation technically more complicated and the argument will be sketched at the end (this seems to be the main difficulty of the present scheme).

3. Melnikov problem (Homological equations)

Consider the Hamiltonian

$$H = \sum_{j=1}^{b} \lambda_j I_j + \sum_{s=1}^{r} \mu_s |y_s|^2 \tag{3.1}$$

$$+Re\Bigg[\sum_{k\in\mathbb{Z}^b, s=1,\dots,r} \widehat{a}_s(k)\, e^{ik.\theta} y_s \Bigg] \tag{3.2}$$

$$+Re\Bigg[\sum_{k\in\mathbb{Z}^b, s,s'=1,\dots,r} \widehat{b}_{s,s'}(k)\, e^{ik.\theta} y_s.y_{s'} \Bigg] + \Bigg[\sum_{k\in\mathbb{Z}^b} \widehat{c}_{s,s'}(k)\, e^{ik.\theta} y_s.\overline{y}_{s'} \Bigg] \tag{3.3}$$

$$+0(|I|^2 + |y|.|I| + |y|^3) \tag{3.4}$$

satisfying (2.8), 2.9). Our aim is to reduce (3.2) by a sequence of approximate canonical transformations. These transformations will however also introduce in particular expressions $H'(I, \theta)$ and at each step the integrable form in the tangential coordinates will have to be recovered by an extra transformation.

Assume (3.2) of "size" δ in an appropriate real analytic function space norm expressed by some exponentially weighted sequence norm of the coefficients $\{\widehat{a}_s(k)\}$. Compose H with a canonical transformation with Hamiltonian

$$F = Re\Big(\sum \widehat{F}_s(k)\, e^{ik.\theta} y_s \Big) \tag{3.5}$$

of size δ^{1-} and to be specified. The new Hamiltonian $\tilde{H}$ is then given by

$$\tilde{H} = H + \{H, F\} + \frac{1}{2!}\{\{H, F\}, F\} + \cdots \tag{3.6}$$

thus adding to H the Poisson bracket $\{H, F\}$ as first order correction. From

(3.1)-(3.4) one finds

$$\{H,F\} = \sum_{j=1}^{b} \frac{\partial H}{\partial I_j}\frac{\partial F}{\partial \theta_j} + i\sum_{s=1}^{r}\left(\frac{\partial H}{\partial \overline{y}_s}\frac{\partial F}{\partial y_s} - \frac{\partial H}{\partial y_s}\frac{\partial F}{\partial \overline{y}_s}\right)$$

$$= Re\left(i\sum_{s,k}\widehat{F}_s(k)\langle k,\lambda\rangle\, e^{ik\theta}y_s\right) \tag{3.7}$$

$$+\,0(|y|^2 + |y|\,|I|) \tag{3.8}$$

$$+\,Im\left(\sum_{s,k}\mu_s\widehat{F}_s(k)\,e^{ik\theta}y_s\right) \tag{3.9}$$

$$+\,\{(3.2),F\} \tag{3.10}$$

$$-\,Im\left[\sum_{s,s',k,k'}\widehat{b}_{ss'}(k)\overline{\widehat{F}_s(k')}\,y_{s'}\,e^{i(k-k').\theta}\right] \tag{3.11}$$

$$+\,Im\left[\sum_{s,s',k,k'}\overline{\widehat{c}_{s,s'}(k)}\,\widehat{F}_s(k')\,y_{s'}\,e^{i(k'-k).\theta}\right] \tag{3.12}$$

$$+\,0(|I|) \tag{3.13}$$

$$+\,0(|y|^2) \tag{3.14}$$

where in particular (3.10) is of size δ^{2-} and depends only on θ and (3.13), (3.14) are of size δ^{1-}. Our first aim is to specify $\widehat{F}_s(k)$ such that

$$(3.2) + (3.7) + (3.9) + (3.11) + (3.12) = 0. \tag{3.15}$$

This will reduce the y-linear term in $\tilde{H}$ to size δ^{2-}.

(3.15) clearly amounts to

$$\begin{aligned}\widehat{a}_s(k) &= i(\langle k,\lambda\rangle - \mu_s)\widehat{F}_s(k) + i\sum_{k',s'}\widehat{b}_{s',s}(k-k')\overline{\widehat{F}_{s'}(-k')}\\ &\quad - i\sum_{k',s'}\widehat{c}_{s,s'}(k-k')\widehat{F}_{s'}(k').\end{aligned} \tag{3.16}$$

Define

$$\widehat{G}_{s'}(k') = \overline{\widehat{F}_{s'}(-k')}. \tag{3.17}$$

Then, taking also the conjugate of (3.16), (3.16) is equivalent to the linear system

$$\frac{1}{i}\widehat{a}_s(k) = (\langle k,\lambda\rangle - \mu_s)\widehat{F}_s(k) - \sum_{s',k'}\widehat{c}_{s,s'}(k-k')\widehat{F}_{s'}(k') + \sum_{s',k'}\widehat{b}_{s',\,s}(k-k')\widehat{G}_{s'}(k')$$

$$\begin{aligned}-\frac{1}{i}\,\overline{\widehat{a}_s(-k)} &= \sum_{s',k'}\overline{\widehat{b}_{s',\,s}(k'-k)}\widehat{F}_{s'}(k') + (-\langle k,\lambda\rangle - \mu_s)\widehat{G}_s(k)\\ &\quad - \sum_{s',k'}\widehat{c}_{s',s}(k-k')\widehat{G}_{s'}(k')\end{aligned}$$

$$= T \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix} \tag{3.18}$$

where

$$T = D + T_1 \tag{3.19}$$

$$D = \begin{pmatrix} (\langle k,\lambda\rangle - \mu_s)\mathbb{1} & 0 \\ 0 & (-\langle k,\lambda\rangle - \mu_s)\mathbb{1} \end{pmatrix} \text{ diagonal} \tag{3.20}$$

and

$$T_1 = \begin{pmatrix} -S_{\widehat{c}_{s,s'}} & S_{\widehat{b}_{s',s}} \\ S^*_{\widehat{b}_{s,s'}} & -S_{\widehat{c}_{s',s}} \end{pmatrix} \tag{3.21}$$

is selfadjoint. Thus T is selfadjoint and preserves the structure (3.17).

Estimates on the inverse of (restrictions of) T will be discussed in the next section (this is the main point of the present analysis) and, essentially, these are similar to the diagonal case $T = D$ from the usual scheme (up to the technicalities required to establish them).

Thus the left member of (3.18) is of size δ (for the considered exponential weight on the sequence $\{\widehat{a}_s(k)\}$) and solving (3.18) in $(\widehat{F}, \widehat{G})$ yields $\|\widehat{F}\| < \delta^{1-}$ (this estimate involves inverting T restricted to $|k| < (\log \frac{1}{\delta})^2$ say and a slight reduction of the convergence radius of the analytic weight, as usual in such procedure).

At this stage, we have thus reduced the y-linear term (3.2) from size δ to size δ^{2-} but the integrability of the part of the Hamiltonian depending only on (I, θ) got lost because of the (3.10) and (3.13) terms. In order to recover this integrability, we perform another canonical transformation in (I, θ). Some care is needed regarding the effect for the y-linear term. Recall that (3.10) is only θ-dependent and of size δ^{2-} while (3.13) is (I, θ)-dependent at least linear in I and of size δ^{1-}. The second transformation is thus defined from a Hamiltonian of the form

$$K(I, \theta) = K_1(\theta) + K_2(I, \theta) \tag{3.22}$$

where K_1 is of size δ^{2-} and K_2 of size δ^{1-} and at least linear in I. Hence the effect of K_1 is harmless while one observes that the Poisson bracket of K_2 and (3.4) yields again terms of the form (3.4).

The conclusion is thus that by a symplectic transformation of size δ^{1-} the Hamiltonian H may be transformed in a Hamiltonian of the same form (3.1)-(3.4) where now (3.2) is reduced from size δ to size δ^{2-}. Iterating the procedure eventually permits to eliminate (3.2) by a convergent sequence of symplectic transformations and hence solve the persistency problem. The frequency vector

λ is taken of the form $\lambda = t\lambda_0$ where $t \approx 1$ is restricted to a set of positive measure, the restriction being such that the required bounds on the inverse of T are fulfilled (see section 4 (iii)). We assume a diophantine property

$$|\langle k, \lambda_0\rangle| \gtrsim (1+|k|)^{-C},\ k \in \mathbb{Z}^b\backslash\{0\} \tag{3.23}$$

and also a nonresonance condition

$$|\langle k, \lambda_0\rangle + \mu_s| \gtrsim (1+|k|)^{-C},\ k \in \mathbb{Z}^b,\ s = 1, \dots, r \tag{3.24}$$

and the perturbation sufficiently small. This enables us in particular to invert restrictions of T in the initial steps (depending on the size of the perturbation). Later on, we rely also on a small variation of the dilation parameter t around 1 in order to control the inverse operators for larger scales.

4. Inversion of the linearized operator

Let $T = D + T_1$, be as in section 3, thus

$$D = \langle k, \lambda\rangle \pm \mu_s,\ k \in \mathbb{Z}^b, s = 1, \dots, r \tag{4.0}$$

where $\lambda = (\lambda_1, \dots, \lambda_b)$ is a diophantine vector and $\mu_1, \dots, \mu_r \neq 0$; T_1 is stationary with respect to k, meaning that the matrix elements $T_1(k, k')$ only depend on the difference $k - k'$, with exponential off-diagonal decay. We will not rely on selfadjointness properties. In this brief exposition we will use ideas and methods from [Bo1, 2, 3] without expliciting all details. The main difficulty compared with [Bo1] is to establish separation properties with λ an essentially fixed (diophantine) vector rather than a b dimensional parameter.

4(i). The multiscale analysis

We outline the construction of the consecutive generations of singular sites and submatrices. Let T_1 be of size ε and take $\delta \gg \varepsilon$.(*) Define

$$P_0 = \{k \in \mathbb{Z}^b\} \qquad Q_0 = \Big\{k \in \mathbb{Z}^b \Big| \min_{1\le s\le r, \pm} |\langle \lambda, k\rangle \pm \mu_s| < \delta\Big\}. \tag{4.1}$$

Thus T_Λ^{-1} may be controlled by a standard Neumann series if $\Lambda \cap Q_0 = \phi$.

Next take a subset P_1 of Q_0 and associate to each $k \in P_1$ a box Ω_k^1 of size N_1 such that

$$\operatorname{dist}(\Omega_k^1, \Omega_{k'}^1) > 10N_1\ (k \neq k') \tag{4.2}$$

(*)The numbers $\delta_0, \delta_1, \dots$ should not be related to "δ" from previous section. The construction described here is an independent process.

$$Q_0 \subset \bigcup_{k \in P_1} \Omega_k^1. \tag{4.3}$$

Denote $\tilde{\Omega}_k^1$ the "doubling" of Ω_k^1. Then, as in [Bo1], the inverse of $T_{\tilde{\Omega}_k^1}$ may be controlled by the reciprocals of expressions $p(\langle k, \lambda \rangle)$, where p runs in a family $\mathcal{P}_1$ of polynomials with

$$\sum_{p \in \mathcal{P}_1} (\text{degree } p) \leq 2r. \tag{4.4}$$

Thus there are numbers

$$\sigma_1^1, \dots, \sigma_{2r}^1 \tag{4.5}$$

such that the inverse of each $T_{\tilde{\Omega}_k^1}$, $k \in \mathcal{P}_1$, is controlled by the reciprocal of

$$\min_{1 \leq s \leq 2r} |\langle \lambda, k \rangle - \sigma_s^1|. \tag{4.6}$$

Preceding construction requires δ sufficiently small depending on N_1, r and the diophantine properties of λ.

We then introduce the first generation of "singular" matrices corresponding to $Q_1 \subset P_1$ where

$$Q_1 = \{k \in P_1 \Big| \min_{1 \leq s \leq 2r} |\langle \lambda, k \rangle - \sigma_s^1| \leq \delta_1\} \tag{4.7}$$

with $\delta_1 \ll \delta$ to be specified. One may control T_Λ^{-1} for all sets $\Lambda \subset \mathbb{Z}^b$ such that $(*)$

$$k \in P_1,\, \Omega_k^1 \cap \Lambda \neq \phi \Rightarrow \tilde{\Omega}_k^1 \subset \Lambda \tag{4.8}$$

$$\Lambda \cap Q_1 = \phi. \tag{4.9}$$

One may indeed write $\Lambda = (\Lambda \backslash Q_0) \cup (\Lambda \cap Q_0)$ and cover $\Lambda \cap Q_0$ with $\bigcup_{k \in P_1 \backslash Q_1} \tilde{\Omega}_k^1 \subset \Lambda$ and apply Pöschel's Lemma (cf. [B2]) based on the resolvent identity.

Take again a subset P_2 of Q_1 and associate to each site $k \in P_2$ a box Ω_k^2 of size N_2 such that

$$k' \in P_1,\, \tilde{\Omega}_{k'}^1 \cap \Omega_k^2 \neq \phi \Rightarrow \tilde{\Omega}_{k'}^1 \subset \Omega_k^2 \tag{4.10}$$

and

$$\text{dist}(\Omega_k^2, \Omega_{k'}^2) > 10 N_2 \ (k \neq k') \tag{4.11}$$

$$Q_1 \subset \bigcup_{k\in P_2} \Omega_k^2. \tag{4.12}$$

Denote $\tilde\Omega_k^2$ a doubling of Ω_k^2 with the same property (4.10).

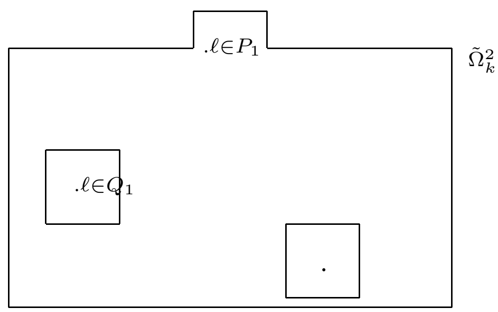
Our aim is again to control the inverses $T^{-1}_{\tilde\Omega_k^2}$, $k \in P_2$, by the reciprocal $p(\langle k,\lambda\rangle)$, $p\in\mathcal{P}_2$ a family of polynomials again satisfying (4.4)

$$\sum_{p\in\mathcal{P}_2} (\text{degree } p) \le 2r. \tag{4.13}$$

Observe that again there are $\le 2r$ types of regions $\tilde\Omega_k^2$ corresponding to a certain partition of (4.5). The construction of the polynomials at this stage will use methods form [Bo2, 3]. Write

$$\tilde\Omega_k^2 = \bigcup_{\ell\in Q_1\cap\tilde\Omega_k^2} \Omega_\ell^1 + \Lambda,\ \Lambda = \tilde\Omega_k^2 \backslash \bigcup_{\ell\in Q_1\cap\tilde\Omega_k^2} \Omega_\ell^1 \tag{4.14}$$

and observe that Λ satisfies (4.8), (4.9), hence T_Λ^{-1} is well-controlled



The inverse of $T_{\tilde\Omega_k^2}$ may then be controlled by the inverse of

$$\bigoplus_{\ell\in Q_1\cap\tilde\Omega_k^2} [T_{\Omega_\ell^1} - (R_{\Omega_\ell^1} T R_\Lambda) T_\Lambda^{-1} (R_\Lambda T^* R_{\Omega_\ell^1})] + \bigoplus_{\ell\neq\ell'} (R'_\ell \cdots R'_{\ell'}) \tag{4.15}$$

$$= \bigoplus_{\ell\in Q_1\cap\tilde\Omega_k^2} [T_{\Omega_\ell^1} - (R_{\Omega_\ell^1} T R_{\tilde\Omega_\ell^1\backslash\Omega_\ell^1}) T^{-1}_{\tilde\Omega_\ell^1\backslash\Omega_\ell^1} (R_{\tilde\Omega_\ell^1\backslash\Omega_\ell^1} T R_{\Omega_\ell^1})] + 0(e^{-c.N_1}) \tag{4.16}$$

taking separation properties and off-diagonal decay into account ($R.$ denoting the usual restriction operator). Taking next for each $\ell\in Q_1\cap\tilde\Omega_k^2$ the determinant of an appropriate submatrix (appearing at previous step when constructing $\mathcal{P}_1$), the inverse of (4.16) is controlled by the reciprocal of an expression of the form

$$\prod_{\ell\in Q_1\cap\tilde\Omega_k^2} \det[\cdots] + 0(e^{-cN_1}) \sim \prod_{\ell\in Q_1\cap\tilde\Omega_k^2} p_\ell(\langle \ell,\ \lambda\rangle) + 0(e^{-cN_1}) \tag{4.17}$$

for some $p_\ell \in \mathcal{P}_1$. Write by factorization

$$p_\ell(\langle \ell, \lambda\rangle) = \prod_{s\in E_\ell} (\langle \ell, \lambda\rangle - \sigma_s^1) \tag{4.18}$$

and retain only those factors $s \in E'_\ell$ that are small

$$s \in E'_\ell \Leftrightarrow |\langle \ell, \lambda\rangle - \sigma_s^1| < \delta_1. \tag{4.19}$$

Thus (4.17) is equivalent to

$$\prod_{\ell\in Q_1\cap\tilde\Omega_k^2} \prod_{s\in E'_\ell} \big(\langle k, \lambda\rangle - (\sigma'_s + \langle k-\ell, \lambda\rangle)\big) + 0(e^{-cN_1}) \tag{4.20}$$

where

$$\bigcup_{\ell\in Q_1\cap\tilde\Omega_k^2} E'_\ell \tag{4.21}$$

labels the subset of $\{\sigma_1^1, \dots, \sigma_{2r}^1\}$ corresponding to $\tilde\Omega_k^2$.

Applying the preparation theorem to (4.20) with $x = \langle k, \lambda\rangle$ as formal variable yields an equivalent polynomial expression

$$p(\langle k, \lambda\rangle); \text{degree } p = \#(4.21).$$

In this way, the inverses of $T_{\tilde\Omega_k^2}$, $k \in P_2$, may again be controlled by the reciprocals $p(\langle k, \lambda\rangle)$, where p runs in a family $\mathcal{P}_2$ of polynomials satisfying (4.13) by construction. One gets again numbers

$$\sigma_1^2, \dots, \sigma_{2r}^2 \tag{4.22}$$

such that the inverse of each $T_{\tilde\Omega_k^2}$, $k \in P_2$, is controlled by the reciprocal of

$$\min_{1\le s\le 2r} |\langle \lambda, k\rangle - \sigma_s^2|. \tag{4.23}$$

Introduce the second generation of "singular" submatrices corresponding to $Q_2 \subset P_2$ where

$$Q_2 = \Big\{k \in P_2 \,\Big|\, \min_{1\le s\le 2r} |\langle \lambda, k\rangle - \sigma_s^2| < \delta_2\Big\}. \tag{4.24}$$

At this stage, we may control T_Λ^{-1} provided $\Lambda \subset \mathbb{Z}^b$ satisfies $(*)$, i.e.

$$k \in P_i,\ \Omega_k^i \cap \Lambda \ne \phi \Rightarrow \tilde\Omega_k^i \subset \Lambda \qquad (i = 1, 2) \tag{4.25}$$

$$\Lambda \cap Q_2 = \phi. \tag{4.26}$$

Writing

$$\Lambda = \bigcup_{\substack{k\in Q_1\\ \Omega_k^1\subset\Lambda}} \Omega_k^1 + \Lambda' \tag{4.27}$$

we claim that Λ' satisfies (4.8), (4.9). Indeed, if $k \in P_1$, $\Omega_k^1 \cap \Lambda' \neq \phi$, then $\tilde{\Omega}_k^1 \subset \Lambda$ by (4.25) and hence $\tilde{\Omega}_k^1 \subset \Lambda'$ since otherwise $\Omega_k^1 \cap \Lambda' = \phi$. Also $\Lambda' \cap Q_1 = \phi$.

For $k \in Q_1$, $\Omega_k^1 \subset \Omega_{k'}^2$ for some $k' \in P_2$, by (4.10), (4.12). Then $\Omega_{k'}^2 \cap \Lambda \neq \phi$ and thus $\tilde{\Omega}_{k'}^2 \subset \Lambda$ (4.25) and $k' \notin Q_2$ (4.26). Consequently

$$\bigcup_{\substack{k\in Q_1\\ \Omega_k^1\subset\Lambda}} \Omega_k^1 \subset \bigcup_{\substack{k'\in P_2\backslash Q_2\\ \tilde{\Omega}_{k'}^2\subset\Lambda}} \tilde{\Omega}_{k'}^2 \tag{4.28}$$

and we may apply again Pöschel's lemma with the $\{\tilde{\Omega}_{k'}^2 | k' \in P_2\backslash Q_2\}$ as separated neighborhoods of the singular islands.

Next, one constructs $P_3 \subset Q_2$ etc.

4(ii). Choice of scale

Take in preceding construction

$$\delta_i = N_i^{-C_1} \tag{4.29}$$

and

$$N_{i+1} \sim \delta_i^{-c_2} \tag{4.30}$$

where c_2 depends on the number r of modes and the diophantine properties of λ. Thus

$$N_{i+1} \sim N_i^{c_2 C_1} \tag{4.31}$$

and we choose C_1 sufficiently large to ensure $c_2 C_1 > 10$ say.

4(iii). Control of the inverse of restrictions of T

We allow variation of λ in a fixed direction, thus $\lambda = t.\lambda_0$ where λ_0 is a given diophantine vector and $t \in \mathbb{R}$, $t \approx 1$. Fix N and consider $T_N = T|_{|k|<N}$. Take i such that $N_i \leq N < N_{i+1}$ and enlarge $[-N, N]^b$ to a region Λ satisfying

$$\Omega_k^j \cap \Lambda \neq \phi \Rightarrow \tilde{\Omega}_k^j \subset \Lambda \tag{4.32}$$

$$\log(\operatorname{diam}\Lambda) \sim \log N. \tag{4.33}$$

If $\Lambda \cap Q_i = \phi$, then Λ satisfies $(*)$ and hence T_Λ^{-1} has desired properties. Otherwise there is a single $k \in P_{i+1}$ such that $\Lambda \cap \Omega_k^{i+1} \neq \phi$, hence $\tilde\Omega_k^{i+1} \subset \Lambda$ and $\Lambda \backslash \Omega_k^{i+1} = \Lambda'$ satisfies $(*)$. The control of T_Λ^{-1} is then achieved from Pöschel's lemma and $T^{-1}_{\tilde\Omega_k^{i+1}}$.

From the discussion of $\tilde\Omega_k^{i+1}$, $T^{-1}_{\tilde\Omega_k^{i+1}}$ may be controlled by a "local determinant" equivalent to a polynomial of the form

$$\left(\prod_{k\in A} \langle \lambda_0, k\rangle\right) t^d + (\text{lower degree}) \tag{4.34}$$

where $d = |A| \leq (2r)^{i+1}$ and $A \subset Q_0$. This polynomial is constructed from consecutive applications of Malgrange's preparation theorem (cf. [Bo2, 3]) for perturbations of polynomials of high degree. At a given step the size of the perturbation is $0(e^{-cN_{j-1}})$, while the degree of the perturbed polynomial is at most $(2r)^j$. Since $(2r)^j < \log\log e^{cN_{j-1}})$, by (4.31), the size of this perturbation is compatible to apply the preparation theorem from [Bo2]. Since $A \subset Q_0$, $|\langle\lambda_0, k\rangle| \approx |\mu_s| > \gamma \neq 0$ for $k \in A$. Thus the leading term in (4.34) has a coefficient of size at least $\gamma^{(2r)^{i+1}}$ and (4.34) may appropriately be kept away from 0 by a small variation of t around 1, considering $\frac{\partial^d}{(\partial t)^d}|_{t=1}$. In the application to the Melnikov problem, also the off-diagonal part of T will have a (weak) dependence on λ, hence on t. This is compatible with the preceding analysis.

5. An example

Our purpose is to exhibit a simple construction illustrating in the Melnikov setting the need of frequency restrictions according to the nonlinearity and in particular the absence of smooth ε-families of perturbed solutions with fixed frequency vector λ for the Hamiltonian $H = H_0 + \varepsilon H_1$.

We consider a phase space of dimension $N = 2b + 2$, where $b = \dim$ tori will be taken sufficiently large. Let

$$H_\varepsilon(I,\, \theta,\, q) = H_0(I) + \mu|q|^2 + \varepsilon\left\{|q|^2 + \delta\left[\cos A\left(\sum_{s=1}^{b} \cos\theta_s\right)\right] Re\, q\right\} \tag{5.1}$$

where $I = (I_1, \dots, I_b)$, $\theta = (\theta_1, \dots, \theta_b)$ are action angle variables and $q \in \mathbb{C}$ is the remaining coordinate (pair). In (5.1), μ is arbitrary; A is an arbitrary large number and we choose δ (depending on A) sufficiently small in order to preserve a bound on H_1 in the analytic function space norm. We will show that we cannot find for any fixed (typical) frequency vector $\lambda = (\lambda_1, \dots, \lambda_b)$ a family $\{u_\varepsilon(t) |\, \varepsilon| < \frac{1}{A}\}$ of quasi-periodic solutions of

$$i\dot u_\varepsilon = \frac{\partial H_\varepsilon}{\partial \overline{u}_\varepsilon}. \tag{5.2}$$

with frequency vector λ, such that

$$|u_\varepsilon(t) - u_0(t)| < C\varepsilon \tag{5.3}$$

(as will result from the argument, even stronger statements are true in fact).

The evolution equations for $u = (I,\, \theta,\, q)$ corresponding to (5.1) are

$$\dot{I}_s = -\varepsilon\delta A\left[\sin A\left(\textstyle\sum_{s=1}^b \cos\theta_s\right)\sin\theta_s . Re\, q\right] \qquad (s = 1, \dots, b) \tag{5.4}$$

$$\dot{\theta}_s = \tfrac{\partial H_0}{\partial I_s} \qquad (s = 1, \dots, b) \tag{5.5}$$

$$i\dot{q} = (\mu + \varepsilon)q + \tfrac{1}{2}\varepsilon\delta\cos\left[A\left(\textstyle\sum_{s=1}^b \cos\theta_s\right)\right] \tag{5.6}$$

and $u_\varepsilon(t)$ is of the form

$$I = I_0 + I'(\varepsilon,\, \lambda t) \tag{5.7}$$

$$\theta = \theta_0 + \lambda t + \theta'(\varepsilon,\, \lambda t) \tag{5.8}$$

$$q = \textstyle\sum_{k\in\mathbb{Z}^b} \widehat{q}(k,\, \varepsilon)\, e^{i(\lambda.k)t} \tag{5.9}$$

where, by (5.3),

$$I' = 0(\varepsilon) \tag{5.10}$$

$$|\theta'| = 0(\varepsilon) \tag{5.11}$$

$$|q| = 0(\varepsilon). \tag{5.12}$$

Substitution of (5.9) in (5.6) yields by (5.11) for each Fourier mode k

$$(\langle\lambda,\, k\rangle + \mu + \varepsilon)\, \widehat{q}\,(k,\, \varepsilon) = -\tfrac{1}{2}\varepsilon\delta\Big[\cos A\Big(\sum_{s=1}^b \cos\theta_s\Big)\Big]^{\wedge}(k) \tag{5.13}$$

$$= -\tfrac{1}{2}\varepsilon\delta\left\{\left[\cos A\left(\sum_{s=1}^b \cos(\theta_{0,\,s} + \lambda_s t)\right)\right]^{\wedge}(k) + 0(A\varepsilon)\right\} \tag{5.14}$$

$$= -\varepsilon\delta\left\{e^{-ik.\theta_0}\int_{\mathbb{T}^b}\cos A\left(\sum_{s=1}^b \cos\varphi_s\right)\cdot e^{ik.\varphi}\, d\varphi_1\cdots d\varphi_b + 0(A\varepsilon)\right\} \tag{5.15}$$

For typical λ, one may choose $k_0 \in \mathbb{Z}^b$, $|k_0| = o(A)$ such that

$$|\langle\lambda,\, k_0\rangle + \mu| < A^{-(b-1)+}. \tag{5.16}$$

Take then ε such that

$$|\varepsilon| < A^{-(b-1)+} \text{ and } \langle\lambda, k_0\rangle + \mu + \varepsilon = 0 \tag{5.17}$$

(the first restriction being compatible with the admissible ε-range for A sufficiently large).

Since for $k = k_0$ the left member of (5.15) vanishes, one obtains that

$$\int_{\mathbb{T}^b} \cos A\left(\sum_{s=1}^{b} \cos \varphi_s\right) e^{ik_0\cdot\varphi} d\varphi_1 \cdots d\varphi_b = 0(A\varepsilon) = 0(A^{-b+2+}) \tag{5.18}$$

from (5.15), (5.17). Now the integral equals however

$$\frac{1}{2}\prod_{s=1}^{b}\left[\int_{\mathbb{T}} e^{i(A\cos\varphi + k_{0,s}\varphi)} d\varphi\right] + \frac{1}{2}\prod_{s=1}^{b}\left[\int_{\mathbb{T}} e^{i(-A\cos\varphi + k_{0,s}\varphi)} d\varphi\right] \tag{5.19}$$

which, by stationary phase, is typically of size $A^{-b/2}$, since $|k_0| \ll A$ (some care is needed in order to ensure that the leading term is not vanishing). In conclusion, we get that $A^{-b/2} < A^{-b+2+}$, a contradiction for $b \geq 5$.

References

[B1] J. Bourgain, *Construction of quasi-periodic solutions of Hamiltonian perturbations of linear equations and applications to nonlinear PDE*, Internat. Math. Res. Notices **N11** (1994), 475–497.

[B2] ———, *Quasiperiodic solutions of Hamiltonian perturbations of 2D-linear Schrödinger equations*, Ann. of Math (to appear).

[B3] ———, *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. **4** (1995), 629–639.

[C-W1] W. Craig, C. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **46** (1993), 1409–1501.

[C-W2] ———, *Periodic solutions of nonlinear Schrödinger equations and the Nash-Moser method*, NATO Adv. Sci. Inst. Ser. B Phys., 331, Plenum, New York, 1994, pp. 103–122.

[E] L. H. Eliasson, *Perturbations of stable invariant tori*, . Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), 115-147.

[Kuk] S. Kuksin, *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes in Math, 1556, Springer-Verlag, Berlin, 1993.

[Pos1] J. Pöschel, *On elliptic lower dimensional tori in Hamiltonian systems*, Math. Z. **202** (1989), 559–608.

[Pos2] ———, *On Frölich-Spencer estimates on Green's function*, Manuscripta Math. **70** (1990), 27–37.

Institute for Advanced Study, School of Mathematics, Princeton, NJ 08540 and University of Illinois at Urbana-Champaign, Urbana, IL 61801

E-mail address: bourgain@math.ias.edu