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BEST BOUNDS FOR THE HILBERT TRANSFORM ON $L^p(\mathbb{R}^1)$

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ABSTRACT. We present a short proof of the boundedness of the Hilbert transform on $L^p(\mathbb{R}^1)$ with the best possible constant. The proof assumes two lemmas known in the literature, but it avoids using the conjugate function on the circle and it is given directly on the line.

Let

$$(Hf)(x) = \text{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x-t)}{t}\,dt\ ,$$

be the usual Hilbert transform on the line. We shall prove the following:

Theorem. (Stylianos Pichorides [*P*], Brian Cole (unpublished)). *H maps $L^p(\mathbb{R}^1)$ into itself with bound $\tan(\pi/2p)$ for $1 < p \le 2$ and $\cot(\pi/2p)$ for $2 \le p < \infty$.*

By duality it suffices to prove the theorem for $1 < p \le 2$. Fix such a p. We will need the following two lemmas:

Lemma 1. *For all a and b real numbers the following inequality is true* :

$$(1) \qquad |b|^p \le \tan^p(\pi/2p)|a|^p - B_p Re\,[(|a|+ib)^p],$$

where $B_p = \sin^{p-1}(\pi/2p)/\sin\big((p-1)\pi/2p\big)$ is a positive constant.

Setting $a = R\cos x$ and $b = R\sin x$, the above reduces to Lemma 2.1 in [P]. Furthermore, we will need Lemma 3.5 in [P] (which can also be found in [G] p. 129):

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Lemma 2. *The function $g(x,y) = Re\,[(|x|+iy)^p]$ defined on $\mathbb{R}^2$ is subharmonic.*

To prove the theorem, take f to be a C^∞ compactly supported function and apply (1) with $a = f(x)$ and $b = (Hf)(x)$. Then integrate (1) with respect to x to obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} |(Hf)(x)|^p dx \le \tan^p\left(\frac{\pi}{2p}\right) &\int_{-\infty}^{+\infty} |f(x)|^p dx \\ &- B_p \int_{-\infty}^{+\infty} \mathrm{Re}\,[(|f(x)| + i(Hf)(x))^p] dx.\end{aligned} \tag{2}$$

Since $B_p \ge 0$, the proof of the theorem will be complete once we show that

$$\int_{-\infty}^{+\infty} \mathrm{Re}\,[(|f(x)| + i(Hf)(x))^p] dx \ge 0\,. \tag{3}$$

Consider the holomorphic extension of $(f + iHf)(x)$ on the upper half space given by

$$u(z) + iv(z) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{z-t}\, dt\,, \qquad u,\ v \text{ real-valued.} \tag{4}$$

Since $u(z) + iv(z)$ is holomorphic and $g(x,y)$ is subharmonic, it follows that $g(u(z), v(z))$ is subharmonic on the upper half space. The boundary values of $g(u(z), v(z))$ are

$$g(u(x+i0), v(x+i0)) = \mathrm{Re}\,[(|f(x)| + i(Hf)(x))^p]. \tag{5}$$

For $R > 100$, let C_R be the circle with center $(0, R)$ and radius $R - R^{-1}$. Denote by C_R^U the upper part of C_R and by C_R^L the lower part of C_R. It follows from the subharmonicity of $g(u(z), v(z))$ that

$$\int_{C_R^U} g(u(z), v(z))\, ds + \int_{C_R^L} g(u(z), v(z))\, ds \ge 2\pi(R - R^{-1})\ g(u(iR), v(iR)). \tag{6}$$

Observe that

$$|u(x+iy)|,\ \ |v(x+iy)| \le \frac{\|f\|_\infty\ |\text{support } f|}{\pi R}, \quad \text{whenever } |y| \ge R, \tag{7}$$

where $|\text{support } f|$ denotes the measure of the support of f. Clearly (7) implies that

$$\begin{aligned}|(R - R^{-1})\, g(u(iR), v(iR))| \le (R - R^{-1})&\left(\frac{2\|f\|_\infty\ |\text{support } f|}{\pi R}\right)^p \to 0 \\ &\text{as } R \to \infty,\end{aligned} \tag{8}$$

and that

$$\Big|\int_{C_R^U} g(u(z),v(z))\,ds\Big| \le \pi(R-R^{-1})\left(\frac{2\|f\|_\infty\ |\text{support } f|}{\pi R}\right)^p \to 0 \quad \text{as } R\to\infty. \tag{9}$$

Letting $R \to \infty$ in (6), and using (5), (8), and (9), we obtain (3).

References

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