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THE SPECTRUM OF THE LAPLACIAN ON A MANIFOLD OF NONNEGATIVE RICCI CURVATURE

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§1 Introduction

The study of the spectrum of the Laplacian on a complete noncompact Riemannian manifold has received much attention during the past decade or so. In particular, it has been conjectured and partially verified that the spectrum of the Laplacian acting on the space of L^2 functions (the L^2 spectrum) is the half line $[0,\infty)$ when the underlying manifold has nonnegative Ricci curvature. J. F. Escobar and A. Freire ([E], [E-F]) have dealt with the case that the manifold has nonnegative sectional curvature and proved among other things that this is true provided that the exponential map from the normal bundle of a soul of the manifold is a diffeomorphism onto the manifold. Later, J. Li in [Li] showed that this is also true for a Ricci nonnegative manifold possessing a pole, namely, the exponential map at some point is a diffeomorphism from the tangent space onto the manifold. Recently, the author was informed that H. Donnelly has considered the Ricci nonnegative manifolds with maximal volume growth. Among other things, he established also that the L^2 spectrum of the Laplacian is $[0,\infty)$ in this case. This last result has also been obtained independently by N. Castañeda in [C]. In this short note, our purpose is to demonstrate the validity of the conjecture without imposing extra assumptions on the manifold. In fact, we shall prove more generally that the L^2 spectrum of the Laplacian on a complete noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature is given by the half line $[0,\infty)$. Recall that a complete manifold (M^n,g) has asymptotically nonnegative Ricci curvature if there exists a small constant $\delta(n)>0$ depending only on n such that for some point $q\in M$, the Ricci curvature satisfies $Ric_M(x)\geq -\delta(n)r^{-2}(x)$, where $r(x)$, the distance from x to q, is sufficiently large. Note that here we do not address the problem whether there is any eigenvalue for the Laplacian.

§2 Proof of the Theorem

In this section, we shall give the proof of the result that the L^2 spectrum of the Laplacian on a complete noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature is given by the half line $[0,\infty)$. The argument

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relies on a result in [S] which is stated below and utilizes the testing functions constructed in [E] and [Li]. We first recall that the Laplacian Δ of a complete manifold (M, g) is defined on $C_c^\infty(M)$, the space of smooth functions with compact support on M, and can be uniquely extended into a self-adjoint operator on the space $L^2(M)$, which is denoted by Δ_2. It is well-known then that the semigroup $\exp(t\Delta_2)$ extends to a strongly continuous contraction semigroup on any $L^p(M)$, $p \in [1, \infty)$. Its generator is denoted by Δ_p. For $p = \infty$, Δ_p is defined to be the adjoint of Δ_1, i.e., $\Delta_\infty = \Delta_1^*$. It is clear that $C_c^\infty(M)$ is contained in the domain of Δ_p and $\Delta_p = \Delta$ on $C_c^\infty(M)$ for all $p \in [1, \infty]$. The volume of the manifold (M, g) is said to grow uniformly subexponentially if for any $\epsilon > 0$, there exists a constant $C < \infty$ such that $V_x(r) \leq Ce^{\epsilon r} V_x(1)$ for all $x \in M$ and $r \geq 1$, where $V_x(r)$ is the volume of the geodesic ball $B_x(r)$ in M. The following result is proved in [S].

Theorem 1. [S] *If the Ricci curvature of a complete noncompact manifold (M, g) is bounded from below and the volume of (M, g) grows uniformly subexponentially, then the spectrum of Δ_p acting on $L^p(M)$ is independent of $p \in [1, \infty]$.*

It has been verified in [S] that the volume of manifold (M, g) grows uniformly subexponentially if its Ricci curvature is bounded from below by a function $K(x)$ on M satisfying $\liminf_{x\to\infty} K(x) \geq 0$. In particular, this is true for a manifold with asymptotically nonnegtaive Ricci curvature. Before we proceed to prove our main result, we collect some facts into the following lemma.

Lemma 2. *Let M^n be a complete noncompact Riemannian manifold of dimension n such that its Ricci curvature satisfies $Ric_M(x) \geq -\delta(n) r^{-2}(x)$, where $r(x)$ is the distance from x to q, on $M \setminus B_q(a)$ and $\delta(n) > 0$ depending only on n is a small constant, and has lower bound $-(n-1)K$ on $B_q(a)$, where $K \geq 0$. Then*

(1) *$\Delta r(x) \leq \frac{c_1(n,a,K)}{r}$ for some constant c_1.*
(2) *there exist constants $c_2(n, a, K) > 1$ and $c_3(n, a, K) > 0$ such that for all $r > 0$ and $1 \leq \beta \leq 2$, $V_q(2r) \leq c_2 V_q(r)$ and $A_q(\beta r) \leq c_3 A_q(r)$, where $V_q(r)$ is the volume of the geodesic ball $B_q(r)$ and $A_q(r)$ the area of the geodesic sphere $\partial B_q(r)$.*
(3) *there exists $\epsilon(n, a, K) > 0$ such that for all $r > 0$ sufficiently large, $V_q(\epsilon r) \leq \frac{1}{2}(\frac{1}{c_2})^2 V_q(r)$ and $A_q(\epsilon r) \leq \frac{c_4(n,a,K)}{r}[V_q(2r) - V_q(\epsilon r)]$ for some positive constant c_4.*

Proof. Part (1) is the standard Laplacian comparison theorem. We refer to the book [S-T] for a proof. Part (2) follows from the well-known volume comparison argument. The details have been given in [L] and [L-T]. We prove part (3) by using the argument in [C-G-T] and [L-S]. In fact, we shall prove more generally that there exists a constant $c(n) > 0$ such that for $a \leq r \leq R$,

$$V_q(r) \leq c\frac{r}{R} V_q(R).$$

Fix a constant $\beta > \frac{2}{2^{1/n}-1}$. For $x \in M$ with $r(x) = T = (2\sum_{j=0}^{k}\beta^j - 1 - \beta^k)r$, let γ be a minimal geodesic joining q and x with $\gamma(0) = q$ and $\gamma(T) = x$. Define $x_i = \gamma(t_i)$, where $t_0 = 0$ and $t_i = (2\sum_{j=0}^{i}\beta^j - 1 - \beta^i)r$, $i = 1, 2, \ldots, k$. Let $R_i = \beta^i r$ for $i = 0, 1, \ldots, k$. Then the relative volume comparison theorem (see [C-G-T] or [L]) implies that

$$V_{x_i}(R_i) \geq T_i(V_{x_i}(R_i + 2R_{i-1}) - V_{x_i}(R_i)) \geq T_i V_{x_{i-1}}(R_{i-1}),$$

where

$$T_i = \frac{\int_0^{R_i} \sinh^{n-1}\sqrt{k(x_i, R_i + 2R_{i-1})}tdt}{\int_{R_i}^{R_i+2R_{i-1}} \sinh^{n-1}\sqrt{k(x_i, R_i + 2R_{i-1})}tdt},$$

and in general $-(n-1)k(x,t)$ denotes the lower bound of the Ricci curvature on $B_x(t)$. The assumption on Ric_M yields

$$\sqrt{k(x_i, R_i + 2R_{i-1})} \leq \delta^{1/2}(n)(\sum_{j=0}^{i-2}\beta^j - 1)^{-1}r^{-1}$$

for sufficiently large i. Since $\beta > 1$ is fixed, we conclude that there is a fixed constant $c(n)$ such that

$$(R_i + 2R_{i-1})\sqrt{k(x_i, R_i + 2R_{i-1})} \leq c(n)\delta^{1/2}(n),$$

which can be made arbitrarily small by the smallness assumption on $\delta(n)$. Hence

$$\begin{aligned}
T_i &= \frac{\int_0^{R_i\sqrt{k(x_i,R_i+2R_{i-1})}} \sinh^{n-1} tdt}{\int_{R_i\sqrt{k(x_i,R_i+2R_{i-1})}}^{(R_i+2R_{i-1})\sqrt{k(x_i,R_i+2R_{i-1})}} \sinh^{n-1} tdt} \\
&\approx \frac{R_i^n}{(R_i + 2R_{i-1})^n - R_i^n} \\
&= \frac{\beta^n}{(\beta+2)^n - \beta^n}
\end{aligned}$$

by simply approximating $\sinh t$ with t. Therefore,

$$V_{x_k}(R_k) \geq V_q(r)\Pi_{i=1}^{k} T_i \geq c\left(\frac{\beta^n}{(\beta+2)^n - \beta^n}\right)^k V_q(r).$$

Hence,

$$\begin{aligned}
V_q(2T) &\geq V_{x_k}(R_k) \\
&\geq c\left(\frac{\beta^n}{(\beta+2)^n - \beta^n}\right)^k V_q(r) \\
&\geq c\frac{2T}{r}V_q(r).
\end{aligned}$$

Since k is arbitrary, we conclude that

$$V_q(R) \geq c\frac{R}{r}V_q(r)$$

for all $R \geq r \geq a$.

Finally, to show the last inequality, we note that from (2), there exists a constant $c_5(n, a, K)$ such that

$$c_5 \epsilon r A_q(\epsilon r) \leq \int_{\frac{1}{2}\epsilon r}^{\epsilon r} A_q(t)dt \leq V_q(\epsilon r).$$

So we have

$$A_q(\epsilon r) \leq \frac{c_6}{r}V_q(\epsilon r).$$

However,

$$V_q(2r) - V_q(\epsilon r) \geq V_q(r) - V_q(\epsilon r) \geq (2c_2^2 - 1)V_q(\epsilon r).$$

Therefore, we conclude that

$$A_q(\epsilon r) \leq \frac{c_4}{r}[V_q(2r) - V_q(\epsilon r)]$$

for some constant c_4 depending on n, a and K. The proof is completed.

Now we are ready to prove our main result. In the following, we shall denote by c a generic constant which may depending on n, a and K but not on k.

Theorem 3. *Let M be a complete noncompact Riemannian manifold. Suppose that the Ricci curvature of M is asymptotiaclly nonnegative. Then the spectrum of the Laplacian Δ_p acting on the space $L^p(M)$ is $[0, \infty)$ for all $p \in [1, \infty]$.*

Proof. Since M has asymptotiaclly nonnegative Ricci curvature, Theorem 1 is applicable and implies that the spectrum of Δ_p is independent of $p \in [1, \infty]$. So we need only to show that the spectrum of Δ_1 is $[0, \infty)$. We may fix a point $q \in M$ and assume that the Ricci curvature of M satisfies $Ric_M(x) \geq -\delta(n)r^{-2}(x)$, where $r(x)$ is the distance from x to q, on $M \setminus B_q(a)$ and $\delta(n) > 0$ depending only on n is a small constant, and has a lower bound $-(n-1)K$ on the ball $B_q(a)$, where a and K are nonnegative constants, and n is the dimension of the manifold. Note that $r(x)$ is a Lipschitz function with $|\nabla r(x)| = 1$ almost everywhere. Also, by Lemma 2, $\Delta r(x) \leq \frac{c}{r(x)}$ in the sense of distribution. Let E be the cut-locus of the point $q \in M$ and $\Omega = M \setminus E$. Since the volume of the set E is zero, by the Fubini's theorem, the area of the set $\partial B_q(t) \cap E$ is zero for almost all t. We call such t to be permissible. By mollifying the function $r(x)$, one obtains a sequence of smooth functions $\{f_k\}$ on M such that f_k converges to $r(x)$ uniformly on compact subsets of M, and $|\nabla f_k| \leq 2$ on M

and $|\nabla f_k - \nabla r| \le \frac{1}{k}$ on Ω, and $\Delta f_k(x) \le \frac{c}{r(x)} + \frac{1}{k}$ on M. We note that in particular $\frac{\partial f_k}{\partial r} \ge 0$ on $\partial B_q(t) \cap \Omega$ for all $t > 0$ and if t is permissible, then

$$0 \le \int_{\partial B_q(t)} \frac{\partial f_k}{\partial r} = \int_{\partial B_q(t)\cap\Omega} \frac{\partial f_k}{\partial r} \le 2A_q(t). \tag{1}$$

We now follow [E] and [Li] and define for any $\lambda \ge 0$ a sequence of functions

$$\phi_k(x) = \eta_k \psi(\frac{f_k(x)}{k}) e^{i\sqrt{\lambda} f_k(x)},$$

where $\eta_k = [V_q(2(k+1)) - V_q(\epsilon(k-1))]^{-1}$, and $\psi(r)$ is a smooth function such that $\psi(r) = 1$ for $2\epsilon \le r \le 1$ and $\psi(r) = 0$ for $r \ge 2$ or $r \le \epsilon$ with $0 \le \psi(r) \le 1$, $|\psi''(r)| \le C(\epsilon)$. Here and in the following, $\epsilon > 0$ is a fixed constant chosen to satisfy Lemma 2. It is clear that $\phi_k(x)$ is a smooth function with compact support on M. Also, we may assume without loss of generality that $2(k+1)$ and $\epsilon(k-1)$ are permissible. A direct computation gives

$$\begin{aligned}(\Delta + \lambda)\phi_k(x) = {} & \eta_k \frac{1}{k^2} \psi''(\frac{f_k}{k}) e^{i\sqrt{\lambda} f_k} |\nabla f_k|^2 + 2i\sqrt{\lambda}\eta_k \frac{1}{k}\psi'(\frac{f_k}{k}) e^{i\sqrt{\lambda} f_k}|\nabla f_k|^2 \\ & + \eta_k \frac{1}{k}\psi'(\frac{f_k}{k}) e^{i\sqrt{\lambda} f_k}\Delta f_k + i\sqrt{\lambda}\eta_k \psi(\frac{f_k}{k}) e^{i\sqrt{\lambda} f_k}\Delta f_k \\ & + \lambda(1 - |\nabla f_k|^2)\phi_k.\end{aligned} \tag{2}$$

Thus, we conclude that

$$\begin{aligned}|(\Delta + \lambda)\phi_k(x)| \le {} & \frac{4}{k^2}\eta_k |\psi''(\frac{f_k}{k})| + \frac{8\sqrt{\lambda}}{k}\eta_k|\psi'(\frac{f_k}{k})| \\ & + \eta_k \frac{1}{k}|\psi'(\frac{f_k}{k})||\Delta f_k| + \sqrt{\lambda}\eta_k|\psi(\frac{f_k}{k})||\Delta f_k| \\ & + \lambda\eta_k \left|1 - |\nabla f_k|^2\right| \psi(\frac{f_k}{k}).\end{aligned} \tag{3}$$

Since

$$\Delta f_k(x) \le \frac{c_1(n, a, K)}{r(x)} + \frac{1}{k},$$

if we let $(\Delta f_k)^+$ be the positive part of Δf_k, then

$$(\Delta f_k)^+ \le \frac{c_1(n, a, K)}{r} + \frac{1}{k}. \tag{4}$$

Note that

$$|\Delta f_k| = 2(\Delta f_k)^+ - \Delta f_k. \tag{5}$$

From (3) and using (1), (4), (5), we obtain that for k sufficiently large

$$\begin{aligned}
&\int_M |(\Delta+\lambda)\phi_k(x)|dx \le \frac{4\eta_k}{k^2}\int_M |\psi''(\frac{f_k}{k})|dx + \frac{8\sqrt{\lambda}}{k}\eta_k \int_M |\psi'(\frac{f_k}{k})|dx \\
&+ \eta_k \frac{1}{k}\int_M |\psi'(\frac{f_k}{k})||\Delta f_k|dx + \sqrt{\lambda}\eta_k \int_M |\psi(\frac{f_k}{k})||\Delta f_k|dx \\
&+ \lambda\eta_k \int_M \left|1-|\nabla f_k|^2\right| \psi(\frac{f_k}{k}) \\
&\le \frac{c}{k}\eta_k \int_{\epsilon(k-1)\le r(x)\le 2(k+1)} dx + c\eta_k \int_{\epsilon(k-1)\le r(x)\le 2(k+1)} |\Delta f_k(x)|dx \\
&\le \frac{c}{k} + c\eta_k \int_{\epsilon(k-1)\le r(x)\le 2(k+1)} [2(\Delta f_k)^+ - \Delta f_k]dx \\
&\le \frac{c}{k} + c\eta_k \int_{\epsilon k-1\le r(x)\le 2k+1} \frac{2c_1}{r(x)}dx - c\eta_k \int_{\epsilon(k-1)\le r(x)\le 2(k+1)} \Delta f_k(x)dx \\
&\le \frac{c}{k} + c\eta_k [\int_{\partial B_q(\epsilon(k-1))} \frac{\partial f_k}{\partial r} - \int_{\partial B_q(2(k+1))} \frac{\partial f_k}{\partial r}] \\
&\le \frac{c}{k} + c\eta_k A_q(\epsilon(k-1)),
\end{aligned} \tag{6}$$

where we have used the Green's formula. By Lemma 2, $\eta_k A_q(\epsilon(k-1)) \le \frac{c}{k}$. Therefore, we conclude from (6) that for k sufficiently large

$$\int_M |(\Delta+\lambda)\phi_k(x)|dx \le \frac{c}{k}. \tag{7}$$

Hence

$$\lim_{k\to\infty} \int_M |(\Delta+\lambda)\phi_k(x)|dx = 0. \tag{8}$$

On the other hand, using the definition of $\phi_k(x)$ and Lemma 2, we get for k sufficiently large

$$\begin{aligned}
\int_M |\phi_k(x)|dx &\ge \frac{V_q(k-1) - V_q(2\epsilon(k+1))}{V_q(2(k+1)) - V_q(\epsilon(k-1))} \\
&\ge \frac{V_q(k-1)}{V_q(2(k+1))} - \frac{V_q(2\epsilon(k+1))}{V_q(2(k+1))} \\
&\ge \frac{V_q(k-1)}{V_q(4(k-1))} - \frac{1}{2}(\frac{1}{c_2})^2 \\
&\ge (\frac{1}{c_2})^2 - \frac{1}{2}(\frac{1}{c_2})^2 \\
&\ge \frac{1}{2}(\frac{1}{c_2})^2.
\end{aligned} \tag{9}$$

From (8) and (9), one concludes that λ is in the spectrum of Δ_1. Since $\lambda \ge 0$ is arbitrary, we conclude that the spectrum of Δ_1 is $[0,\infty)$. The same assertion then also holds for the spectrum of Δ_p for $p \in [1,\infty]$. The theorem is proved.

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