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ON THE SEMI-SIMPLICITY OF THE QUANTUM COHOMOLOGY ALGEBRAS OF COMPLETE INTERSECTIONS

Gang Tian and Geng Xu

0. Introduction

In this short note, we study the semi-simplicity of the quantum cohomology algebras of smooth Fano complete intersections in projective space.

Let V be a smooth Fano manifold with a symplectic form ω. The quantum cohomology on V is the cohomology $H^*(V, \mathbb{Z}\{H_2(V)\})$ with a ring structure defined by the GW-invariants. The Novikov ring $\mathbb{Z}\{H_2(V)\}$ can be described as follows: choose a basis $q_1, \cdots, q_s$ of $H_2(V, \mathbb{Z})$, we identify the monomial $q^d = q_1^{d_1} \cdots q_s^{d_s}$ with the sum $\sum_{i=1}^s d_i q_i$. This turns $H_2(V)$ into a multiplicative ring, i.e., $q^d \cdot q^{d'} = q^{d+d'}$. This multiplicative ring has a natural grading defined by $\deg(q^d) = 2c_1(V)(\sum d_i q_i)$. Then $\mathbb{Z}\{H_2(V)\}$ is the graded homogeneous ring generated by all formal power series $\sum_{d=(d_1,\cdots,d_s)} n_d q^d$ satisfying: $n_d \in \mathbb{Z}$, all q^d with $n_d \neq 0$ have the same degree, and the number of n_d with $\omega(\sum d_i q_i) \leq c$ is finite for any $c > 0$.

Now we can define a ring structure on $H^*(V, \mathbb{Z}\{H_2(V)\})$. For any α^*, β^* in $H^*(V, \mathbb{Z})$, we define the quantum multiplication $\alpha^* \bullet \beta^*$ by

$$\alpha^* \bullet \beta^*(\gamma) = \sum_{A \in H_2(V,\mathbb{Z})} \Psi^V_{(A,0,3)}(\alpha, \beta, \gamma) q^A,$$

where $\gamma \in H_*(V, \mathbb{Z})$, and $\Psi^V_{(A,0,3)}(\alpha, \beta, \gamma)$ are the Gromov-Witten invariants (cf. [RT]).

In fact, there is a family of quantum multiplications. Let $\{\beta_a\}_{1 \leq a \leq L}$ be an integral basis of $H_*(V, \mathbb{Z})$ modulo torsions. Any $w \in H^*(V, \mathbb{C})$ can be written as $\sum t_a \beta_a^*$. Clearly, $w \in H^*(V, \mathbb{Z})$ if all t_a are integers. We define the quantum multiplication $\bullet_w$ by

$$\alpha^* \bullet_w \beta^*(\gamma) = \sum_A \sum_{k \geq 0} \frac{\epsilon(\{a_i\})}{k!} \Psi^V_{(A,0,k+3)}(\alpha, \beta, \gamma, \beta_{a_1}, \cdots, \beta_{a_k}) t_{a_1} \cdots t_{a_k}\, q^A$$

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where $\alpha, \beta, \gamma \in H_*(V, \mathbb{Z})$, $\epsilon(\{a_i\})$ is the sign of the induced permutation on odd dimensional β_a, and $\Psi^V_{(A,0,k+3)}(\alpha, \beta, \gamma, \beta_{a_1}, \cdots, \beta_{a_k})$ are the Gromov-Witten invariants. Obviously, this multiplication reduces to $\bullet$ at $w = 0$. It was shown in [RT] that the quantum multiplications $\bullet_w$ are associative.

Now we restrict ourselves to the case $w \in W = H^{\text{even}}(V, \mathbb{C})$. In this case, Dubrovin [D] observed that the quantum multiplications $\bullet_w$ induce the structure of a Frobenius algebra on W. For any $w \in W, w = \sum t_a \beta_a^*$, put

$$X(w) = \sum_{a \leq N} (\deg(\beta_a^*) - 2) t_a \partial_a w - 2c_1(V);$$

here we arrange the basis $\{\beta_a\}$ so that β_a is an even class if and only if $a \leq N$. We say that V is semi-simple in the sense of Dubrovin, if for a generic w, the quantum multiplication $X(w) \bullet_w$ on $H^*(V, \mathbb{C})$ has only simple eigenvalues. The question we are interested in is the following conjecture [T]:

Conjecture.(Tian) *Any Fano manifold is semi-simple in the sense of Dubrovin.*

The conjecture can be checked for complex projective spaces and quadratic hypersurfaces in projective spaces (cf. [KM], [T]).

There is a weaker version of the above conjecture. Let $H^*_{inv}(V, \mathbb{C})$ be the subring of $H^*(V, \mathbb{C})$ with the cup product, generated by $H^2(V, \mathbb{C})$. For any algebraic variety V, it is believed (cf. [T]) that for any $w \in H^*_{inv}(V, \mathbb{C})$, the quantum multiplication $\bullet_w$ preserves the subspace $H^*_{inv}(V, \mathbb{C})$, namely, if α, β are in $H^*_{inv}(V, \mathbb{C})$, so is $\alpha \bullet_w \beta$. Let us define $X_{inv}(w)$ be the restriction of X to $H^*_{inv}(V, \mathbb{C})$. Assuming that the above belief is true, then X_{inv} acts on $H^*_{inv}(V, \mathbb{C})$, and we can expect that $X(w) \bullet_w$ is semi-simple for a generic w in $H^*_{inv}(V, \mathbb{C})$, whenever V is Fano (cf. [T]).

Now, let $V \subset \mathbf{P}^{n+r}$ be a Fano complete intersection of dimension $n \geq 3$. Then the ordinary cohomology algebra $H^*(V, \mathbb{C})$ is generated by the hyperplane class H and the primitive cohomology $H^n(V, \mathbb{C})_o$. In particular, $H^*_{inv}(V, \mathbb{C})$ is the subspace of $H^*(V, \mathbb{C})$ generated by the hyperplane class H. It was observed in [T] that for any $w \in H^*_{inv}(V, \mathbb{C})$, the quantum multiplication $\bullet_w$ preserves the subspace $H^*_{inv}(V, \mathbb{C})$. Then X_{inv} acts on $H^*_{inv}(V, \mathbb{C})$ for any w in $H^*_{inv}(V, \mathbb{C})$. An important case of the above conjecture is to prove the semi-simplicity of $X(w) \bullet_w$ for any complete intersection V and a generic w in $H^*_{inv}(V, \mathbb{C})$. The result we have is the following:

Theorem 1. *Let $V \subset \mathbf{P}^{n+r} (n \geq 3)$ be a smooth complete intersection of degree $(d_1, d_2, \cdots, d_r)$. If $n > 2\sum_{i=1}^r (d_i - 1) - 1$ (except $n = 7$ and $\sum_{i=1}^r (d_i - 1) = 2$), then $X(w) \bullet_w$, which acts on $H^*_{inv}(V, \mathbb{C})$, is semi-simple in the sense of Dubrovin for a generic $w \in H^*_{inv}(V, \mathbb{C})$.*

Here we use Beauville's computation of quantum cohomology algebra for complete intersections satisfying the condition in the above theorem (cf. [B]). It seems that this condition on degree can be removed by using recent results in [G].

Our motivation for the study of semi-simplicity comes from the following result of Dubrovin [D]: if $\{\alpha_1, \cdots, \alpha_m\}$ is a basis of $H^*(V, \mathbb{C})$ such that the quantum multiplication $X(w)\bullet_w$ on $H^*(V, \mathbb{C})$ with respect to this basis has only simple eigenvalues, then the integrable system defined by using the Gromov-Witten prepotential Φ^V (see section 2) can be extented meromorphically to $(\mathbf{P}^1)^m$.

The main tool of this paper is an elementary lemma in section 1, the rest are standard computations. The method should also apply to some other Fano complete intersections. In this note, by a rational curve on V we mean a simple genus 0 J-holomorphic curve for some generic almost complex structure J on V.

Throughout this paper we work over the complex number field $\mathbb{C}$.

1. An algebraic lemma

The main tool of our computations is the following elementary lemma.

Lemma 1. *Assume that*

$$g(y, z) = y^m + g_1(z)y^{m-1} + \cdots + g_m(z)$$

is a polynomial of y, and $g_1(z), \cdots, g_m(z)$ are holomorphic functions of $z = (z_1, z_2, \cdots, z_N)$ defined in a neighborhood of $0 = (0, \cdots, 0) \in \mathbb{C}^N$. If the only repeated root of the polynomial $g(y, 0) = 0$ is $y = 0$, and the polynomial $g(y, z) = 0$ does not have distinct roots for generic $z \in \mathbb{C}^N$, then

$$g_m(z) = 0 \quad \text{mod } (z_1, z_2, \cdots, z_N)^2,$$

that is, the constant term and linear terms of $g_m(z)$ are all 0.

Proof. Let $\mathcal{O}_0$ denote the ring of holomorphic functions defined in some neighborhood of $(0, 0) \in \mathbb{C}^N \times \mathbb{C}$. Since all the repeated roots of $g(y, z) = 0$ appear nearby $y = 0$, we will study $g(y, z)$ in $\mathcal{O}_0$. By [GH] (page 8-11), $\mathcal{O}_0$ is a unique factorization domain module units, we can write

$$g(y, z) = (f_1(y, z))^{i_1} \cdots (f_h(y, z))^{i_h},$$

where $f_i(y, z) \in \mathcal{O}_0$ are distinct and irreducible Weierstrass polynomials in y.

Now we claim that $i_j > 1$ for some j. Otherwise, let $V_j = \{f_j(y, z) = 0\}$, then

$$g(y, z) = f_1(y, z) \cdots f_h(y, z), \quad \{g(y, z) = 0\} = V_1 \cup \cdots \cup V_h.$$

Since the Weierstrass polynomial $f_i(y, z)$ is irreducible, and $\frac{\partial f_i}{\partial y}(y, z)$ has degree lower than $f_i(y, z)$ in y, we conclude that $f_i(y, z)$ and $\frac{\partial f_i}{\partial y}(y, z)$ are relative prime in $\mathcal{O}_0$. Hence the analytic variety

$$\{f_i(y, z) = 0\} \cap \Big\{\frac{\partial f_i}{\partial y}(y, z) = 0\Big\} \subset \mathbb{C}^N \times \mathbb{C}$$

has dimension $N-1 < \dim \mathbb{C}^N$. Therefore the polynomial $f_i(y,z)=0$ has distinct roots for generic $z \in \mathbb{C}^N$.

Similarly, the analytic variety $\{f_i=0\} \cap \{f_j=0\} \subset \mathbb{C}^N \times \mathbb{C}$ has dimension $N-1<N$ for $i \neq j$ by Weak Nullstellensatz ([GH] p. 11). Hence the polynomials

$$f_i(y,z)=0, \quad f_j(y,z)=0$$

do not have common roots for generic $z \in \mathbb{C}^N$. This contradicts our assumption that $g(y,z)=0$ has repeated roots for generic z.

We conclude from above that $i_j > 1$ for some j. Now $f_j(y,z)$ is not a unit in $\mathcal{O}_0$, we have $f_j(0,0)=0$, so $f_j(0,z) \in (z_1,\cdots,z_N)$, that is,

$$g(0,z)=g_m(z)=f_1^{i_1}\cdots f_h^{i_h}(0,z) \in (z_1,\cdots,z_N)^2$$

because $i_j>1$. So the lemma is proved.

2. The computations

We now start our computations. For simplicity of notations, we will assume that $q=(q_1,\cdots,q_s)=q_1=1$ in the rest of the paper.

Let $V \subset \mathbf{P}^{n+r}$ be a smooth complete intersection of degree $(d_1,d_2,\cdots,d_r)$, and

$$\beta_a = H^{n+1-a} \in H_{2(a-1)}(V,\mathbb{C}), \quad 1 \le a \le n+1$$

the $(n+1-a)$-th power of the hyperplane H, then $\{\beta_a\}$ is a basis for the homology group $H_{inv}(V,\mathbb{C})$. Any cohomology class $w \in H^*_{inv}(V,\mathbb{C})$ can be written as

$$w = t_1\beta_1^* + t_2\beta_2^* + \cdots + t_{n+1}\beta_{n+1}^*,$$

here $\beta_a^* = H^{n+1-a} \in H^{2(n+1-a)}(V,\mathbb{C})(1 \le a \le n+1)$ in the ordinary cohomology. Then

$$\Phi^V_{inv}(t_1,\cdots,t_{n+1}) = \frac{d}{6}\sum_{a+b+c=2n+3} t_a t_b t_c + \sum_{k \ge 1}\sum_{\{k_a\}\in S_{V,k}} \frac{\sigma_k^V(k_1,\cdots,k_{n-1})t_1^{k_1}\cdots t_{n-1}^{k_{n-1}}}{k_1!\cdots k_{n-1}!} e^{kt_n},$$

$$S_{V,k} = \{\{k_a\}_{1\le a\le n-1} | \sum_{i=1}^{n-1} i k_{n-i} = (n+r+1-\sum_{i=1}^{r} d_i)k + n-3 > 0\},$$

$$(\beta_a^* \bullet_w \beta_b^*)(\beta_c) = \frac{\partial^3 \Phi^V}{\partial t_a \partial t_b \partial t_c}(w),$$

where $d=d_1d_2\cdots d_r$ is the degree of V, Φ^V is the Gromov-Witten prepotential, and $\sigma_k^V(k_1,\cdots,k_{n-1})$ is the number of degree k rational curves in V through k_1 points, $\cdots, k_{n-1}$ subspaces of dimension $n-2$ in general position.

Proposition 2, 3 in [B] and its higher degree analogy imply that there is a small positive number $\varepsilon = \varepsilon(d_1, \cdots, d_r, n) > 0$, such that $\Phi^V_{inv}(t_1, \cdots, t_{n+1})$ is well-defined and holomorphic in $t_1, \cdots, t_{n+1}$ when $|t_1| < \varepsilon, \cdots, |t_{n+1}| < \varepsilon$ (also cf. [G],[J], [T]).

Since $X(w)\bullet_w$ is a linear operator on $H^*_{inv}(V, \mathbb{C})$, its matrix $A(t_1, \cdots, t_{n+1})$ with respect to the basis $H^0 = 1, H^1, \cdots, H^n$ of $H^*_{inv}(V, \mathbb{C})$ in the ordinary cohomology is a $(n+1) \times (n+1)$ matrix. Denote

$$G(\lambda, t_1, \cdots, t_{n+1}) = \det(\lambda I - A(t_1, \cdots, t_{n+1})),$$

$$e = 1 + \sum_{i=1}^{r} (d_i - 1).$$

In [B],[J],[G],[CJ], it was proved that

$$G(\lambda, 0, \cdots, 0) = \lambda^{n+1} - d_1^{d_1} \cdots d_r^{d_r} \lambda^{e-1},$$

when $e < n+1$ and $n \geq 3$. Therefore, we have

Lemma 2. *The only repeated root of the polynomial* $G(\lambda, 0, \cdots, 0) = 0$ *is* $\lambda = 0$ *when* $e < n+1$ *and* $n \geq 3$.

Moreover, it is easy to see that $G(\lambda, 0, \cdots, 0)$ has distinct roots when $e = 1, 2$ (cf. [KM],[T]), that is, $X(w)\bullet_w$ is semi-simple when $e = 1, 2$ with $w = 0$. Hence we may assume that $e \geq 3$.

We choose

$$w = t_{2e-3}\beta^*_{2e-3} = t_{2e-3} H^{n-2e+4} \in H^*_{inv}(V, \mathbb{C})$$

when $n > 2e - 3$, and assume that t_{2e-3} is very close to 0. The rest of the computation will be made mod $(t_{2e-3})^2$.

Consider

$$(n-j)k_j + (n-m)k_m + (n-2e+3)k_{2e-3} = (n+2-e)k + (n-3).$$

If $k_{2e-3} = 0$, then we have $k = 1$ and $j + m = e + 1$ because of the assumption $n > 2e - 3$. If $k_{2e-3} = 1$, then we have either $k = 1$ and $j + m = n - e + 4$, or $k = 2$ and $j + m = 2$.

Hence the $(n+1) \times (n+1)$-matrix for $H^1\bullet_w$ with respect to the basis $H^0 = 1, H^1, \cdots, H^n$ of $H^*_{inv}(V, \mathbb{C})$ in the ordinary cohomology is

$$\begin{pmatrix}
0 & 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & 0 \\
a_1 t_{2e-3} & 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\
0 & a_2 t_{2e-3} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots & 0 \\
b_1 & 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\
0 & b_2 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots & 1 \\
2c_1 t_{2e-3} & 0 & 0 & \dots & b_e & \dots & a_{n-e+3} t_{2e-3} & \dots & 0
\end{pmatrix}.$$

Here $e < n-e+3 \le n$ because of the assumption $n > 2e-3$ and $e \ge 3$. When $e = 3$, we have $a_1 = a_{n-e+3} = a_n = 0$ as k_j is only defined for $1 \le j \le n-1$.

We next compute the matrix for $H^{n-2e+4}\bullet_w$. We only need to do the computation mod (t_{2e-3}) here.

Consider

$$(n-2e+3)k_{2e-3} + (n-j)k_j + (n-m)k_m = (n+2-e)k + (n-3).$$

We choose $k_{2e-3} = 1$ here. We have either $k = 1$ and $j+m = n-e+4$, or $k = 2$ and $j+m = 2$. Hence the matrix for $H^{n-2e+4}\bullet_w$ is

$$\begin{pmatrix} 0 & 0 & \dots & 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_1 & 0 & \dots & 0 & 0 & \dots & 1 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots \\ c_1 & 0 & \dots & 0 & 0 & \dots & a_{n-e+3} & \dots & 0 \end{pmatrix}.$$

Here the 1 in the first row appears in the $(n-2e+5)$-th column.

Now

$$X(w) = (n-2e+3)t_{2e-3}H^{n-2e+4} - (n-e+2)H^1,$$

$$-\frac{1}{n-e+2}X(w) = H^1 - \delta t_{2e-3}H^{n-2e+4},$$

here $\delta = \frac{n-2e+3}{n-e+2}$, and $0 < \delta < 1$ because $n > 2e-3$ and $e \ge 3$.

Therefore the matrix for $(H^1 - \delta t_{2e-3}H^{n-2e+4})\bullet_w$ with respect to the basis $H^0 = 1, H^1, \cdots, H^n$ of $H^*_{inv}(V, \mathbb{C})$ in the ordinary cohomology is

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & -t & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & -t & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ a_1^*t & 0 & 0 & \dots & 0 & 0 & \dots & 1 & \dots & -t & \dots & 0 \\ 0 & a_2^*t & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ b_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & 1 \\ c_1^*t & 0 & 0 & \dots & 0 & 0 & \dots & b_e & \dots & a_{n-e+3}^*t & \dots & 0 \end{pmatrix},$$

here $t = \delta t_{2e-3}$, $a_i^* = (\delta^{-1} - 1)a_i$, and $c_1^* = (2\delta^{-1} - 1)c_1$.

Now let $A_i(t)$ be the matrix obtained from $A(t)$ by replacing the i−th column of $A(t)$ by its derivative with respect to t. A detailed computation shows that

$$\begin{aligned} \det A_1(0) &= (-1)^n(c_1^* - a_1^* b_e), \\ \det A_{n-e+3}(0) &= (-1)^{n+1} b_1(a_{n-e+3}^* + b_e), \\ \det A_j(0) &= 0 \qquad \text{for } j \neq 1, n-e+3. \end{aligned}$$

As a result,

$$\begin{aligned} \det\, A(t) &= (\det A_1(0) + \det A_2(0) + \cdots + \det A_{n+1}(0))t \\ &= (-1)^n(c_1^* - a_1^* b_e - a_{n-e+3}^* b_1 - b_1 b_e)t \qquad \mod (t^2). \end{aligned}$$

Following Beauville [B] we use dl_j to denote the number of lines in V meeting two general linear space of codimension $n-j$ and $n+1-e+j$ respectively. We know that the varieties of lines and conics (with respect to the complex structure of V) contained in V have the expected dimensions when V is general [B].

In the case $e > 3$, we have $b_1 = b_e = l_0$. By proposition 2 in [B], we have

$$a_1 = a_{n-e+3} = l_0.$$

By the corollary of proposition 3 in [B], we have $c_1 = \frac{1}{2} l_0^2$. Therefore

$$\begin{aligned} c_1^* - a_1^* b_e - a_{n-e+3}^* b_1 - b_1 b_e &= c_1^* - 2a_1^* b_1 - b_1^2 \\ &= -(\delta^{-1} - \frac{1}{2}) l_0^2 \\ &< 0, \end{aligned}$$

because $0 < \delta < 1$. As a result, $\det\, A(t) \neq 0 \mod (t^2)$.

In case $e = 3$, $a_1 = a_{n-e+3} = a_n = 0$, we have

$$\begin{aligned} c_1^* - a_1^* b_e - a_{n-e+3}^* b_1 - b_1 b_e &= (\delta^{-1} - \frac{3}{2}) l_0^2 \\ &= (\frac{n-1}{n-3} - \frac{3}{2}) l_0^2 \\ &\neq 0 \end{aligned}$$

except $n = 7$ and $e = 3$. Again, we have $\det A(t) \neq 0 \mod (t^2)$.

Finally, we give

Proof of Theorem 1. By the above computation, we have

$$\begin{aligned} G(0, \cdots, 0, t_{2e-3}, 0, \cdots, 0) &= \det(-A(0, \cdots, 0, t_{2e-3}, 0, \cdots, 0)) \\ &= \det(-A(t)) \\ &\neq 0 \qquad \mod (t_{2e-3}^2). \end{aligned}$$

Therefore, Lemma 1 and 2 implies that the polynomial $G(\lambda, 0, \cdots, 0, t_{2e-3}, 0, \cdots, 0) = 0$ has distinct roots for generic $t_{2e-3} \in \mathbb{C}$. However, being semi-simple is an open conditions, we conclude that the polynomial

$$G(\lambda, t_1, \cdots, t_{n+1}) = 0$$

has distinct roots for generic $(t_1, \cdots, t_{n+1}) \in \mathbb{C}^{n+1}$, that is, $X(w) \bullet_w$ is semi-simple for generic w.

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Department of Mathematics, M. I. T., Cambridge, MA 02139
E-mail address: tian@math.mit.edu

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218
E-mail address: geng@math.jhu.edu