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HARDY'S INEQUALITIES FOR SOBOLEV FUNCTIONS

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Abstract. The fractional maximal function of the gradient gives a pointwise interpretation of Hardy's inequality for functions $u \in W_0^{1,p}(\Omega)$. With mild assumptions on Ω Hardy's inequality holds for a function $u \in W^{1,p}(\Omega)$ if and only if $u \in W_0^{1,p}(\Omega)$.

1. Introduction

The fractional maximal function of a locally integrable function $f\colon \mathbb{R}^n \to [-\infty,\infty]$ is defined by

$$\mathcal{M}_\alpha f(x) = \sup_{r>0} r^{\alpha-n} \int_{B(x,r)} |f(y)|\,dy, \qquad 0 \le \alpha \le n. \tag{1.1}$$

The fractional maximal function is a classical tool in harmonic analysis, but it is also useful in studying Sobolev functions and partial differential equations. The fundamental fact is that the oscillation of a Sobolev function is controlled by the fractional maximal function of the gradient. To be more precise, suppose that $u \in W^{1,p}(\mathbb{R}^n)$ and let $0 \le \alpha < p$. Then there is a constant c, depending only on n, such that

$$|u(x)-u(y)| \le c\,|x-y|^{1-\alpha/p}\big(\mathcal{M}_{\alpha/p}|Du|(x) + \mathcal{M}_{\alpha/p}|Du|(y)\big) \tag{1.2}$$

for every $x, y \in \mathbb{R}^n \setminus N$ with $|N| = 0$. The proof of this elegant inequality essentially is due to Hedberg [He]. Recently Bojarski and Hajłasz have employed (1.2) in studying Sobolev functions, see [BH], [Ha1] and [Ha3]. Lewis [Le1] has also used (1.2) to construct a Lipschitz continuous test function for elliptic systems of partial differential equations. Several properties of Sobolev functions, including pointwise behaviour, approximation by Hölder continuous functions, Sobolev's, Poincaré's and Morrey's lemmas, follow from (1.2). We discuss some of these applications in Section 2.

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In Section 3 we apply (1.2) to Sobolev functions with zero boundary values. To this end, let Ω be an open set such that $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat. This means that there is a constant $\gamma > 0$ such that

$$\mathrm{C}_p\left((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x,r), B(x,2r)\right) \geq \gamma\, \mathrm{C}_p\left(\overline{B}(x,r), B(x,2r)\right)$$

for every $x \in \mathbb{R}^n \setminus \Omega$ and for all radii $r > 0$. Here C_p refers to the ordinary variational $p-$capacity, see [HKM, Ch. 2]. This requirement is not very restrictive, since all Lipschitz domains or domains satisfying the exterior cone condition are uniformly $p-$fat for every p, $1 < p < \infty$. If $p > n$, then the complement of every open set $\Omega \neq \mathbb{R}^n$ is uniformly $p-$fat. Suppose that $u \in W_0^{1,p}(\Omega)$. If $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat and $0 \leq \alpha < p$, then we show that there is a constant c, depending only on p and n and the geometry of Ω, such that

$$|u(x)| \leq c\, \mathrm{dist}(x, \partial\Omega)^{1-\alpha/p} \big(\mathcal{M}_\alpha |Du|^p(x)\big)^{1/p} \tag{1.3}$$

for almost every $x \in \Omega$. In particular, if $p > n$, then (1.3) holds provided $\Omega \neq \mathbb{R}^n$. By integrating (1.3) over Ω and using the Hardy–Littlewood–Wiener maximal theorem we obtain a new proof for Hardy's inequality

$$\int_\Omega \Big(\frac{|u(x)|}{\mathrm{dist}(x,\partial\Omega)}\Big)^p\, dx \leq c \int_\Omega |Du(x)|^p\, dx. \tag{1.4}$$

The constant c depends only on p and n and the geometry of Ω. It has come to our attention that Piotr Hajłasz [Ha2] has obtained another proof for Hardy's inequality (1.4) using similar ideas.

We also show that Hardy's inequality is a necessary and sufficient condition for a function in $W^{1,p}(\Omega)$ to belong to $W_0^{1,p}(\Omega)$ provided $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat. In fact, we prove a stronger result which generalizes results in [K, p. 74] and [EE, p. 223]. Hardy's inequalities have been studied extensively under various conditions on Ω. The classical one–dimensional Hardy's inequality can be found in [HLP, p. 240]. Higher dimensional versions have been investigated in [A], [EH], [Le2], [K], [Maz, 2.3.3], [Mi], [OK], [W1] and [W2].

Our notation is standard. The function u belongs to the Sobolev space $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, $1 \leq p \leq \infty$, provided $u \in L^p(\Omega)$ and the first weak partial derivatives also belong to $L^p(\Omega)$. We endow the Sobolev space $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{1,p,\Omega} = \|u\|_{p,\Omega} + \|Du\|_{p,\Omega}.$$

If $\Omega = \mathbb{R}^n$, we denote $\|u\|_{1,p,\mathbb{R}^n} = \|u\|_{1,p}$. We recall that $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{1,p,\Omega}$. For properties of the Sobolev functions we refer to the monograph [Z]. Various positive constants throughout the paper are denoted by c and they may differ even on the same line. The dependence on parameters is expressed, for example, by $c = c(n,p)$.

2. Maximal function inequalities

For $0 \le \alpha \le n$ the fractional maximal function of a locally integrable function $f\colon \mathbb{R}^n \to [-\infty, \infty]$ is defined by

$$\mathcal{M}_\alpha f(x) = \sup_{r>0} r^{\alpha-n} \int_{B(x,r)} |f(y)|\, dy.$$

For $\alpha = 0$ we obtain the Hardy–Littlewood maximal function and we write $\mathcal{M}_0 = \mathcal{M}$. The set

$$E_\lambda = \big\{ x \in \mathbb{R}^n \colon \mathcal{M}_\alpha f(x) > \lambda \big\}, \qquad \lambda \ge 0,$$

is open and hence the fractional maximal function is lower semicontinuous. Next we give some estimates for the measure of E_λ. To this end, the Hausdorff $d-$measure, $0 < d < \infty$, of $E \subset \mathbb{R}^n$ is

$$\mathcal{H}^d(E) = \lim_{\delta \to 0} \mathcal{H}^d_\delta(E),$$

where

$$\mathcal{H}^d_\delta(E) = \inf \Big\{ \sum_{i=1}^\infty r_i^d \colon E \subset \bigcup_{i=1}^\infty B(x_i, r_i),\, r_i \le \delta \Big\}, \qquad 0 < \delta \le \infty.$$

$\mathcal{H}^d_\infty(E)$ is the Hausdorff $d-$content of E. Clearly $\mathcal{H}^d_\infty(E) \le \mathcal{H}^d_\delta(E) \le \mathcal{H}^d(E)$ for every $0 < \delta \le \infty$.

The standard Vitali covering argument, see [BZ, Lemma 3.2], yields the following weak type inequality for the fractional maximal function.

2.1. Lemma. *Suppose that $f \in L^1(\mathbb{R}^n)$ and $0 \le \alpha < n$. Then there is a constant $c = c(n, \alpha)$ such that*

$$\mathcal{H}^{n-\alpha}_\infty(E_\lambda) \le \frac{c}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, dx, \qquad \lambda > 0. \tag{2.2}$$

If $\alpha = 0$, then the Hausdorff content may be replaced by the Lebesgue measure.

Next we recall a pointwise inequality for a smooth function in terms of the fractional maximal function. This estimate is well–known, the proof relies on an argument due to Hedberg, see [He] and [Ha1].

2.3. Theorem. *Suppose that $u \in C^\infty(\mathbb{R}^n)$ and let $1 \le p < \infty$, $0 \le \alpha < p$. Let χ be the characteristic function of the ball $B(x_0, R)$. Then there is a constant $c = c(n)$ such that*

$$|u(x) - u(y)| \le c\, |x - y|^{1-\alpha/p} \big(\mathcal{M}_{\alpha/p}(|Du|\chi)(x) + \mathcal{M}_{\alpha/p}(|Du|\chi)(y) \big) \tag{2.4}$$

for every $x, y \in B(x_0, R)$.

Sketch of the proof. The proof follows easily from inequalities

$$\begin{aligned}\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} |u(x)-u(y)|\,dy &\le c(n)\int_{B(x_0,R)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \\ &\le c(n)\,R^{1-\alpha/p}\mathcal{M}_{\alpha/p}(|Du|\chi)(x),\end{aligned} \tag{2.5}$$

for every $x \in B(x_0, R)$.

Inequality (2.4) has turned out to be very useful in studying the Sobolev spaces. We take an opportunity to briefly describe some of the main developments here; some of our observations are folklore, but there are also new aspects.

Since smooth functions are dense in $W^{1,p}(\Omega)$, we see that for an arbitrary Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$ inequality (2.4) holds whenever $x, y \in B(x_0, R) \setminus N$ with $|N| = 0$. In this case we say that (2.4) holds for almost every x and y.

The Sobolev embedding theorems and the Poincaré inequalities are easy consequences of (2.4), see [He] and [Z]. Suppose that $u \in W^{1,p}(\mathbb{R}^n)$. Hölder's inequality implies

$$\mathcal{M}_{\alpha/p}|Du|(x) \le c(n,p)\,\big(\mathcal{M}_\alpha|Du|^p(x)\big)^{1/p}, \qquad x \in \mathbb{R}^n. \tag{2.6}$$

If $n < p < \infty$, then we may take $\alpha = n$ and

$$\big(\mathcal{M}_n|Du|^p(x)\big)^{1/p} \le \|Du\|_p < \infty, \qquad x \in \mathbb{R}^n. \tag{2.7}$$

By combining this observation to (2.4) we see that

$$|u(x)-u(y)| \le c(n,p)\|Du\|_p\,|x-y|^{1-n/p} \tag{2.8}$$

for almost every $x, y \in \mathbb{R}^n$. This shows that $u \in C^{1-n/p}(\mathbb{R}^n)$ after redefinition on a set of measure zero. This is the Sobolev embedding theorem in the case $n < p < \infty$.

Suppose that $1 \le p \le n$ and $0 \le \alpha < \min(p, n)$. If $\mathcal{M}_\alpha|Du|^p$ is bounded, then after redefinition on a set of measure zero, $u \in C^{1-\alpha/p}(\mathbb{R}^n)$ by (2.4). This is Morrey's lemma [Mo, Theorem 3.5.2]. Even if $\mathcal{M}_\alpha|Du|^p$ is unbounded, then

$$|u(x)-u(y)| \le c(n,p)\lambda\,|x-y|^{1-\alpha/p}, \tag{2.9}$$

for almost every $x, y \in \mathbb{R}^n \setminus E_\lambda$, where

$$E_\lambda = \big\{x \in \mathbb{R}^n \colon \mathcal{M}_\alpha|Du|^p(x) > \lambda^p\big\}, \qquad \lambda \ge 0.$$

This means that the function $u \in W^{1,p}(\mathbb{R}^n)$ is Hölder continuous in the set $\mathbb{R}^n \setminus E_\lambda$ after redefinition on a set of measure zero. Then we may extend it to a Hölder continuous function in $\mathbb{R}^n$ with the same exponent and the same constant by defining

$$u_\lambda(x) = \inf_{y\in\mathbb{R}^n\setminus E_\lambda} \big(u(y) + c(n,p)\lambda\,|x-y|^{1-\alpha/p}\big), \qquad x \in \mathbb{R}^n.$$

Here $c(n,p)$ is the constant in (2.9). This is the classical McShane extension [Mc] of $u|_{\mathbb{R}^n\setminus E_\lambda}$. Using estimate (2.2) we see that for every $\lambda > 0$ there is an open set E_λ and a function u_λ such that

(1) $u_\lambda \in C^{1-\alpha/p}(\mathbb{R}^n)$,
(2) $u_\lambda(x) = u(x)$ for a.e. $x \in \mathbb{R}^n \setminus E_\lambda$,
(3) $\mathcal{H}^{n-\alpha}_\infty(E_\lambda) \le c(n,\alpha)\lambda^{-p}\,\|Du\|_p^p$.

In particular, if $\alpha = 0$, then the function u_λ is Lipschitz continuous. This has been previously studied by Malý [Mal]. His proof is based on the representation of Sobolev functions as Bessel potentials.

Usually a Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$ is defined only up to a set of measure zero, but following [BH] we define u pointwise by

$$\widetilde{u}(x) = \limsup_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy. \tag{2.10}$$

By Lebesgue's theorem not only limit superior but limit exists and equals to u almost everywhere. In fact, the limit exists except on a set of capacity zero, but we do not need this refinement here. Hence the pointwise definition coincides with u almost everywhere and represents the same element in $W^{1,p}(\mathbb{R}^n)$. Then we may proceed as in [BH] and show that inequality (2.4) holds everywhere for every function $u \in W^{1,p}(\mathbb{R}^n)$ defined pointwise by (2.10). If $p > n$, then by (2.7) every $\widetilde{u} \in W^{1,p}(\mathbb{R}^n)$ belongs to $C^{1-n/p}(\mathbb{R}^n)$ and hence all points are Lebesgue points for $\widetilde{u}$. Suppose then that $1 \le p \le n$ and $0 \le \alpha < p$. Using (2.4) it is easy to see that x is a Lebesgue point of $\widetilde{u}$ provided $\mathcal{M}_{\alpha/p}|Du|(x) < \infty$. This observation is a sharpening of [BH, Lemma 4]. The set of non-Lebesgue points of $\widetilde{u}$ is contained in

$$E_\infty = \big\{x \in \mathbb{R}^n \colon \mathcal{M}_\alpha|Du|^p(x) = \infty\big\}.$$

The weak type estimate (2.2) implies that

$$\mathcal{H}^{n-\alpha}_\infty(E_\infty) \le c(n,\alpha)\lambda^{-p}\,\|Du\|_p^p, \qquad \lambda > 0.$$

Letting $\lambda \to \infty$ we see that $\mathcal{H}^{n-\alpha}_\infty(E_\infty) = 0$. Hence the set of non-Lebesgue points has $(n-\alpha)-$Hausdorff content zero for every $\alpha < p$. Since the Hausdorff content and the Hausdorff measure have the same null sets, also the $(n-\alpha)-$dimensional Hausdorff measure of E_∞ is zero. In general, a function $u \in W^{1,p}(\mathbb{R}^n)$ has Lebesgue points outside a set of capacity zero, which is of course a stronger result, but our approach gives a concrete method to check whether a given point is a Lebesgue point.

For further applications of (2.4) we refer to [BH], [Ha1], [Ha3] and [Le1].

3. Hardy's inequality

Suppose first that $p > n$, $n < q < p$, $0 \le \alpha < q$, and let $\Omega \ne \mathbb{R}^n$ be an open set. Let $u \in C_0^\infty(\Omega)$. Fix $x \in \Omega$ and take $x_0 \in \partial\Omega$ such that $|x - x_0| = \operatorname{dist}(x, \partial\Omega) = R$. Then we use (2.4) together with (2.6) and obtain

$$
\begin{aligned}
|u(x)| &\le c(n,q)\,|x-x_0|^{1-n/q}\Big(\int_{B(x,2R)} |Du(y)|^q\,dy\Big)^{1/q} \\
&\le c(n,q)\,R^{1-\alpha/q}\Big(R^{\alpha-n}\int_{B(x,2R)} |Du(y)|^q\,dy\Big)^{1/q} \\
&\le c(n,q)\operatorname{dist}(x,\partial\Omega)^{1-\alpha/q}\big(\mathcal{M}_\alpha|Du|^q(x)\big)^{1/q}.
\end{aligned}
\tag{3.1}
$$

For $u \in W_0^{1,p}(\Omega)$ inequality (3.1) holds almost everywhere. Integrating (3.1) with $\alpha = 0$ over Ω and using the Hardy–Littlewood–Wiener maximal function theorem [S, Theorem I.1] we arrive at

$$
\begin{aligned}
\int_\Omega \Big(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)}\Big)^p dx &\le c\int_\Omega \big(\mathcal{M}|Du|^q(x)\big)^{p/q}\,dx \\
&\le c\int_\Omega |Du(x)|^p\,dx, \qquad c = c(n,p,q).
\end{aligned}
\tag{3.2}
$$

Since the constant in (3.2) is independent of $u \in C_0^\infty(\Omega)$, a simple approximation argument shows that (3.2) holds for every $u \in W_0^{1,p}(\Omega)$.

This gives a proof for the well–known Hardy's inequality for all open sets with non-empty complements if $n < p < \infty$. The case $1 < p \le n$ is more involved, since then extra conditions must be imposed on Ω, see [Le2, Theorem 3]. However, there is a sufficient condition in terms of the capacity density of the complement.

3.3. Definition. A closed set $E \subset \mathbb{R}^n$ is uniformly p–fat, $1 < p < \infty$, if there is a constant $\gamma > 0$ such that

$$
\mathrm{C}_p\big(E \cap \overline{B}(x,r), B(x,2r)\big) \ge \gamma\,\mathrm{C}_p\big(\overline{B}(x,r), B(x,2r)\big)
\tag{3.4}
$$

for every $x \in E$ and for all radii $r > 0$.

Here $\mathrm{C}_p(K,\Omega)$ denotes the variational p–capacity

$$
\mathrm{C}_p(K,\Omega) = \inf\Big\{\int_\Omega |Du(x)|^p\,dx \colon\ u \in C_0^\infty(\Omega),\ u(x) \ge 1 \text{ for } x \in K\Big\}
$$

of the condenser (K,Ω). Here Ω is open and K is a compact subset of Ω. For information on the capacity we refer to [HKM, Ch. 2]. We recall that

$$
\mathrm{C}_p\big(\overline{B}(x,r), B(x,2r)\big) = c(n,p)\,r^{n-p}.
\tag{3.5}
$$

3.6. Examples.

(1) If $p > n$, then all non–empty closed sets are uniformly $p-$fat.
(2) All closed sets satisfying the interior cone condition are uniformly $p-$fat for every p, $1 < p < \infty$.
(3) The complements of the Lipschitz domains are uniformly $p-$fat for every p, $1 < p < \infty$.
(4) If there is a constant $\gamma > 0$ such that

$$|B(x,r) \cap E| \geq \gamma\, |B(x,r)|$$

for every $x \in E$ and $r > 0$, then E is uniformly $p-$fat for every p, $1 < p < \infty$.

The fundamental property of uniformly fat sets is the following deep result due to Lewis [Le2, Theorem 1]. For another proof see [Mi, Theorem 8.2].

3.7. Theorem. *Let $E \subset \mathbb{R}^n$ be a closed uniformly $p-$fat set. Then there is q, $1 < q < p$ such that E is uniformly $q-$fat.*

If $\Omega \subset \mathbb{R}^n$ is an open set such that $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat, then Lewis proved [Le2, Theorem 2] that Hardy's inequality holds. We have already seen that Hardy's inequality follows from pointwise inequalities involving the Hardy–Littlewood maximal function if $p > n$. We show that this is the case also when $1 < p \leq n$. In the proof we need the following version of Poincaré's inequality. For the proof see [Mi, Lemma 8.11], [KK, 3.1] or [Maz, Ch. 10].

3.8. Lemma. *Suppose that $1 < p < \infty$, let $u \in C^\infty(\mathbb{R}^n)$ and denote*

$$Z = \big\{x \in \mathbb{R}^n \colon u(x) = 0\big\}.$$

Then there is $c = c(n,p)$ such that

$$\begin{aligned}\Big(\frac{1}{|B(x_0,2R)|}\int_{B(x_0,R)} |u(y)|^p\,dy\Big)^{1/p} \\ \leq c\,\Big(\mathrm{C}_p\,\big(Z \cap \overline{B}(x_0,R), B(x_0,2R)\big)^{-1}\int_{B(x_0,2R)} |Du(y)|^p\,dy\Big)^{1/p}.\end{aligned}$$

Now we are ready to prove a pointwise Hardy's inequality.

3.9. Theorem. *Let $1 < p \leq n$, $0 \leq \alpha < p$, let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat and suppose that $u \in C_0^\infty(\Omega)$. Then there is a constant c, depending only on p, n and γ in (3.4), such that*

$$|u(x)| \leq c\,\operatorname{dist}(x,\partial\Omega)^{1-\alpha/p}\big(\mathcal{M}_\alpha|Du|^p(x)\big)^{1/p}, \qquad x \in \Omega. \tag{3.10}$$

Proof. Let $x \in \Omega$ and choose $x_0 \in \partial\Omega$ such that $|x-x_0| = \operatorname{dist}(x,\partial\Omega) = R$. We define u and Du to be zero in the complement of Ω. Then by (2.5) and (2.6) we have

$$|u(x) - u_{B(x_0,2R)}| \le c(n,p)\, R^{1-\alpha/p}\big(\mathcal{M}_\alpha |Du|^p(x)\big)^{1/p}, \qquad x \in B(x_0,2R),$$

and hence

$$\begin{aligned}|u(x)| &\le |u(x) - u_{B(x_0,2R)}| + |u_{B(x_0,2R)}| \\ &\le c\, R^{1-\alpha/p}\big(\mathcal{M}_\alpha |Du|^p(x)\big)^{1/p} + |u|_{B(x_0,2R)}, \qquad x \in B(x_0,2R).\end{aligned}$$

Here we use the familiar notation

$$u_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy.$$

Using Lemma 3.8, (3.4) and (3.5) we arrive at

$$\begin{aligned}&\frac{1}{|B(x_0,2R)|}\int_{B(x_0,2R)} |u(y)|\,dy \\ &\qquad \le c\Big(\operatorname{C}_p\big(Z\cap \overline{B}(x_0,2R), B(x_0,4R)\big)^{-1}\int_{B(x_0,4R)} |Du(y)|^p\,dy\Big)^{1/p} \\ &\qquad \le c\Big(\operatorname{C}_p\big((\mathbb{R}^n\setminus\Omega)\cap \overline{B}(x_0,2R), B(x_0,4R)\big)^{-1}\int_{B(x_0,4R)} |Du(y)|^p\,dy\Big)^{1/p} \\ &\qquad \le c\Big(R^{p-n}\int_{B(x,8R)} |Du(y)|^p\,dy\Big)^{1/p} \\ &\qquad \le c\, R^{1-\alpha/p}\big(\mathcal{M}_\alpha |Du|^p(x)\big)^{1/p}, \qquad c = c(n,p,\gamma).\end{aligned}$$

This completes the proof.

If $\mathbb{R}^n\setminus\Omega$ is $p-$fat, then by Theorem 3.7 it is $q-$fat for some $1<q<p\le n$. Using (3.10) with $\alpha = 0$ we get

$$|u(x)| \le c(n,q)\operatorname{dist}(x,\partial\Omega)\big(\mathcal{M}|Du|^q(x)\big)^{1/q}, \qquad x\in\Omega.$$

Integrating and using the Hardy–Littlewood–Wiener theorem exactly the same way as in (3.2), we get the proof for Hardy's inequality also in the case $1<p\le n$. Again, an approximation argument shows that Hardy's inequality holds for every $u\in W_0^{1,p}(\Omega)$. Observe that the existence of $q<p$ given by Theorem 3.7 is essential here, since the Hardy–Littlewood–Wiener theorem does not hold in $L^1(\mathbb{R}^n)$. Thus we have given a new proof for the following result.

3.11. Corollary. *Let $1 < p < \infty$ and suppose that $\Omega \subset \mathbb{R}^n$ is an open set such that $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat. If $u \in W_0^{1,p}(\Omega)$, then there is a constant c, depending on p, n and γ in (3.4), such that*

$$\int_\Omega \Big(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)}\Big)^p\,dx \le c\int_\Omega |Du(x)|^p\,dx. \tag{3.12}$$

In particular, if $p > n$, then (3.12) holds provided $\Omega \neq \mathbb{R}^n$.

For different proofs of Corollary 3.11 we refer to [A] (in the case $p = n = 2$) [Le2] and [Mi]. If $p = n$, then Hardy's inequality is equivalent to the fact that $\mathbb{R}^n \setminus \Omega$ is uniformly $p-$fat, see [Le2, Theorem 3]. Another necessary and sufficient condition is given in [Maz 2.3.3].

It is known that $u \in W^{1,p}(\Omega)$ belongs to $W_0^{1,p}(\Omega)$ if $u(x)/\operatorname{dist}(x,\partial\Omega) \in L^p(\Omega)$ without any restrictions on Ω, see [EE, p. 223]. In particular, if Hardy's inequality holds, this is true. We improve this result here. For this end, we record that a function $f\colon \Omega \to [-\infty,\infty]$ belongs to the weak $L^p(\Omega)$ if there is a constant c so that

$$\big|\{x \in \Omega\colon |f(x)| > \lambda\}\big| \le c\,\lambda^{-p}, \qquad \lambda > 0.$$

3.13. Theorem. *Let Ω be an open set and suppose that $u \in W^{1,p}(\Omega)$ with $1 < p < \infty$. Then $u \in W_0^{1,p}(\Omega)$ provided $u(x)/\operatorname{dist}(x,\partial\Omega)$ belongs to the weak $L^p(\Omega)$.*

Proof. First we suppose that Ω is bounded. Then it is easy to see that $u(x)/\operatorname{dist}(x,\partial\Omega)$ belongs to $L^q(\Omega)$ for every $q < p$. Using Theorem 3.4 in [EE, p. 223] we conclude that $u \in W_0^{1,q}(\Omega)$ for every $q < p$. In particular, this implies that u has the generalized gradient Du in $\mathbb{R}^n$. Moreover, $u(x) = 0$ and $Du(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. We define u and Du to be zero in $\mathbb{R}^n \setminus \Omega$ and the above discussion implies that $u \in W^{1,p}(\mathbb{R}^n)$. One can also check directly, without using the result in [EE], that the zero extension to the complement belongs to $W^{1,p}(\mathbb{R}^n)$. This can be done, for example, by using the ACL–characterization of the Sobolev functions.

Let $N \subset \mathbb{R}^n$, $|N| = 0$, be the exceptional set for inequality (2.4) and denote

$$\begin{aligned} F_\lambda = \big\{x \in \Omega \setminus N\colon |u(x)| \le \lambda,\ \mathcal{M}|Du|^p(x) \le \lambda^p \\ \text{and } |u(x)|/\operatorname{dist}(x,\partial\Omega) \le \lambda\big\} \cup (\mathbb{R}^n \setminus \Omega), \qquad \lambda > 0. \end{aligned} \tag{3.14}$$

We show that $u|_{F_\lambda}$ is $c(n,p)\lambda-$Lipschitz continuous.

Suppose that $x, y \in \Omega \cap F_\lambda$. Then by (2.4) and (3.14) we get

$$\begin{aligned} |u(x) - u(y)| &\le c(n,p)\,|x-y|\big((\mathcal{M}|Du|^p(x))^{1/p} + (\mathcal{M}|Du|^p(y))^{1/p}\big) \\ &\le c(n,p)\lambda\,|x-y|. \end{aligned}$$

If $x \in \Omega \cap F_\lambda$ and $y \in \mathbb{R}^n \setminus \Omega$, we have

$$|u(x) - u(y)| = |u(x)| \le \lambda \operatorname{dist}(x, \partial\Omega) \le \lambda\, |x - y|.$$

If $x, y \in \mathbb{R}^n \setminus \Omega$, then the claim is clear. Since all the other cases follow by symmetry, it follows that $u|_{F_\lambda}$ is Lipschitz continuous with the constant $c(n,p)\lambda$.

Then we extend $u|_{F_\lambda}$ to a Lipschitz continuous function $\overline{u}_\lambda$ to $\mathbb{R}^n$ with the same constant by defining

$$\overline{u}_\lambda(x) = \inf_{y \in F_\lambda} \bigl(u(y) + c(n,p)\lambda\, |x - y|\bigr)$$

and finally we set $u_\lambda(x) = \operatorname{sgn} u(x) \min(|\overline{u}_\lambda(x)|, \lambda)$. This is a slight modification of the classical McShane extension [Mc]. Then u_λ enjoys the following properties:

(1) $u_\lambda(x) = u(x)$ for every $x \in F_\lambda$.
(2) $|u_\lambda(x)| \le \lambda$ for every $x \in \mathbb{R}^n$.
(3) $|Du_\lambda(x)| \le c(n,p)\lambda$ for every $x \in \mathbb{R}^n$.
(4) $u_\lambda(x) = 0$ when $x \in \mathbb{R}^n \setminus \Omega$.
(5) $Du_\lambda(x) = Du(x)$ for almost every $x \in F_\lambda$.

We write

$$\begin{aligned} F_\lambda^1 &= \bigl\{x \in \Omega \colon |u(x)| > \lambda\bigr\}, \\ F_\lambda^2 &= \bigl\{x \in \Omega \colon \mathcal{M}|Du|^p(x) > \lambda^p\bigr\}, \\ F_\lambda^3 &= \bigl\{x \in \Omega \colon |u(x)|/\operatorname{dist}(x, \partial\Omega) > \lambda\bigr\}. \end{aligned}$$

Then

$$\begin{aligned} \int_\Omega |Du_\lambda(x)|^p \, dx \le \int_{F_\lambda \cap \Omega} |Du(x)|^p \, dx &+ \int_{F_\lambda^1} |Du_\lambda(x)|^p \, dx \\ + \int_{F_\lambda^2} |Du_\lambda(x)|^p \, dx &+ \int_{F_\lambda^3} |Du_\lambda(x)|^p \, dx. \end{aligned} \tag{3.15}$$

We estimate the integrals on the right side separately. Using the fact that $|Du_\lambda(x)| \le c(n,p)\lambda$ we see that

$$\begin{aligned} \int_{F_\lambda^1} |Du_\lambda(x)|^p \, dx &\le c(n,p)\lambda^p \, \bigl|\{x \in \Omega \colon |u(x)| > \lambda\}\bigr| \\ &\le c(n,p) \int_\Omega |u(x)|^p \, dx \end{aligned}$$

and by the weak type inequality (2.2) we find

$$\begin{aligned} \int_{F_\lambda^2} |Du_\lambda(x)|^p \, dx &\le c(n,p)\lambda^p \, \bigl|\{x \in \mathbb{R}^n \colon \mathcal{M}|Du|^p(x) > \lambda^p\}\bigr| \\ &\le c(n,p) \int_\Omega |Du(x)|^p \, dx. \end{aligned}$$

Finally, using the definition of F^3_λ and the hypothesis that $u(x)/\operatorname{dist}(x,\partial\Omega)$ belongs to the weak $L^p(\Omega)$ we obtain

$$\int_{F^3_\lambda}|Du_\lambda(x)|^p\,dx\le c(n,p)\lambda^p\left|\{x\in\Omega\colon |u(x)|/\operatorname{dist}(x,\partial\Omega)>\lambda\}\right|\le c(n,p,u)<\infty.$$

The obtained estimates and (3.15) imply

$$\begin{aligned}&\int_\Omega|Du_\lambda(x)|^p\,dx\\ &\le c\int_\Omega|u(x)|^p\,dx+c\int_\Omega|Du(x)|^p\,dx+c,\qquad c=c(n,p,u),\end{aligned}\tag{3.16}$$

for every $\lambda>0$ and hence the family Du_λ is bounded in $L^p(\Omega)$ uniformly in λ. Using the weak type inequality (2.2) in the same manner as in estimating the gradient we see that also u_λ is bounded in $L^p(\Omega)$ uniformly in λ and

$$\int_\Omega|u_\lambda(x)|^p\,dx\le c\int_\Omega|u(x)|^p\,dx+c\int_\Omega|Du(x)|^p\,dx+c,\qquad c=c(n,p,u),\tag{3.17}$$

for every $\lambda>0$. Since u_λ is Lipschitz and $u_\lambda(x)=0$ when $x\in\mathbb{R}^n\setminus\Omega$ we conclude that $u_\lambda\in W^{1,p}_0(\Omega)$ for every $\lambda>0$. Estimates (3.16) and (3.17) show that u_λ is uniformly bounded family in $W^{1,p}_0(\Omega)$ and

$$\|u_\lambda\|_{1,p,\Omega}\le c(n,p)\,\|u\|_{1,p,\Omega}+c(n,p,u)$$

for every $\lambda>0$. Finally, since $|\Omega\setminus F_\lambda|\to 0$ as $\lambda\to\infty$ and u_λ coincides with u in F_λ, we see that $u_\lambda\to u$ almost everywhere in Ω. Since u_λ is uniformly bounded family in $W^{1,p}_0(\Omega)$ and $u_\lambda\to u$ almost everywhere in Ω, a weak compactness argument shows that $u\in W^{1,p}_0(\Omega)$.

Suppose then that Ω is unbounded. Let $x_0\in\partial\Omega$. Choose a cut–off function $\phi_i\in C^\infty_0(\Omega)$, $i=1,2,\dots$, be such that $0\le\phi_i(x)\le 1$ for every $x\in\mathbb{R}^n$, $\phi_i(x)=1$ if $x\in B(x_0,i)$, $\phi_i(x)=0$ if $x\in\mathbb{R}^n\setminus B(x_0,2i)$ and $|D\phi_i(x)|\le c$ with c independent of i. Let $v_i=\phi_i u$ and denote $\Omega_i=\Omega\cap B(x_0,4i)$ for $i=1,2,\dots$ Then $v_i\in W^{1,p}(\Omega)$ and $v_i\to u$ in $W^{1,p}(\Omega)$ as $i\to\infty$. Clearly $|v_i(x)|/\operatorname{dist}(x,\partial\Omega_i)$ belongs to the weak $L^p(\Omega_i)$, $i=1,2,\dots$ Since Ω_i is bounded, we obtain $v_i\in W^{1,p}_0(\Omega_i)$ and hence $v_i\in W^{1,p}_0(\Omega)$. Since $v_i\to u$ in $W^{1,p}(\Omega)$, we conclude that $u\in W^{1,p}_0(\Omega)$. This completes the proof.

3.18. *Remark.* It is known, [K, p. 74], that if Ω is a Lipschitz-domain, then $u\in W^{1,p}(\Omega)$ belongs to $W^{1,p}_0(\Omega)$ if and only if $u(x)/\operatorname{dist}(x,\partial\Omega)$ belongs to $L^p(\Omega)$. We show that the condition on Ω can be considerably weakened.

Let $p>n$, $\Omega\ne\mathbb{R}^n$ and $u\in W^{1,p}(\Omega)$. Then by (3.2) and Theorem 3.13 $u\in W^{1,p}_0(\Omega)$ if and only if u satisfies Hardy's inequality with the exponent p.

If $1<p\le n$ and $\mathbb{R}^n\setminus\Omega$ is uniformly p–fat, then $u\in W^{1,p}(\Omega)$ belongs to $W^{1,p}_0(\Omega)$ if and only if $u(x)/\operatorname{dist}(x,\partial\Omega)$ is in weak $L^p(\Omega)$.

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