

## A LEFSCHETZ FORMULA FOR FLOWS

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### Introduction

Let  $M$  be a smooth compact manifold and  $f : M \rightarrow M$  a diffeomorphism with nondegenerate fixed points. Suppose we are given a flat vector bundle  $E \rightarrow M$  and that  $f$  lifts linearly to  $E$ . Then  $f$  acts by pullback on the  $E$ -valued differential forms and on the de Rham cohomology  $H^\cdot(E)$  with coefficients in  $E$ . The theorem of Atiyah-Bott-Lefschetz now tells us that

$$(1) \quad \sum_{a \in \text{Fix}(f)} \text{ind}_f(a) \text{tr} (f|_{E_a}) = \sum_{j=0}^{\dim M} (-1)^j \text{tr} (f^*|_{H^j(E)}).$$

We want to formulate an analogous statement for a flow  $\Phi$  instead of a diffeomorphism.

By looking at the local side of (1) the first idea would be to consider the fixed points of the flow and to assume they are nondegenerate. This way we get the Hopf formula expressing the Euler characteristic on the global side.

More subtle information is supposed to be obtained by considering the closed orbits of the flow. The non-degeneracy of the latter translates to a hyperbolicity condition of the flow. Assume the flow lifts to the flat bundle  $E$  then on each closed orbit  $c$  with length  $l(c)$  and on any point  $y$  on  $c$  we have a trace  $\text{tr} (\Phi_{l(c)}|_{E_y})$  which is independent of  $y$ . The index is at the first glance to be replaced by the Lefschetz index  $\text{ind}_L(c)$  of the Poincaré map around the orbit, but it turns out that one has to take into account the multiplicity  $\mu(c)$  so we define the Fuller index to be  $\text{ind}_F(c) := \text{ind}_L(c)/\mu(c)$ . Our local side would then be the sum over all closed orbits of  $\text{ind}_F(c) \text{tr} (\Phi_{l(c)}|_E)$ . Unfortunately this sum does not converge. So we have to replace it by a zeta-regularized version  $\hat{\sum}_c \text{ind}_F(c) \text{tr} (\Phi_{l(c)}|_E)$ .

D. Fried [7] has shown that in case of geodesic flows on hyperbolic spaces we have an identity

$$-\hat{\sum}_c \text{ind}_F(c) \text{tr} (\Phi_{l(c)}|_E) = \log \tau(E) \pmod{2\pi i\mathbb{Z}},$$

where  $\tau(E)$  is the analytic torsion of the bundle  $E$ . This formula is an analogue of the Hopf formula but not of the Atiyah-Bott-Lefschetz formula since on the global side the presence of the flow is not visible.

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In this paper we present an analogue of (1). Let  $H$  be the generator of the flow. We define a twisted version  $\bar{H}(\mathcal{C}_\varphi)$  of the reduced tangential cohomology of the stable foliation. Then our Lefschetz formula is

$$\sum_c \widehat{\text{ind}}_F(c) \text{tr} (\Phi_{l(c)}|E) = \sum_{p=0}^{\dim Y - 1} (-1)^{p+1} \log \det(H|\bar{H}^P(\mathcal{C}_\varphi)) \pmod{2\pi i\mathbb{Z}},$$

where  $\det$  means the regularized determinant in the sense of D. Ray and I. Singer [10].

### 1. The main theorem

Let  $X$  denote a compact hyperbolic manifold of dimension  $d$ , i.e.  $X$  is a Riemannian manifold of constant negative curvature which we may normalize to be  $-1$ . We will further restrict to the case that the dimension  $d$  of  $X$  is odd.

Now suppose  $\Phi$  is the geodesic flow of  $X$ . Then  $\Phi$  acts on the sphere bundle  $Y = SX$  of  $X$ . It is known that  $\Phi$  is Anosov, i.e. there is a smooth splitting of the tangent bundle of  $Y$  as

$$TY = T_0 \oplus T_u \oplus T_s,$$

where  $T_0$  is spanned by the flow, the flow acts contractingly on the *stable part*  $T_s$ , as the time tends to  $+\infty$  and contractingly on the *unstable part*  $T_u$ , as the time tends to  $-\infty$ . The bundles  $T_u$  and  $T_s$  are integrable but their sum  $T_s \oplus T_u$  is not.

Let the *periodic set*  $\mathcal{P}$  of  $\Phi$  be the subset of (space  $\times$  time)  $= Y \times (0, \infty)$  consisting of points  $(y, t)$  such that  $\Phi_t y = y$ . For such a point  $(y, t)$  let  $c = \{(\Phi_s y, s) | 0 \leq s \leq t\}$  be the underlying *closed orbit* of  $\Phi$ , then  $l(c) := t$  will be called the *length* or *period* of  $c$ .

For any closed orbit  $c$  of period  $l(c)$  consider the linear Poincaré map

$$P_{c,y} := D\Phi_{l(c)}|_{T_{s,y} \oplus T_{u,y}}$$

for some point  $(y, t)$  in  $c$ . We are only interested in the index of this map, so the dependence on the point will vanish. Let

$$\text{ind}_L(c) := \text{sign} \det(1 - P_{c,y})$$

be the *Lefschetz index* of the orbit  $c$ .

For a closed orbit  $c$  let  $m(c)$  be its *multiplicity*, i.e. the largest natural number  $m$  such that  $t/m$  is a period of  $y$ , where  $(y, t) \in c$ . Now the *Fuller index* of  $c$  is by definition

$$\text{ind}_F(c) := \frac{\text{ind}_L(c)}{m(c)}.$$

The closed orbits of  $\Phi$  project down to closed geodesics on  $X$ . To a geodesic  $c$  we attach its free homotopy class of closed paths  $[c]$  in  $X$ . The set of free homotopy classes of closed paths stands in bijection to the set of conjugacy

classes of the fundamental group  $\Gamma$ . We get a map  $\{\text{closed orbits of } \Phi\} \rightarrow \Gamma/\text{conjugation}$ .

Fix a finite dimensional unitary representation  $(\varphi, V_\varphi)$  of  $\Gamma$ . Via the above map it makes sense to speak of the number  $\text{tr } \varphi(c)$  for a closed orbit  $c$ . There is also a geometric interpretation of  $\text{tr } \varphi(c)$ : the representation  $\varphi$  defines a flat vector bundle  $E_\varphi$  over  $X$  which lifts to a flat vector bundle  $F_\varphi$  over  $Y$ . Each orbit  $c$  lifts via parallel transport to a linear map  $\varphi(c)_y$  on the fibre  $F_{\varphi,y}$  for any point  $y$  on the orbit. Then the trace  $\text{tr } \varphi(c)_y$  does not depend on the point  $y$  and equals the above  $\text{tr } \varphi(c)$ .

In the following we will sometimes assume the representation  $\varphi$  to be *acyclic*, i.e.  $H^q(\Gamma, \varphi) = 0$  for all  $q$ . Under these circumstances we will call the bundle  $E_\varphi$  acyclic as well.

For  $a, b$  nonnegative integers consider the space of smooth sections:

$$\mathcal{C}_\varphi^{a,b} := \Gamma^\infty(\wedge^a T_s^* \otimes \wedge^b T_u^* \otimes F_\varphi).$$

Let the differential  $d_1 : \mathcal{C}_\varphi^{a,b} \rightarrow \mathcal{C}_\varphi^{a+1,b}$  be the projection of the exterior differential given by the flat connection. Let  $d_2 : \mathcal{C}_\varphi^{a,b} \rightarrow \mathcal{C}_\varphi^{a,b+1}$  be the zero differential. Form the corresponding bicomplex  $(\mathcal{C}_\varphi^{*,*}, d_1 + d_2)$  and its total complex

$$\mathcal{C}_\varphi^p := \bigoplus_{a+b=p} \mathcal{C}_\varphi^{a,b}.$$

Let  $H^p(\mathcal{C}_\varphi)$  denote the  $p$ -th cohomology space. Note that all the spaces  $\mathcal{C}_\varphi^p$  carry natural structures of Fréchet spaces. We cannot guarantee that the differentials have closed image so we define the *reduced tangential cohomology*  $\bar{H}^p(\mathcal{C}_\varphi)$ , i.e. the greatest Hausdorff quotient of  $H^p(\mathcal{C}_\varphi)$ .

For any complex  $E = E^0 \rightarrow \dots \rightarrow E^n$  over some additive category let  $\chi(E) := \sum_{p=0}^n (-1)^p H^p(E)$  be the corresponding *Euler object*. Most generally you might interpret it as a virtual object of the category, i.e. an element of the free abelian group generated by isomorphism classes of objects. Since everything we are going to do with the Euler object factors over exact sequences we can also consider it as an element of the Grothendieck group.

For operators on infinite dimensional spaces we have the notion of a *regularized determinant* (see [6]). On a Hilbert space it is defined as follows: Suppose for an operator  $A$  that  $A^{-s}$  is defined for  $\Re(s) \gg 0$  and is of trace class. Suppose further the zeta function  $\zeta_A(s) := \text{tr } (A^{-s})$  extends to a meromorphic function on  $\mathbb{C}$ , holomorphic at  $s = 0$ . Then, extending the finite case, the *determinant* of  $A$  is defined as

$$\det(A) := \exp(-\zeta'_A(0)).$$

Note that by definition the determinant  $\det(A)$  comes with a well defined logarithm:  $\log \det(A) = -\zeta'_A(0)$ .

We would like to form the sum over all closed orbits of the Fuller indices but this sum does not converge. So we define the *regularized sum* as follows:

Consider the sum

$$\zeta_{\Phi, \varphi}(s) := \sum_c \text{ind}_F(c) \text{tr } \varphi(c) e^{-sl(c)},$$

which converges for  $\Re(s) \gg 0$ . It turns out that  $\zeta_{\Phi, \varphi}(s)$  extends to  $\mathbb{C}$  with logarithmic singularities and that in case  $\varphi$  is acyclic,  $\zeta_{\Phi, \varphi}(s)$  is regular at  $s = 0$ . This means that the number

$$\widehat{\sum}_c \text{ind}_F(c) \text{tr } \varphi(c) := \zeta_{\Phi, \varphi}(0)$$

is well defined modulo  $2\pi i\mathbb{Z}$ .

**Theorem 1.1.** (*Lefschetz formula*) *The unit generator  $H$  of the flow  $\Phi$  acts on  $\bar{H}^p(\mathcal{C}_\varphi)$  and for acyclic  $\varphi$  we have*

$$\widehat{\sum}_c \text{ind}_F(c) \text{tr } \varphi(c) = \sum_{p=0}^{\dim Y - 1} (-1)^{p+1} \log \det(H|H^p(\mathcal{C}_\varphi)) \pmod{2\pi i\mathbb{Z}}.$$

Let  $\Delta_{\varphi, p}$  be the  $p$ -th Laplacian for the flat bundle  $E_\varphi$  over  $X$ . Ray and Singer [10] defined the *analytic torsion* of  $\varphi$  to be

$$\tau(\varphi) := \prod_{p=0}^{\dim X} \det(\Delta_{\varphi, p})^{p(-1)^{p+1}}.$$

D. Fried [7] has proven that:

$$\log \tau(\varphi) = \widehat{\sum}_c \text{ind}_F(c) \text{tr } \varphi(c) \pmod{2\pi i\mathbb{Z}}.$$

Putting these things together we conclude

$$\prod_{p=0}^{\dim X} \det(\Delta_{\varphi, p})^{p(-1)^p} = \prod_{q=0}^{2 \dim X - 2} \det(H|\bar{H}^q(\mathcal{C}_\varphi))^{(-1)^{q+1}}.$$

### 2. A determinant formula

In this section we will consider the *Ruelle zeta function*

$$R_\varphi(s) := \prod_{c \text{ prime}} \det \left( 1 - e^{-sl(c)} \varphi(c) \right),$$

where the product is taken over all prime closed geodesics (i.e. those with  $\mu(c) = 1$ ). Note that in the notation of section 1 we have  $\log R_\varphi(s) = \zeta_{\Phi, \varphi}(s)$ .

Our Lefschetz formula will follow from

**Theorem 2.1.** (*determinant formula*) *The function  $R_\varphi(s)$  extends to a meromorphic function on  $\mathbb{C}$ . For  $\varphi$  acyclic it is regular at  $s = 0$  and it satisfies*

$$R_\varphi(s) = e^{P(s)} \prod_{p=0}^{2 \dim X - 2} \det(H + s|\bar{H}^p(\mathcal{C}_\varphi))^{(-1)^p},$$

where  $P(s)$  is an odd polynomial.

The relation (RS) in ([7], p.532) represents  $R_\varphi$  as an alternating product of Selberg zeta functions  $Z_l$ . So our determinant formula will follow from a corresponding one for the Selberg function which we will state in greater generality in the next section.

### 3. A remark on Selberg zeta functions

Let  $\tilde{X}$  be the universal covering of  $X$  and let  $G$  denote the identity component of the isometry group of  $\tilde{X}$ . Then  $G$  is a semisimple Lie group which acts transitively on  $\tilde{X}$ . Let  $\tilde{Y} := S\tilde{X}$  be the sphere bundle of  $\tilde{X}$  then via the differential the action of  $G$  lifts to  $\tilde{Y}$ , since  $X$  is a rank one space this action still is transitive. So let  $M$  be the stabilizer of some point in  $\tilde{Y}$  and  $(\sigma, V_\sigma)$  a finite dimensional representation of  $M$  then  $\sigma$  defines a  $G$ -homogeneous vector bundle  $\tilde{\mathcal{F}}_\sigma$  and all such arise this way. The action of the flow  $\Phi$  on  $\tilde{Y}$  lifts to a  $G$ -equivariant action on the vector bundle and hence pushes down to an action of the induced bundle  $\mathcal{F}_\sigma$  over  $Y$ . For a closed orbit  $c$  and  $(y, t) \in c$  let  $\sigma(c)_y$  be the induced automorphism of the fibre  $\mathcal{F}_{\sigma,y}$ . We will speak of the trace or the determinant of  $\sigma(c)$  since they do not depend on the point  $y$ . Likewise for  $(y, t)$  in  $c$  let  $P_{c,y}^s$  and  $P_{c,y}^u$  be the stable and unstable parts of the linear Poincaré map.

Now we are ready to define the *Selberg zeta function* for  $\Re(s) \gg 0$  as

$$Z_{\sigma,\varphi}(s) := \prod_{c \text{ prime}} \prod_{N \geq 0} \det\left(1 - e^{-sl(c)} \sigma(c) \otimes \varphi(c) \otimes S^N((P_c^u)^{-1})\right),$$

where  $S^N$  means the  $N$ -th symmetric power. In [1] it is shown that  $Z_{\sigma,\varphi}$  admits meromorphic continuation, that it satisfies a functional equation and a Riemann hypothesis.

Consider the space of smooth sections:

$$\mathcal{C}_{\sigma,\varphi}^a := \Gamma^\infty(\wedge^a T_s^* \otimes \mathcal{F}_\sigma \otimes F_\varphi).$$

Now  $F_\varphi$  is equipped with a flat connection,  $\tilde{\mathcal{F}}_\sigma$  has a unique  $G$ -invariant connection which pushes down to  $\mathcal{F}_\sigma$ . So we have a canonical connection on the tensor product  $\mathcal{F}_\sigma \otimes F_\varphi$  and we define  $d : \mathcal{C}_{\sigma,\varphi}^a \rightarrow \mathcal{C}_{\sigma,\varphi}^{a+1}$  as the projection of the exterior differential given by the connection. This projection can be expressed in terms of Lie algebra cohomology (see [5]) which shows  $d^2 = 0$ . So the spaces  $\mathcal{C}_{\sigma,\varphi}$  form a complex.

**Theorem 3.1.** *Assume that the representation  $\sigma$  is self-dual, then for the Selberg zeta function we have*

$$Z_{\sigma,\varphi}(s) = \prod_{p=0}^{\dim X - 1} \det(H + s|\bar{H}^p(\mathcal{C}_{\sigma,\varphi}))^{(-1)^p}.$$

*Proof.* We say that a function  $f$  on  $\mathbb{C}$  is of *determinant type* if  $f(z) = e^{P(z)} \det(A + z)$  for a polynomial  $P(z)$  and an operator  $A$ . The determinant formula in [1] shows that  $Z_{\sigma,\varphi}$  is of determinant type. By Proposition 2.3 in [5] we get

$$Z_{\sigma,\varphi}(s) = e^{P_\sigma(s-d+1)} \det(H + s|\bar{\chi}(\mathcal{C}_{\sigma,\varphi}))^{(-1)^{\dim X - 1}}$$

for some polynomial  $P_\sigma$ . Note that the above citation only gives the desired identity if we take the determinant on the smaller space of  $K$ -finite vectors, but the comparison theorem of Casselman allows us to deduce the claim.

It is worthwhile noting that there also is a comparison theorem for the real analytic setting due to Bunke and Olbrich [2] or [3] such that our results also hold for the real analytic sheaves.

The absence of discrete series and the vanishing theorem of  $\mathfrak{n}$ -cohomology [8] show that the space  $\bar{H}^p(\mathcal{C}_{\sigma,\varphi})$  are finite dimensional unless  $p = \dim X - 1$ . Hence the existence of the determinant on  $\chi(\mathcal{C}_{\sigma,\varphi})$  implies the existence of the determinant on each single  $\bar{H}^p(\mathcal{C}_{\sigma,\varphi})$ .

It remains to show that the polynomial  $P_\sigma$  vanishes. To this end we consider the asymptotics of  $\log Z_{\sigma,\varphi}(s)$  as  $s$  tends to  $+\infty$ . By definition it is clear that  $\log Z_{\sigma,\varphi}(s)$  tends to zero for  $s \rightarrow +\infty$ . By [1] or [5] the divisor of  $Z_{\sigma,\varphi}(s)$  lies in  $(-\infty, 2d - 2] \cup (d - 1 + i\mathbb{R})$  for some  $a, b > 0$ . Therefore the function  $Z_{\sigma,\varphi}(s)$  which is of determinant type is actually a product of two determinants:

$$Z_{\sigma,\varphi}(s + d - 1) = e^{P_\sigma(s)} Q(s) \det(A + is) \det(A - is),$$

where  $Q$  is a polynomial and the operator  $A$  has spectrum contained in  $[0, \infty)$ .

From [6] we take that there are constants  $\alpha_\nu$  and  $c_\nu$  for  $\nu \geq 0$  with  $\alpha_\nu$  real and tending to  $+\infty$ , such that

$$\begin{aligned} -\log \det(A \pm is) &= \sum_{\alpha_\nu \neq 0, -1, \dots} c_\nu \Gamma(\alpha_\nu) (\pm is)^{-\alpha_\nu} \\ &+ \sum_{\alpha_\nu = -k \in \{0, -1, \dots\}} c_\nu \frac{(-1)^k}{k!} \left( \sum_{j=1}^k \frac{1}{j} - \log(\pm is) \right) (\pm is)^k \\ &+ (\dim \ker A) \log(\pm is) + o(1), \end{aligned}$$

as  $s \rightarrow \infty$ . Recall that the constants  $c_\nu$  are the coefficients of the asymptotic expansion of the heat kernel of the operator  $A$ . Recall further that, by changing the polynomial  $Q$  the operator  $A$  is only defined up to finite rank operators. Since  $\sigma$  is self dual it lies in the image of the restriction map  $\text{Rep}K \rightarrow \text{Rep}M$ ,

[4]. The argumentations in [4] give that up to a finite rank operator  $A$  is given as  $\sqrt{D_\sigma + c_\sigma}$  for a constant  $c_\sigma$  and a virtual elliptic differential operator  $D_\sigma$  of order two on  $X$ . (In [4] we did assume the dimension to be even, but for odd dimensions the same arguments apply, they even become easier by the lack of discrete series.) Since  $\dim X$  is odd it is known that in the asymptotics of the heat kernel of  $D_\sigma$  and hence of  $A$  there is no constant term. So we see that in the above asymptotic expansion the summand with  $\alpha_\nu = 0$  vanishes. It follows that if  $\log Q(s) + \log(A + is) + \log(A - is)$  is asymptotic to a polynomial it must be the zero polynomial.  $\square$

To conclude the proof of Theorem 2.1 and hence of Theorem 1.1 recall from [5] that we have

$$R_\varphi(s) = \prod_{l=0}^{d-1} Z_{\wedge^l \mathfrak{n}, \varphi}(s + 2l)^{(-1)^l}.$$

Now we have that as  $M$ -representations  $\wedge^l \mathfrak{n} \cong \wedge^{d-1-l} \mathfrak{n}$  and hence  $Z_{\wedge^l \mathfrak{n}, \varphi} = Z_{\wedge^{d-1-l} \mathfrak{n}, \varphi}$ . Further the  $M$ -representation  $\wedge^l \mathfrak{n}$  is self dual. We conclude

$$R_\varphi(s) = \prod_{p=0}^{2 \dim X - 2} \det(H + s|\bar{H}^p(\mathcal{C}_\varphi))^{(-1)^p},$$

which implies the claim.  $\square$

**Note added in Proof:** Unfortunately the proof of Casselman’s comparison theorem hasn’t become available yet. There is a version of our result which does not depend on this theorem. Just replace the spaces of smooth sections  $\mathcal{C}_\varphi^{a,b}$  by the corresponding spaces  $\mathcal{A}_\varphi^{a,b}$  of *real analytic sections*. Due to the comparison theorem of Bunke and Olbrich [2] one can then get the results as in the text.

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