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SHARP BOUNDS FOR THE GREEN'S FUNCTION AND THE HEAT KERNEL

Peter Li, Luen-fai Tam, and Jiaping Wang

§0 Introduction

The main purpose of this note is two-fold. The first is to give simple proofs to some recent theorems of Colding-Minicozzi in [C-M] on the asymptotic behavior of the Green's function on manifolds with non-negative Ricci curvature and maximal volume growth. The second is to give sharp upper and lower estimates for the heat kernel on this class of manifolds. In fact, after integrating these estimates with respect to the time variable, they yield sharp pointwise bounds for the Green's function. The interested reader should refer to [C-M] for a detailed history of this subject.

Throughout this paper, we assume that M^n is an n-dimensional complete noncompact manifold with non-negative Ricci curvature. With the exception of Theorem 2.1 and Corollary 2.3, we will assume that $n \geq 3$. Let us denote the distance between $x, y \in M$ by $d(x, y)$. If $B_x(r)$ denotes the geodesic ball of radius r centered at x, then we denote $V_x(r)$ and $A_x(r)$ to be the volume and the area of $B_x(r)$ and $\partial B_x(r)$, respectively. In all the theorems, with the exception of Theorem 1.2, we will assume that M has maximal volume growth. This means that for some fixed point $p \in M$,

$$\liminf_{r\to\infty} r^{-n}\, V_p(r) > 0.$$

If we define

$$\theta_p(r) = n^{-1} r^{1-n}\, A_p(r),$$

then the Bishop volume comparison theorem readily [B-C] asserts that there exists $\theta > 0$ independent of p, such that

$$\theta_p(r) \searrow \theta \tag{0.1}$$

and

$$r^{-n}\, V_x(r) \searrow \theta \tag{0.2}$$

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as $r \to \infty$. Let $G(x,y)$ be the minimal positive Green's function on M, whose existence is guaranteed by [V] and [L-Y]. It also follows from the estimates in [L-Y] and (0.2) that there exists a positive constant $C(n,\theta)$ depending only on n and θ such that

$$C^{-1}(n,\theta)\, d^{2-n}(x,y) \le G(x,y) \le C(n,\theta)\, d^{2-n}(x,y). \tag{0.3}$$

Moreover, the Laplacian comparison theorem [G-W] implies that, for any $x \in M$,

$$\Delta_y d^{2-n}(x,y) \ge 0 \tag{0.4}$$

in the sense of distribution. Finally, the gradient estimate in [Cg-Y] asserts that there exists a positive $C(n) > 0$ depending only on n, such that, if u is a positive harmonic function defined on $B_x(R)$ then

$$|\nabla \log u|(x) \le C(n)\, R^{-1}. \tag{0.5}$$

§1 Asymptotic behavior of the Green's function

We are now ready to give a simple proof to Theorem 0.1 of [C-M].

Theorem 1.1. *Let M^n be an n-dimensional ($n \ge 3$) Riemannian manifold with non-negative Ricci curvature. If M has maximal volume growth, then the Green's function $G(p,x)$ with a pole at any fixed point $p \in M$ must satisfy the asymptotic behavior*

$$\lim_{x\to\infty} \frac{n(n-2)\theta\, G(p,x)}{d^{2-n}(p,x)} = 1.$$

Proof. Let us denote the cut locus to the point $p \in M$ by E, and let Ω be the complement of E in M. Since E is of measure zero, Fubini's theorem asserts that the set $\partial B_p(r) \cap E$ has $(n-1)$-dimensional measure zero for almost all r. We say such r is permissible. By mollifying the function $d_p(\cdot) = d(p,\cdot)$, one obtains a sequence of smooth functions $\{f_k\}$ on M such that f_k converges to $d_p(x)$ uniformly on compact subsets of M. Moreover, $|\nabla f_k| \le 2$ on M and ∇f_k converges to ∇d_p on compact subsets of Ω. It is also well known that (for example, see [C]) it is possible to choose f_k so that $\Delta f_k \le \frac{n-1}{d_p} + 1$ on M, and that Δf_k converges to Δd_p in the sense of distribution. Let $h_k(x) = f_k^{2-n}(x)$ and $h(x) = d_p(x)^{2-n}$. Then for any permissible r, the fact that $|\nabla f_k| \le 2$ and ∇f_k converges to ∇d_p on $\partial B_p(r) \cap \Omega$ implies that

$$\begin{aligned}
\lim_{k\to\infty} \int_{B_p(r)} \Delta h_k(y)\, dy &= \lim_{k\to\infty} \int_{\partial B_p(r)} \frac{\partial}{\partial \nu}\left(h_k(y)\right)\, dy \\
&= \lim_{k\to\infty} \int_{\partial B_p(r)\cap\Omega} \frac{\partial}{\partial \nu}\left(h_k(y)\right)\, dy \\
&= \int_{\partial B_p(r)\cap\Omega} \frac{\partial}{\partial \nu}\left(h(y)\right)\, dy \\
&= -(n-2)\, r^{1-n}\, A_p(r) \\
&= -n(n-2)\, \theta_p(r).
\end{aligned} \tag{1.1}$$

For permissible R and R_0 with $1 < R_0 < R$, let us denote $B_0 = B_p(R_0)$. Let x satisfies $d_p(x) = r$ with $R_0 \le r \le \frac{R}{2}$, then the divergence theorem implies that

$$h_k(x) = -\int_{B_p(R)} G(x,y)\,\Delta h_k(y)\,dy + \int_{\partial B_p(R)} \left(G(x,y)\frac{\partial h_k}{\partial \nu_y} - h_k \frac{\partial}{\partial \nu_y} G(x,y)\right),$$

where ν_y is the unit outward normal of $\partial B_p(R)$. This can be rewritten as

$$\begin{aligned} &h_k(x) + G(p,x) \int_{B_0} \Delta h_k(y)\,dy \\ &\quad= \int_{B_0} (G(p,x) - G(x,y))\,\Delta h_k(y)\,dy - \int_{B_p(R)\setminus B_0} G(x,y)\,\Delta h_k(y)\,dy \\ &\qquad+ \int_{\partial B_p(R)} \left(G(x,y)\frac{\partial h_k}{\partial \nu_y} - h_k \frac{\partial}{\partial \nu_y} G(x,y)\right), \end{aligned} \tag{1.2}$$

Clearly, the upper bound of G implies that

$$\limsup_{k\to\infty} \left| \int_{\partial B_p(R)} \left(G(x,y)\frac{\partial h_k}{\partial \nu_y} - h_k \frac{\partial}{\partial \nu_y} G(x,y)\right)\right| \le C(n) R^{2-n}. \tag{1.3}$$

Also (0.3) and (0.5) imply that

$$|G(p,x) - G(x,y)| \le C(n)\, R_0\, (r - R_0)^{1-n},$$

hence we conclude that

$$\begin{aligned} &\left|\limsup_{k\to\infty} \int_{B_p(R_0)} (G(p,x) - G(x,y))\,\Delta h_k(y)\,dy\right| \\ &\le \limsup_{k\to\infty} \left(\int_{B_p(R_0)} |\nabla G(x,y)|\,|\nabla h_k(y)| + \right. \\ &\qquad\qquad \left. \int_{\partial B_p(R_0)} |G(p,x) - G(x,y)| \left|\frac{\partial h_k(y)}{\partial \nu}\right| \right) \\ &= \int_{B_p(R_0)} |\nabla G(x,y)|\,|\nabla h(y)| + \int_{\partial B_p(R_0)} |G(p,x) - G(x,y)| \left|\frac{\partial h(y)}{\partial \nu_y}\right| \\ &\le C(R_0)(r - R_0)^{1-n}. \end{aligned} \tag{1.4}$$

For any $1 > \epsilon > 0$, noting that $|\nabla h_k|$ are uniformly bounded, and $|\nabla_y G|$ is integrable in $B_x(\epsilon r)$, we have

$$
\begin{aligned}
&\limsup_{k\to\infty}\left|\int_{B_x(\epsilon r)} G(x,y)\Delta h_k dy\right| \\
&\qquad\le \limsup_{k\to\infty}\left|\int_{B_x(\epsilon r)} < \nabla_y G, \nabla h_k >\right| + \limsup_{k\to\infty}\left|\int_{\partial B_x(\epsilon r)} G\frac{\partial h_k}{\partial \nu_y}\right| \\
&\qquad= \left|\int_{B_x(\epsilon r)} < \nabla_y G, \nabla h >\right| + \left|\int_{\partial B_x(\epsilon r)} G\frac{\partial h}{\partial \nu_y}\right| \\
&\qquad\le C(n)\left(r^{1-n}\int_0^{\epsilon r}\rho^{1-n}A_x(\rho)\,d\rho + \epsilon r^{2-n}\right) \\
&\qquad\le \epsilon\, C(n)\, r^{2-n}.
\end{aligned}
\tag{1.5}
$$

On the other hand, (1.1) together with $\Delta h_k(y) \ge -C\, d_p^{1-n}(y)$ on $B_p(R)\setminus B_0$ and the negative part $(\Delta h_k(y))_- = \max\{-\Delta h_k(y), 0\}$ of $\Delta h_k(y)$ converges to zero almost everywhere, we conclude from (1.5) that

$$
\begin{aligned}
&\left|\limsup_{k\to\infty}\int_{B_p(R)\setminus B_p(R_0)} G(x,y)\,\Delta h_k(y)\,dy\right| \\
&\le \limsup_{k\to\infty}\int_{B_p(R)\setminus(B_p(R_0)\cup B_x(\epsilon r))} G(x,y)\,|\Delta h_k(y)|\,dy \\
&\qquad + \limsup_{k\to\infty}\left|\int_{B_x(\epsilon r)} G(x,y)\Delta h_k(y)\,dy\right| \\
&\le \limsup_{k\to\infty} C_2(\epsilon r)^{2-n}\int_{B_p(R)\setminus B_p(R_0)} |\Delta h_k(y)|\,dy + \epsilon\, C(n)\, r^{2-n} \\
&= \limsup_{k\to\infty} C_2(\epsilon r)^{2-n}\int_{B_p(R)\setminus B_p(R_0)} \Delta h_k(y)\,dy \\
&\qquad + \limsup_{k\to\infty} 2C_2(\epsilon r)^{2-n}\int_{B_p(R)\setminus B_p(R_0)} (\Delta h_k(y))_-\,dy + \epsilon\, C(n)\, r^{2-n} \\
&= \limsup_{k\to\infty} C_2(\epsilon r)^{2-n}\left(\int_{\partial B_p(R)}\frac{\partial}{\partial\nu}\left(h_k(y)\right)\,dy - \int_{\partial B_0}\frac{\partial}{\partial\nu}\left(h_k(y)\right)\,dy\right) \\
&\qquad + \epsilon\, C(n)\, r^{2-n} \\
&= C_2(\epsilon r)^{2-n} n(n-2)\left(\theta_p(R_0) - \theta_p(R)\right) + \epsilon\, C(n)\, r^{2-n}.
\end{aligned}
\tag{1.6}
$$

Hence, combining (1.1), (1.3), (1.4) and (1.6), we conclude from (1.2) that

$$
\begin{aligned}
&|h(x) - n(n-2)\,\theta_p(R_0)\,G(p,x)| \\
&\qquad\le CR^{2-n} + C(\epsilon r)^{2-n}(\theta_p(R_0) - \theta_p(R)) + C\epsilon r^{2-n} + C(R_0)(r-R_0)^{1-n}.
\end{aligned}
$$

Letting $R \to \infty$ and choosing $\epsilon = \eta_p^{\frac{1}{n-1}}(R_0)$, where $\eta_p(R_0) = \theta_p(R_0) - \theta$, this becomes

$$|h(x) - n(n-2)\,\theta_p(R_0)\,G(p,x)| \le C\left(C(R_0)(r-R_0)^{1-n} + \eta_p^{\frac{1}{n-1}}(R_0)\,r^{2-n}\right).$$

Multiplying both sides by $r^{n-2} = d_p^{n-2}(x)$, and letting $r \to \infty$, we have

$$\limsup_{x\to\infty}\left|1 - n(n-2)\,\theta_p(R_0)\,d^{n-2}(p,x)\,G(p,x)\right| \le C(n,\theta)\,\eta_p^{\frac{1}{n-1}}(R_0).$$

The theorem now follows by letting $R_0 \to \infty$.

We should remark that when $n = 2$, any manifold with non-negative curvature is parabolic. Hence any Green's function must change sign. However, if M^2 has non-negative curvature and has maximal volume growth, then M must be conformally equivalent to $\mathbb{R}^2$. In particular, we can write the metric on M as $ds^2 = e^{2u}\,ds_0^2$ where ds_0^2 is the Euclidean metric on $\mathbb{R}^2$ and u is a smooth function. Let us denote the Euclidean distant between any two points p, $x \in M$ by $d_0(p,x)$. Using the fact that the Laplacian is conformally invariant in dimension 2, we see that the Euclidean Green's function $G(p,x) = \frac{1}{2\pi}\log d_0(p,x)$ is also a Green's function with respect to the metric ds^2. It was shown in [L-T 1] (Corollary 3.3) that

$$\lim_{x\to\infty}\frac{\log d(p,x)}{\log d_0(p,x)} = \frac{\theta}{\pi}.$$

Hence, in this case, we also have the asymptotic behavior

$$\lim_{x\to\infty}\frac{2\theta\,G(p,x)}{\log d(p,x)} = 1.$$

In fact, in [L-T 1] the author considered the larger class of surfaces, namely, surfaces with finite total curvature. Estimates similar to the one stated above holds for surfaces with finite total curvature and quadratic volume growth. In our next theorem, we prove a slightly stronger result than Theorem 0.3 in [C-M].

Theorem 1.2. *Let M^n be a complete noncompact manifold with non-negative Ricci curvature. Assume that $n \ge 3$ and let $p \in M$ be a fixed point. Suppose M admits a positive Green's function $G(x,y)$ and let u be any positive harmonic function defined on $M \setminus B_p(1)$. Then there exist constants a, b, and $C > 0$, and a function v which is harmonic on $M \setminus B_p(3/2)$ satisfying*

$$|v(x)| \le C\,d^{-1}(p,x)\,G(p,x),$$

such that,

$$u(x) = a + b\,G(p,x) + v(x)$$

on $M \setminus B_p(2)$. In fact, the constant b is given by $b = -\int_{\partial B_p(2)} \frac{\partial u}{\partial r}$.

Proof. Note that an argument of Milnor [M] asserts that M has finite first Betti number. The assumption that M is non-parabolic ([L-Y] and [C-G]) implies that M has only one end. It was proved in [L-T 2] that the limit $\lim_{x\to\infty} u(x) = a$ exists, and must be finite. Let us first extend u to a smooth function on M such that u remains unchanged outside $B_p(3/2)$. Define $h = \Delta u$ which has support in $B_p(3/2)$ and

$$u(x) = a - \int_M G(x,y)h(y)dy.$$

In particular, for $x \neq p$,

$$\begin{aligned} u(x) &= a - \int_{B_p(3/2)} G(x,y)h(y)dy \\ &= a - G(p,x)\int_{B_p(3/2)} h(y)dy + \int_{B_p(3/2)} \left(G(p,x) - G(x,y)\right)h(y)dy \\ &= a + b\,G(p,x) + v(x), \end{aligned}$$

where $b = -\int_{B_p(3/2)} h(y)dy$, and $v(x) = \int_{B_p(3/2)} \left(G(p,x) - G(x,y)\right)h(y)dy$. One checks readily that $v(x)$ is harmonic on $M \setminus B_p(3/2)$. As in the proof of the previous theorem, one can show that there exists $C > 0$, such that,

$$|v(x)| \leq C\,d^{-1}(p,x)\,G(p,x)$$

for $x \notin B_p(2)$. This completes the proof of the theorem.

§2 Heat kernel estimates

In this section, we will continue to assume that M has non-negative Ricci curvature and maximal volume growth. The goal of the next theorem is to prove sharp pointwise upper and lower bounds for the heat kernel on such a manifold. The heat kernel estimates are valid for all dimensions. When $n \geq 3$, after integrating with respect to time, we obtain sharp pointwise estimate for the Green's function. Moreover, if we let $d(p,q) \to \infty$, then these upper and lower bounds will reduce to the asymptotic estimates proved in Theorem 1.1.

Theorem 2.1. *Let M be a complete manifold with non-negative Ricci curvature and maximal volume growth. For any $\delta > 0$, the heat kernel of M must satisfy the estimate*

$$\begin{aligned} \frac{\omega_n}{\theta_p(\delta d(p,q))}&(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1+9\delta}{4t} d^2(p,q)\right) \\ &\leq H(p,q,t) \\ &\leq (1 + C_{14}(\delta+\beta))\frac{\omega_n}{\theta}(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1-\delta}{4t} d^2(p,q)\right), \end{aligned}$$

where $C_{14} > 0$ is a constant depending only on n and θ, and β is given by

$$\beta = \delta^{-2n} \max_{r \geq (1-\delta)\, d(p,q)} \left\{ 1 - \frac{\theta_p(r)}{\theta_p(\delta^{2n+1} r)} \right\}.$$

Proof. Using the Cheeger-Yau [C-Y] comparison theorem, Li-Yau [L-Y] showed that for any $R > 0$,

$$\int_{B_p(R)} H(x,q,t)\, dx \geq \int_{B_{\bar{p}}(R)} \bar{H}(\bar{x},\bar{q},t)\, d\bar{x} \tag{2.1}$$

where $\bar{H}$ is the heat kernel on $\mathbb{R}^n$ and $d(p,q) = d(\bar{p},\bar{q})$. On the other hand, the Harnack inequality of [L-Y] implies that for $\eta > 0$,

$$H(x,q,t) \leq (1+\eta)^{\frac{n}{2}} \exp\left(\frac{d^2(x,p)}{4\eta t} \right) H(p,q,(1+\eta)t). \tag{2.2}$$

Hence, applying this to both H and $\bar{H}$, we have

$$\begin{aligned} &H(p,q,(1+\eta)^2 t) \\ &\qquad \geq V_p(R)^{-1} (1+\eta)^{-\frac{n}{2}} \exp\left(\frac{-R^2}{4\eta\,(1+\eta)t} \right) \int_{B_{\bar{p}}(R)} \bar{H}(\bar{x},\bar{q},(1+\eta)t)\, d\bar{x} \\ &\qquad \geq \omega_n R^n V_p(R)^{-1} (1+\eta)^{-n} \exp\left(\frac{-(2+\eta)R^2}{4\eta\,(1+\eta)t} \right) \bar{H}(\bar{p},\bar{q},t). \end{aligned}$$

Setting $R = \eta\, d(p,q)$ and writing $V_p(R) = \theta_p(R)\, R^n$, we have

$$H(p,q,(1+\eta)^2 t) \geq \frac{\omega_n}{\theta_p(\eta\, d(p,q))} (1+\eta)^{-n} \exp\left(\frac{-\eta\,(2+\eta)\, d^2(p,q)}{4(1+\eta)t} \right) \bar{H}(\bar{p},\bar{q},t).$$

Rewriting this and letting $s = (1+\eta)^2 t$, we conclude that for any $\eta > 0$,

$$H(p,q,s) \geq \frac{\omega_n}{\theta_p(\eta\, d(p,q))} (4\pi s)^{-\frac{n}{2}} \exp\left(-\frac{(1+9\eta)\, d^2(p,q)}{4s} \right), \tag{2.3}$$

which is the desired lower estimate.

To prove the upper bound, let us first recall that it was proved in [L] that for a fixed point $p \in M$, the function $t^{\frac{n}{2}} H(p,p,t)$ is monotonically non-decreasing with

$$t^{\frac{n}{2}} H(p,p,t) \nearrow \frac{\omega_n}{(4\pi)^{n/2}\theta}.$$

Let $\delta > 0$ be sufficiently small, and for $d(p,q) \leq \delta\sqrt{t}$, (2.2) implies that

$$\begin{aligned} H(p,q,t) &\leq (1+\delta)^{n/2} \exp\left(\frac{d^2(p,q)}{4\delta t}\right) H(p,p,(1+\delta)t) \\ &\leq (4\pi t)^{-n/2} \frac{\omega_n}{\theta} \exp\left(\frac{d^2(p,q)}{4\delta t}\right) \\ &\leq (4\pi t)^{-n/2} \frac{\omega_n}{\theta} \exp\left(-\frac{d^2(p,q)}{4t}\right) \exp\left(\frac{\delta^2+\delta}{4}\right) \\ &\leq (1 + C_{15}\,\delta)\,(4\pi t)^{-n/2} \frac{\omega_n}{\theta} \exp\left(-\frac{d^2(p,q)}{4t}\right) \end{aligned} \tag{2.4}$$

where $C_{15} > 0$ is some positive constant. Let us now consider the case $d(p,q) \geq \delta\sqrt{t}$, and let $R = (1-\delta)d(p,q)$. Since

$$\int_M H(p,x,s)\,dx = \int_{\mathbb{R}^n} \bar{H}(\bar{p},\bar{x},s)\,d\bar{x} = 1,$$

inequality (2.1) implies

$$\int_{M\setminus B_p(R)} H(p,x,(1+\delta)t)\,dx \leq \int_{\mathbb{R}^n\setminus B_{\bar{p}}(R)} \bar{H}(\bar{p},\bar{x},(1+\delta)t)\,d\bar{x}.$$

Using the fact that $B_p(R) \cap B_q(\delta R) = \emptyset$, we have

$$\begin{aligned} &\int_{B_q(\delta R)} H(p,x,(1+\delta)t)\,dx \\ &\qquad \leq \int_{\mathbb{R}^n\setminus B_{\bar{p}}(R)} \bar{H}(\bar{p},\bar{x},(1+\delta)t)\,d\bar{x} \\ &\qquad\quad - \int_{M\setminus(B_p(R)\cup B_q(\delta R))} H(p,x,(1+\delta)t)\,dx. \end{aligned}$$

Invoking the lower bound (2.3) of $H(p,q,s)$ and noting that $\theta_p(r) \searrow \theta$, we get

$$
\begin{aligned}
&\int_{B_q(\delta R)} H(p,x,(1+\delta)t)dx \\
&\quad \le \int_{\mathbb{R}^n \setminus B_{\bar{p}}(R)} \bar{H}(\bar{p},\bar{x},(1+\delta)t)d\bar{x} - (4\pi(1+\delta)t)^{-\frac{n}{2}} \\
&\qquad \times \int_{M\setminus(B_p(R)\cup B_q(\delta R))} \frac{\omega_n}{\theta_p(\eta\, d(p,x))} \exp\left(-\frac{(1+9\eta)\, d^2(p,x)}{4(1+\delta)t}\right) dx \\
&\quad \le n\omega_n(4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1}\exp\left(-\frac{r^2}{4(1+\delta)t}\right) dr \\
&\qquad - (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_{M\setminus B_p(R)} \frac{\omega_n}{\theta_p(\eta\, d(p,x))} \exp\left(-\frac{(1+9\eta)\, d^2(p,x)}{4(1+\delta)t}\right) dx \\
&\qquad + (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_{B_q(\delta R)} \frac{\omega_n}{\theta_p(\eta\, d(p,x))} \exp\left(-\frac{(1+9\eta)\, d^2(p,x)}{4(1+\delta)t}\right) dx \\
&\quad \le n\omega_n\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1}\exp\left(-\frac{r^2}{4(1+\delta)t}\right) dr \\
&\qquad - n\omega_n\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right) \frac{A_p(r)}{n\,\theta_p(\eta\, r)}\, dr \\
&\qquad + \frac{\omega_n}{\theta}(4\pi(1+\delta)t)^{-\frac{n}{2}} \exp\left(-\frac{1+9\eta}{4(1+\delta)t}\,(d(p,q)-\delta R)^2\right) V_q(\delta R).
\end{aligned}
$$

Rewriting the first two terms on the right hand side as

$$
\begin{aligned}
&n\omega_n\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1} \left(\exp\left(-\frac{r^2}{4(1+\delta)t}\right) - \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right)\right) dr \\
&\qquad + n\omega_n\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1}\, \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right) \left(1-\frac{\theta_p(r)}{\theta_p(\eta r)}\right) dr
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int_{B_q(\delta R)} H(p,x,(1+\delta)t)dx \\
&\quad \le n\omega_n\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1}\, \exp\left(-\frac{r^2}{4(1+\delta)t}\right) \frac{9\eta\, r^2}{4(1+\delta)t}\, dr \\
&\qquad + n\omega_n\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \max_{r\ge R}\left\{1-\frac{\theta_p(r)}{\theta_p(\eta\, r)}\right\} \\
&\qquad\quad \times \int_R^\infty r^{n-1}\exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right) dr \\
&\qquad + \frac{\omega_n}{\theta}\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \exp\left(-\frac{1+9\eta}{4(1+\delta)t}(d(p,q)-\delta R)^2\right) V_q(\delta R)
\end{aligned}
\tag{2.5}
$$

Direct computation shows that

$$
\begin{aligned}
(2.6)\qquad & (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1} \exp\left(-\frac{r^2}{4(1+\delta)t}\right) \frac{r^2}{4(1+\delta)t}\, dr \\
&\leq C_{16}\, (1+R^n(4\pi(1+\delta)t)^{-\frac{n}{2}})\, \exp\left(-\frac{R^2}{4(1+\delta)t}\right)
\end{aligned}
$$

for some constant $C_{16} > 0$ depending only on n. Also, there exists constants C_{17} and C_{18} depending on n, such that,

$$
\begin{aligned}
(2.7)\qquad & (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1} \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right) dr \\
&= (4\pi(1+\delta)t)^{-\frac{n}{2}} \frac{1}{n} \left[r^n \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right)\right]_R^\infty \\
&\quad + (4\pi(1+\delta)t)^{-\frac{n}{2}} \frac{1}{n} \int_R^\infty r^n \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right) \frac{2(1+9\eta)\, r}{4(1+\delta)t}\, dr \\
&\leq C_{17}\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \int_R^\infty r^{n-1} \exp\left(-\frac{(1+9\eta)\, r^2}{4(1+\delta)t}\right) \frac{r^2}{4(1+\delta)t}\, dr \\
&\leq C_{18}\, (1+R^n(4\pi(1+\delta)t)^{-\frac{n}{2}})\, \exp\left(-\frac{R^2}{4(1+\delta)t}\right).
\end{aligned}
$$

Combining (2.6) and (2.7) with (2.5), we conclude that

$$
\begin{aligned}
(2.8)\qquad & \int_{B_q(\delta R)} H(p,x,(1+\delta)t)\, dx \\
&\leq \frac{\omega_n}{\theta}\, (4\pi(1+\delta)t)^{-\frac{n}{2}} \exp\left(-\frac{1+9\eta}{4(1+\delta)t}(d(p,q)-\delta R)^2\right) V_q(\delta R) \\
&\quad + C_{18}\, (1+R^n(4\pi(1+\delta)t)^{-\frac{n}{2}})\, (\eta+\alpha(\eta,R)) \exp\left(-\frac{R^2}{4(1+\delta)t}\right)
\end{aligned}
$$

where

$$
\alpha(\eta, R) = \max_{r\geq R} \left\{ 1 - \frac{\theta_p(r)}{\theta_p(\eta\, r)} \right\}.
$$

Note that $\theta_p(R) \searrow \theta$ implies $\alpha(\eta, R) \to 0$. Using (2.2) and (2.8), we obtain

$$\begin{aligned}
&H(p,q,t) \\
&\quad \le (1+\delta)^{\frac{n}{2}} \exp\left(\frac{\delta^2 R^2}{4\delta t}\right) V_q^{-1}(\delta R) \int_{B_q(\delta R)} H(p,x,(1+\delta)t)\,dx \\
&\quad \le \frac{\omega_n}{\theta}(1+\delta)^{\frac{n}{2}} \exp\left(\frac{\delta R^2}{4t}\right)(4\pi(1+\delta)t)^{-\frac{n}{2}} \\
&\qquad\qquad\qquad\qquad \exp\left(-\frac{1+9\eta}{4(1+\delta)t}(d(p,q)-\delta R)^2\right) \\
&\qquad + C_{19}\,(\eta+\alpha(\eta,R))\exp\left(-\frac{R^2}{4(1+\delta)t}\right) \\
&\qquad \times \left(V_q^{-1}(\delta R) + \frac{R^n}{V_q(\delta R)}(4\pi(1+\delta)t)^{-\frac{n}{2}}\right).
\end{aligned} \tag{2.9}$$

Notice that $V_q(\delta R) \ge \theta\,(\delta R)^n$ and also

$$R = (1-\delta)\,d(p,q) \ge \delta(1-\delta)\sqrt{t}.$$

Therefore, we conclude from (2.9) that

$$\begin{aligned}
&H(p,q,t) \\
&\quad \le \frac{\omega_n}{\theta}(4\pi t)^{-\frac{n}{2}} \exp\left(\left(-\frac{(1+9\eta)(1-\delta)^2}{4(1+\delta)t} + \frac{\delta(1-\delta)^2}{4t}\right) d^2(p,q)\right) \\
&\qquad + C_{14}\,(\eta+\alpha(\eta,R))\,\delta^{-2n}(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{(1-\delta)^2}{4(1+\delta)t} d^2(p,q)\right).
\end{aligned} \tag{2.10}$$

Choosing $\eta = \delta^{2n+1}$ and writing

$$\beta = \delta^{-2n}\,\alpha(\eta,(1-\delta)\,d(p,q)),$$

we derive from (2.10) that

$$H(p,q,t) \le (1+C_{14}(\delta+\beta))\,\frac{\omega_n}{\theta}\,(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1-\delta}{4t} d^2(p,q)\right). \tag{2.11}$$

Combining (2.4) and (2.11), we obtain the desired upper estimate.

Corollary 2.2. *Let M satisfy the same assumption as Theorem 2.1, and $n \ge 3$. Then the minimal positive Green's function must satisfy*

$$\begin{aligned}
&(1+9\delta)^{1-\frac{n}{2}} \frac{d^{2-n}(p,q)}{n(n-2)\,\theta_p(\delta\,d(p,q))} \\
&\quad \le G(p,q) \\
&\quad \le (1+C_{14}\,(\delta+\beta))\,(1-\delta)^{1-\frac{n}{2}} \frac{d^{2-n}(p,q)}{n(n-2)\,\theta}\,.
\end{aligned}$$

Proof. Let us first observe that

$$\begin{aligned}\int_0^\infty (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-\gamma\, d^2(p,q)}{4t}\right) dt &= \gamma^{1-\frac{n}{2}} \int_0^\infty (4\pi s)^{-\frac{n}{2}} \exp\left(\frac{-d^2(p,q)}{4s}\right) ds \\ &= \frac{1}{n(n-2)\omega_n}\, \gamma^{1-\frac{n}{2}}\, d^{2-n}(p,q).\end{aligned}$$

Integrating over $0 < t < \infty$ and using the lower bound of $H(p,q,t)$, we have

$$\begin{aligned}G(p,q) &= \int_0^\infty H(p,q,t)\, dt \\ &\geq \frac{\omega_n}{\theta_p(\delta\, d(p,q))} \int_0^\infty (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-(1+9\delta)\, d^2(p,q)}{4t}\right) dt \\ &= \frac{1}{n(n-2)\,\theta_p(\delta\, d(p,q))}\, (1+9\delta)^{1-\frac{n}{2}}\, d^{2-n}(p,q).\end{aligned}$$

Similarly, if we use the upper bound estimate of $H(p,q,t)$, then

$$\begin{aligned}G(p,q) &= \int_0^\infty H(p,q,t)\, dt \\ &\leq (1 + C_{14}\,(\delta+\beta))\, \frac{\omega_n}{\theta} \\ &\quad \times \int_0^\infty (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-(1-\delta) d^2(p,q)}{4t}\right) dt \\ &= (1 + C_{14}(\delta+\beta))\, \frac{1}{n(n-2)\,\theta}\, (1-\delta)^{1-\frac{n}{2}}\, d^{2-n}(p,q).\end{aligned}$$

Note that by letting $q \to \infty$, and using the fact that $\theta_p(\delta\, d(p,q)) \to \theta$ we conclude that

$$\lim_{q\to\infty} d^{n-2}(p,q)\, G(p,q) \geq \frac{1}{n(n-2)\theta}\, (1+9\delta)^{1-\frac{n}{2}}.$$

The sharp asymptotic lower bound follows by letting $\delta \to 0$. Similarly, the sharp asymptotic upper bound also follows by using the fact that $\beta \to 0$ and then $\delta \to 0$.

The following corollary gives a slightly stronger result concerning the asymptotic behavior in the time variable for the heat kernel as was proved in [L].

Corollary 2.3. *Let M be as in Theorem 2.1. If $\gamma(t) = (q(t), t)$ is any path on $M \times (0,\infty)$ satisfying $d^2(q(t), p) = O(t)$ as $t \to \infty$, then*

$$\lim_{t\to\infty} V_p(\sqrt{t}) \exp\left(\frac{d^2(p,q(t))}{4t}\right) H(p,q(t),t) = \omega_n (4\pi)^{-\frac{n}{2}}.$$

Proof. If $d^2(q(t),p) = o(t)$ as $t \to \infty$, then the result follows from [L]. Without loss of generality, we may assume that $d(q(t),p) \to \infty$ as $t \to \infty$. In particular, $\theta_p(\delta d(p,q(t))) \to \theta$ and $\beta \to 0$ as $t \to \infty$. Applying the heat kernel bounds of Theorem 2.1 and letting $t \to \infty$, we have

$$\begin{aligned}
&\liminf_{t\to\infty} V_p(\sqrt{t})\ \exp\left(\frac{d^2(p,q(t))}{4t}\right) H(p,q(t),t)\\
&\qquad \geq \omega_n(4\pi)^{-\frac{n}{2}} \lim_{t\to\infty} \frac{V_p(\sqrt{t})}{\theta\,(\sqrt{t})^n}\ \exp\left(-\frac{9\delta\, d^2(p,q(t))}{4t}\right)\\
&\qquad \geq \omega_n(4\pi)^{-\frac{n}{2}}\ \exp(-C_{20}\,\delta)
\end{aligned}$$

for some constant $C_{20} > 0$. Letting $\delta \to 0$, we conclude that

$$\liminf_{t\to\infty} V_p(\sqrt{t})\ \exp\left(\frac{d^2(p,q(t))}{4t}\right) H(p,q(t),t) \geq \omega_n(4\pi)^{-\frac{n}{2}}.$$

Similarly, we have

$$\begin{aligned}
&\limsup_{t\to\infty} V_p(\sqrt{t})\ \exp\left(\frac{d^2(p,q(t))}{4t}\right) H(p,q(t),t)\\
&\qquad \leq (1 + C_{14}\,(\delta+\beta))\omega_n(4\pi)^{-\frac{n}{2}} \lim_{t\to\infty} \frac{V_p(\sqrt{t})}{\theta\,(\sqrt{t})^n}\ \exp\left(\frac{\delta\, d^2(p,q(t))}{4t}\right)\\
&\qquad \leq \omega_n(4\pi)^{-\frac{n}{2}}\ \exp(C_{21}\,\delta),
\end{aligned}$$

where $C_{21} > 0$ is a constant. Thus, by taking $\delta \to 0$, we have

$$\limsup_{t\to\infty} V_p(\sqrt{t})\ \exp\left(\frac{d^2(p,q(t))}{4t}\right) H(p,q(t),t) \leq \omega_n(4\pi)^{-\frac{n}{2}},$$

and the corollary follows.

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References

[B-C] R. Bishop and R. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

[C-G] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **6** (1971), 119–128.

[C-Y] J. Cheeger and S. T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math. **34** (1981), 465–480.

[Cg-Y] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[C] T. H. Colding, *Ricci curvature and volume convergence*, preprint.

[C-M] T. H. Colding and W. P. Minicozzi II, *Large scale behavior of kernels of Schrödinger operators*, preprint.

[G-W] R. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Springer-Verlag Lecture Notes in Mathematics, 699, Berlin, 1979.

[L] P. Li, *Large time behavior of the heat equation on complete manifolds with non-negative Ricci curvature*, Ann. of Math. **124** (1986), 1–21.

[L-T 1] P. Li and L. F. Tam, *Complete surfaces with finite total curvature*, J. Diff. Geom. **33** (1991), 139–168.

[L-T 2] ———, *Green's functions, harmonic functions and volume comparison*, J. Diff. Geom. **41** (1995), 277–318.

[L-Y] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[M] J. Milnor, *A note on curvature and fundamental group*, J. Diff. Geom. **2** (1968), 1–7.

[V] N. Varopoulos, *The Poisson kernel on positively curved manifolds*, J. Funct. Anal. **44** (1981), 359–380.

Department of Mathematics, University of California, Irvine, Irvine, CA 92697-3875

E-mail address: pli@math.uci.edu

Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong

E-mail address: lftam@math.cuhk.hk,

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307

E-mail address: jwang@math.mit.edu