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ON AN ANALOGUE OF THE ROZENBLUM-LIEB-CWIKEL INEQUALITY FOR THE BIHARMONIC OPERATOR ON A RIEMANNIAN MANIFOLD

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0. Introduction

Let Δ be the Laplacian on $\mathbb{R}^d$, $l > 0$ - an integer and $V \geq 0$ a measurable function ("weight-function"). Consider the eigenvalue problem

$$\lambda(-\Delta)^l u = Vu. \tag{0.1}$$

The following result was proved by Rosenblum [Roz]:

Theorem 0.1. *Let $2l < d$ and $V \in L_{\frac{d}{2l}}(\mathbb{R}^d)$. Then the non-zero spectrum of the problem (0.1) consists of positive eigenvalues λ_k (counted according to their multiplicities), and for their distribution function*

$$n(\lambda) = \#\{k : \quad \lambda_k > \lambda\}, \quad \lambda > 0,$$

the following estimate and asymptotics hold:

$$n(\lambda) \leq C(d,l)\lambda^{-d/2l} \int_{\mathbb{R}^d} V^{d/2l}\, dx, \quad d > 2l, \quad \text{any} \quad \lambda > 0; \tag{0.2}$$

$$\lim_{\lambda \to 0} \lambda^{d/2l} n(\lambda) = c(d) \int_{\mathbb{R}^d} V^{d/2l} dx, \quad c(d) = (2\sqrt{\pi})^{-d} \left(\Gamma\left(1 + d/2\right)\right)^{-1}. \tag{0.3}$$

It immediately follows from Theorem 0.1 that the same estimate and asymptotics are valid for the problem

$$\lambda((-\Delta)^l u + u) = Vu. \tag{0.4}$$

The problem (0.1) was extensively investigated for $l = 1$, since it is closely related to the Schrödinger operator $-\Delta - V$. Other proofs of (0.2) for $l = 1$

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were given in [Lb], [Cw], [LY] and [Con]. Note that the general result obtained in [Cw] actually applies to any $l < d/2$.

The following question arises naturally in connection with Theorem 0.1: what is the class of Riemannian manifolds to which the estimate (0.2) and the asymptotics (0.3) can be extended? The question on the estimate is the basic here, because the asymptotic formula can be easily deduced from the estimate in a rather standard way.

For $l = 1$ this problem is completely solved. The methods of [Lb], [LY] apply to a Riemannian manifold, provided the global Sobolev inequality for functions $u \in C_0^\infty(M)$ is satisfied

$$\left(\int_M |u|^{\frac{2d}{d-2}}\, dv\right)^{\frac{d-2}{d}} \le C \int_M |\nabla u|^2\, dv.$$

The corresponding analog of (0.2) (for $l = 1$) is

$$n(\lambda) \le C(M)\lambda^{-d/2} \int_M V^{d/2}\, dv, \quad d > 2. \tag{0.5}$$

Here and in the sequel dv stands for the Riemannian volume element on M. In [LS], [RS] the estimates of the type (0.5) were extended to a wider class of problems.

The approaches of [Lb], [LY] and also of [LS], [RS] do not apply to the higher order case ($l > 1$). For higher order operators it is natural to use localization, reducing the problem to $\mathbb{R}^d$. However, it is unclear in advance what are the suitable assumptions on a manifold. We analyze this problem only for $l = 2$, because all the main difficulties manifest themselves already for this case.

More exactly, we consider the problem (0.4) (for $l = 2$). We prove that for manifolds of bounded geometry, i.e. if the Ricci curvature is bounded from below and the injectivity radius is positive, the analog of (0.2) takes place:

$$n(\lambda) \le C(M)\lambda^{-d/4} \int_M V^{d/4}\, dv, \quad d > 4.$$

So, these conditions turn out to be the same as for the problem (0.4) with $l = 1$, see [LS, Theorem 3.3]. As for an analog of (0.2) for the problem (0.1), it is not clear at the moment whether these assumptions on M are sufficient in order to well define the operator, corresponding to the problem (0.1).

1. Main results

We study the non-zero spectrum of the problem

$$\lambda(\Delta^2 u + u) = Vu \quad \text{on} \quad M. \tag{1.1}$$

Here Δ is the Laplacian on M and $V \geq 0$ is a given measurable weight-function. The Laplacian has the following form in local coordinates:

$$\Delta = \sum_{i,j} \mathbf{g}^{-\frac{1}{2}} \frac{\partial}{\partial x^j} \left(\mathbf{g}^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^i} \right),$$

where g^{ij} are the components of the Riemannian tensor, $g_{ij} = \{g^{ij}\}^{-1}$ and $\mathbf{g} = det(g_{ij})_{i,j=1,\dots,d}$. Everywhere the Einstein summation convention is adopted.

We begin with some preliminaries. The Sobolev space $H^2(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|_{H^2(M)} = \sqrt{\int_M (|\nabla^2 u|^2 + |u|^2)\, dv}, \tag{1.2}$$

where $|\nabla^2 u|^2$ is defined in an invariant way as

$$|\nabla^2 u|^2 = \nabla^l \nabla^k u\ \nabla_l \nabla_k u = g^{pl} g^{kq} \left(\frac{\partial^2 u}{\partial x^k \partial x^l} - \Gamma^m_{kl} \frac{\partial u}{\partial x^m} \right) \left(\frac{\partial^2 u}{\partial x^p \partial x^q} - \Gamma^n_{pq} \frac{\partial u}{\partial x^n} \right).$$

Lemma 1.1. *Let (M, g) be a Riemannian d-manifold, $d > 1$. Suppose that there is a constant $K \geq 0$ such that*

$$Ric \geq -Kg \quad (\textit{as quadratic forms}). \tag{1.3}$$

Then the metrics (1.2) and

$$|\!|\!| u |\!|\!|_{H^2(M)} = \sqrt{\int_M (|\Delta u|^2 + |u|^2)\, dv} \tag{1.4}$$

are equivalent on $C_0^\infty(M)$.

Remind that both the Riemannian tensor g and the Ricci curvature Ric are (2,0)-tensors, so the inequality in (1.3) has an invariant meaning.

Proof. The classical Bochner-Lichnerowicz-Weitzenböck formula (see [Lic]) reads

$$\int_M |\nabla^2 u|^2\, dv = \int_M |\Delta u|^2\, dv - \int_M Ric\,(\nabla u, \nabla u)\, dv, \quad \text{any} \quad u \in C_0^\infty(M).$$

Together with (1.3) it implies that

$$\int_M |\nabla^2 u|^2\, dv \leq \int_M |\Delta u|^2\, dv + K \int_M |\nabla u|^2\, dv.$$

One derives from the imbedding theorem (see [Au, Theorem 3.69], where we put $p = q = r = 2$) that for any d-manifold M

$$\|\nabla u\|^2_{L_2(M)} \le \sqrt{d}\, \|u\|_{L_2(M)} \|\nabla^2 u\|_{L_2(M)}, \quad \text{any} \quad u \in C_0^\infty(M).$$

It follows from the two last inequalities that

$$\int_M (|\nabla^2 u|^2 + |u|^2)\, dv \le C(d, K) \int_M \left(|\Delta u|^2 + |u|^2\right)\, dv. \tag{1.5}$$

To prove the converse, we use an algebraic inequality for $d \times d$-matrices:

$$d\, Tr(CC^\star) \ge |Tr\, C|^2. \tag{1.6}$$

Applying it to the matrix with entries

$$C_l^q = g^{kq} \left(\frac{\partial^2 u}{\partial x^k \partial x^l} - \Gamma^m_{kl} \frac{\partial u}{\partial x^m} \right),$$

we come to

$$d|\nabla^2 u|^2 \ge |\Delta u|^2.$$

Together with (1.5) this finishes the proof. □

Let $V(x)$ be a non-negative measurable function on M. If the quadratic form $b[u] = \int_M V|u|^2\, dv$ is bounded on $H^2(M)$, i.e.

$$\int_M V|u|^2\, dv \le C \int_M (|\Delta u|^2 + |u|^2)\, dv,$$

then it generates a bounded self-adjoint operator B_V on $H^2(M)$. Under some additional conditions on V, the operator B_V is compact. Its eigenvalues and eigenfunctions have the following "weak" description:

$$\lambda \int_M (\Delta u \cdot \Delta \overline{w} + u\overline{w})\, dv = \int_M V u \overline{w}\, dv, \quad \text{any} \quad w \in H^2(M).$$

We denote by $n(\lambda, B_V)$ the eigenvalue distribution function:

$$n(\lambda, B_V) = \#\{k : \lambda_k > \lambda\},$$

where λ_k are the eigenvalues of B_V (counting multiplicities).

Our basic result is as follows:

Theorem 1.2. *Let (M, g) be a complete Riemannian d-manifold $(d > 4)$, which has a lower bound on the Ricci curvature and on the injectivity radius:*

$$Ric \geq -Kg, \quad K \geq 0 \quad (\textit{as bilinear forms}),$$

$$inj \geq i_0 > 0.$$

Then the following estimate is valid:

$$n(\lambda, B_V) \leq C(d, K, i_0)\lambda^{-d/4} \int_M V^{d/4}\, dv. \tag{1.7}$$

For the operator B_V the same assumptions yield also the Weyl-type asymptotics of $n(\lambda, B_V)$.

Theorem 1.3. *Under the assumptions of Theorem 1.2, the asymptotic formula is satisfied:*

$$\lim_{\lambda\to 0} \lambda^{d/4} n(\lambda, B_V) = c(d) \int_M V^{d/4}\, dv, \quad c(d) = (2\sqrt{\pi})^{-d} \left(\Gamma(1 + d/2)\right)^{-1}. \tag{1.8}$$

Under the condition (1.3) the problem (1.1) is equivalent to a study of the negative spectrum of the operator $A_{\alpha V} = \Delta^2 - \alpha V$ on M. By the definition, $A_{\alpha V}$ is the self-adjoint operator on $L^2(M)$ associated with the quadratic form

$$a_{\alpha V}[u] = \int_M \left(|\Delta u|^2 - \alpha V|u|^2\right)\, dv, \tag{1.9}$$

considered on $H^2(M)$. According to the Birman-Schwinger principle, compactness of B_V yields that for any $\alpha > 0$ the form (1.9) is bounded from below and closed. If we denote by $N_{-1}(A_{\alpha V})$ the number of eigenvalues of $A_{\alpha V}$ lying to the left of -1, then

$$N_{-1}(A_{\alpha V}) = n(\alpha^{-1}, B_V). \tag{1.10}$$

Theorems 1.2 and 1.3 together with (1.10) lead to the following statement:

Theorem 1.4. *Let the conditions of Theorem 1.2 be satisfied. Then*

$$N_{-1}(\Delta^2 - \alpha V) \leq C(d, K, i_0)\alpha^{d/4} \int_M V^{d/4}\, dv$$

and

$$\lim_{\alpha\to\infty} \alpha^{-d/4} N_{-1}(\Delta^2 - \alpha V) = c(d) \int_M V^{d/4}\, dv.$$

In the next section we give necessary technical information. The proof of Theorem 1.2 is contained in Section 3. Theorem 1.3 will be proved in Section 4.

2. Auxiliary information

2.1 Harmonic coordinates.

Definition 2.1. A coordinate system $\{(y^1, \dots, y^d)\}$, defined on an open subset of M is called *harmonic coordinate system*, if $\Delta y^i = 0$, $i = 1, \dots, d$. For each point $x \in M$ there is always a harmonic coordinate system defined on some neighborhood of x. In harmonic coordinates the expression for the Laplacian considerably simplifies:

$$\Delta u = g^{ij} \frac{\partial^2 u}{\partial y^i \partial y^j}. \tag{2.1}$$

Definition 2.2. Given $0 < \alpha < 1$ and $Q > 1$, the C^α-harmonic radius at $x \in M$ is the largest number $R_H = R_H(Q, \alpha, x)$ such that on the geodesic ball $B = B_x(R_H)$ of radius R_H, there is a harmonic coordinate chart $F_x = \{y^i\}_1^d : B \to \mathbb{R}^d$ such that the metric tensor g_{ij} is C^α-controlled in these coordinates, i.e.

$$Q^{-1}|\xi|^2 \le g_{ij}\xi^i\xi^j \le Q|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \tag{2.2}$$

and

$$R_H^\alpha \|g_{ij}\|_{C^\alpha} \le Q - 1. \tag{2.3}$$

Here

$$\|g_{ij}\|_{C^\alpha} = \sup_{y \ne z} \frac{|g_{ij}(y) - g_{ij}(z)|}{d_g(y,z)^\alpha}$$

is a seminorm taken with respect to the coordinates $\{y^i\}_1^d$ and d_g is the distance associated to g. The "global" harmonic radius $r_H(Q, \alpha)$ of M is defined as

$$r_H(Q, \alpha) = \inf_{x \in M} R_H(Q, \alpha, x).$$

It is easy to see that the estimates (2.2) and (2.3) imply the similar inequalities for g^{ij}:

$$Q^{-1}|\xi|^2 \le g^{ij}\xi_i\xi_j \le Q|\xi|^2, \quad \forall \xi \in \mathbb{R}^d; \tag{2.4}$$

and

$$R_H^\alpha \|g^{ij}\|_{C^\alpha} \le C(Q, d)(Q - 1). \tag{2.5}$$

The following proposition which plays an essential role in the proof of Theorem 1.2 may be found in [AC, Theorem 0.3], or [Hb, Theorem 1.3].

Proposition 2.1. *Under the hypotheses of Theorem 1.2, for any $Q > 1$ and $0 < \alpha < 1$ the C^α-harmonic radius $r_H = r_H(Q, \alpha)$ is bounded away from zero by some constant $C = C(Q, \alpha, d, K, i_0) > 0$.*

Clearly, (2.2) gives for any $r \le r_H$

$$B^e(r/\sqrt{Q}) \subset F_x(B_x(r)) \subset B^e(r\sqrt{Q}), \tag{2.6}$$

where $B^e(r)$ denotes the euclidean ball of radius r.

In the Definition 2.2 we fix some value of α and put $Q = 4$. Then it follows from (2.2) that

$$2^{-d} \le \sqrt{\mathbf{g}(y)} \le 2^d. \tag{2.7}$$

2.2 Variational technique. Let H be a complete Hilbert space, $a[u] > 0$ - a form defining a metric in H. Assume that $b[u]$ is a real quadratic form defined on H and bounded:

$$|b[u]| \le Ca[u].$$

Then it generates a bounded self-adjoint operator $B = B(a, b, H)$. If this operator is compact in H, then we also say that the quadratic form $b[u]$ is compact in H. We usually deal with a non-negative B. According to the variational principle, the distribution function $n(\lambda, B)$ can be expressed, e.g., by the well-known Glazman's lemma, in terms of the variational (Rayleigh) quotient

$$b[u]/a[u], \quad u \in H. \tag{2.8}$$

We say that B is the operator, corresponding to the variational quotient (2.8). For its distribution function we sometimes use a simplified and expressive notation, like

$$n(\lambda, (2.8)) = n(\lambda, B).$$

The two propositions given below can be found in [BS1, Lemmas 1.15, 1.16].

Proposition 2.2. *Let us consider two compact and non-negative operators B_1 and B_2 on Hilbert spaces H_1 and H_2 respectively, corresponding to the variational quotients $b_1[u]/a_1[u]$, $u \in H_1$ and $b_2[u]/a_2[u]$, $u \in H_2$. Suppose that there exists a bounded operator $S : H_2 \to H_1$ such that*

$$b_2[u] = b_1[Su], \quad u \in H_2. \tag{2.9}$$

If for some $t > 0$ and for all $u \in H_2$ one has

$$a_2[u] \ge t a_1[Su],$$

then for any $\lambda > 0$

$$n(\lambda, B_2) \le n(t\lambda, B_1).$$

Proposition 2.3. *Let us consider a compact operator B_1, corresponding to the variational quotient $b[u]/a_1[u]$, $u \in H$. Let $a_2[u]$ be a form on H such that $a_2[u] > 0$ for $u \in H \setminus \{0\}$. Suppose that the form $a_1 - a_2$ is compact in H. Then the forms a_1 and a_2 define equivalent metrics on H, the operator $B_2 = B_2(a_2, b, H)$ is compact, and moreover,*

$$\limsup_{\lambda\to 0} \lambda^q n(\lambda, B_1) = \limsup_{\lambda\to 0} \lambda^q n(\lambda, B_2),$$
$$\liminf_{\lambda\to 0} \lambda^q n(\lambda, B_1) = \liminf_{\lambda\to 0} \lambda^q n(\lambda, B_2).$$

3. Localization

We now move on to the construction of an appropriate covering of M. Fix $R = \min\{1, r_H/4\} > 0$ and take any maximal family of mutually disjoint closed balls $\bar{B}_{x_n}(R/2) \subset M$. Maximality means that any closed ball of radius $R/2$ on M has non-empty intersection with at least one ball $\bar{B}_{x_n}(R/2)$. It follows that the open balls $B_{x_n}(R)$ constitute a covering of M and the multiplicity N_0 of the covering $(B_{x_n}(8R))_{n=1,2,\dots}$ can be easily estimated. Indeed, suppose that $x_0 \in M$ belongs to $B_{x_{n_j}}(8R)$, where $j = 1, \dots, N$. Then for any j

$$B_{x_{n_j}}(R/2) \subset B_{x_0}(17R/2) \subset B_{x_{n_j}}(33R/2).$$

By the classical Bishop-Gromov volume comparison theorem (see [Ch, Theorems 3.9, 3.10])

$$|B_{x_0}(17R/2)| \le |B_{x_{n_j}}(33R/2)| \le 33^d e^{\frac{33R}{2}\sqrt{(d-1)K}} |B_{x_{n_j}}(R/2)|.$$

The balls $B_{x_{n_j}}(R/2)$ are mutually disjoint, and it follows from the last two inequalities that

$$N \le N_0 = 33^d e^{\frac{33R}{2}\sqrt{(d-1)K}}. \tag{3.1}$$

Let $\omega \in C_0^\infty(\mathbb{R}^d)$ be such that

$$0 \le \omega \le 1, \quad \omega = 1 \quad \text{on} \quad B^e(2R), \quad \omega = 0 \quad \text{on} \quad \mathbb{R}^d \setminus B^e(4R).$$

As a consequence of (2.6), we obtain for any $x \in M$ that $\omega \circ F_x \in C_0^\infty(M)$ satisfies

$$0 \le \omega \circ F_x \le 1, \quad \omega \circ F_x = 1 \quad \text{on} \quad B_x(R), \quad \omega \circ F = 0 \quad \text{on} \quad M \setminus B_x(8R).$$

Let $F_n = F_{x_n} : B_{x_n}(r_H) \to \mathbb{R}^d$ and $\psi_n = \omega \circ F_n$. Applying (1.6) to the matrix with entries

$$C_l^q = g^{kq} \frac{\partial^2 u}{\partial x^k \partial x^l},$$

we see that

$$|\Delta u|^2 \le d\, g^{ij} g^{kl} \frac{\partial^2 u}{\partial y^i \partial y^k} \frac{\partial^2 u}{\partial y^l \partial y^j}.$$

The following elementary fact from tensor algebra will be used below.

Proposition 3.1. *Let* $\{h^{ij}\}_{i,j=1,\dots,d}$ *be a non-negative real symmetric* $d \times d$ *matrix,*

$$0 < h^{ij}\xi_i\xi_j \le Q|\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

Then for any other real symmetric $d \times d$ *matrix* $\{\eta_{ij}\}_{i,j=1,\dots,d}$

$$h^{ij}h^{kl}\eta_{ik}\eta_{lj} \le Q^2 \sum_{i,k} \eta_{ik}^2.$$

In what follows, we denote by $\boldsymbol{\nabla}_e^2$ and $\boldsymbol{\nabla}_e$ the Euclidean second gradient and the Euclidean gradient. Applying Proposition 3.1 to

$$\eta_{ik} = \frac{\partial^2 \psi_n}{\partial y^i \partial y^k} \quad \text{and} \quad h^{ik} = g^{ik},$$

and taking into account that $Q = 4$, we obtain

$$|\Delta \psi_n| \le 4\sqrt{d}|\boldsymbol{\nabla}_e^2 \omega| \le P_1. \tag{3.2}$$

From (2.4) we conclude

$$|\nabla \psi_n|^2 = g^{ij}\frac{\partial \psi_n}{\partial y^i}\frac{\partial \psi_n}{\partial y^j} \le 4|\boldsymbol{\nabla}_e \omega|^2 \le P_2. \tag{3.3}$$

In (3.2) and (3.3) P_1 and P_2 are some constants.

In view of (3.1), the covering $U_n = F_n^{-1}(B^e(4R))$ is uniformly locally finite because $U_n \subset B_{x_n}(8R)$ for any n. Now, the functions

$$\phi_n(x) = \frac{\psi_n(x)}{\sqrt{\sum_l \psi_l^2(x)}},$$

are supported in U_n and satisfy $\sum_n \phi_n^2(x) = 1$.

Lemma 3.2. *The following estimates take place*:

$$|\Delta \phi_n| \le P_3, \quad |\nabla \phi_n|^2 \le P_4, \tag{3.4}$$

where P_3 *and* P_4 *are some constants which do not depend on* n.

Proof. It follows from the construction that

$$1 \le \sum_k \psi_k^2(x) \le N_0 \qquad \text{on} \qquad M. \tag{3.5}$$

Indeed, since each point $x \in M$ is covered by some ball $B_{x_n}(R)$, there exists $n = n(x)$, such that $\psi_n(x) \ge 1$, and the left inequality in (3.5) holds true. Further, the right inequality follows from (3.1).

While verifying (3.4), we apply the following elementary identities, which do not depend on the choice of local coordinates:

$$\left|\nabla\left(h^{-1/2}\right)\right|^2=\frac{1}{4h^3}|\nabla h|^2; \tag{3.6}$$

and

$$\Delta\left(h^{-1/2}\right)=-\frac{\Delta h}{2h^{3/2}}+\frac{3|\nabla h|^2}{4h^{5/2}}. \tag{3.7}$$

Note also that

$$\left|\nabla\left(\sum_k\psi_k^2\right)\right|^2=4\sum_k\psi_k^2|\nabla\psi_k|^2\leq 4N_0P_2$$

and

$$|\Delta\left(\sum_k\psi_k^2\right)|\leq\sum_k|\Delta(\psi_k^2)|\leq 2\sum_k\left(\psi_k|\Delta\psi_k|+|\nabla\psi_k|^2\right)\leq 2N_0(P_1+P_2).$$

We apply (3.6) and (3.7) to $h=\sum_k\psi_k^2$:

$$|\nabla\phi_n|^2\leq|\nabla\left(\psi_n h^{-1/2}\right)|^2\leq 2|\nabla\psi_n|^2\frac{1}{h}+2\left|\nabla\left(h^{-1/2}\right)\right|^2\psi_n^2\leq 2P_2+2N_0P_2=:P_3.$$

Similarly,

$$\begin{aligned}|\Delta\phi_n|\leq|\Delta\left(\psi_n h^{-1/2}\right)|&\leq|\Delta\psi_n|h^{-1/2}+2|\nabla\psi_n||\nabla\left(h^{-1/2}\right)|+\psi_n|\Delta\left(h^{-1/2}\right)|\\&\leq P_1+2\sqrt{N_0}P_2+N_0P_1+4N_0P_2=:P_4.\end{aligned}$$

□

Proof of Theorem 1.2. The operator B_V corresponds to the variational quotient

$$\frac{\int\limits_M V(x)|u|^2\,dv}{\int\limits_M(|\Delta u|^2+|u|^2)\,dv},\quad u\in H^2(M). \tag{3.8}$$

At first, we estimate the denominator of (3.8) from below through the similar expressions for the functions $u\phi_n$. We have

$$\frac{1}{3}\int_{U_n}|\Delta(u\phi_n)|^2\,dv\leq\int_{U_n}\phi_n^2|\Delta u|^2\,dv+4\int_{U_n}|\nabla u|^2|\nabla\phi_n|^2\,dv+\int_{U_n}|u|^2|\Delta\phi_n|^2\,dv.$$

Now, add the term $\int_{U_n} |u\phi_n|^2\, dv$ and take the summation over all n. Granting (3.4), we obtain

$$\frac{1}{3}\sum_n \int_{U_n} \left(|\Delta\,(u\phi_n)\,|^2 + |u\phi_n|^2\right)\, dv \leq$$

$$\leq \sum_n \int_{U_n} |\Delta u|^2 \phi_n^2\, dv + 4P_4 \sum_n \int_{U_n} |\nabla u|^2\, dv + P_3^2 \sum_n \int_{U_n} |u|^2\, dv + \frac{1}{3}\int_M |u|^2\, dv$$

$$\leq \int_M |\Delta u|^2\, dv + 4P_4 N_0 \int_M |\nabla u|^2\, dv + \left(\frac{1}{3} + P_3^2 N_0\right)\int_M |u|^2\, dv. \tag{3.9}$$

The equivalence of metrics (1.2) and (1.4) implies that the middle term of (3.9) can be estimated by the sum of two others:

$$\int_M |\nabla u|^2\, dv \leq C_1(d,K)\int_M |\Delta u|^2\, dv + C_2(d,K)\int_M |u|^2\, dv.$$

Therefore,

$$\int\limits_M (|\Delta u|^2 + |u|^2)\, dv \geq L \sum_n \int\limits_{U_n} \left(|\Delta(u\phi_n)|^2 + |u\phi_n|^2\right)\, dv,$$

where L depends only on the structural constants d, K and i_0.

The numerator of (3.8) can be written as a sum:

$$\int_M V|u|^2\, dv = \int_M V|u|^2 \sum_n \phi_n^2\, dv = \sum_n \int_{U_n} V|u\phi_n|^2\, dv.$$

It follows directly from the two last formulas that

$$L\frac{\int\limits_M V|u|^2\, dv}{\int\limits_M (|\Delta u|^2 + |u|^2)\, dv} \leq \frac{\sum\limits_n \int\limits_{U_n} V|u\phi_n|^2\, dv}{\sum\limits_n \int\limits_{U_n} \left(|\Delta(u\phi_n)|^2 + |u\phi_n|^2\right)\, dv}.$$

Denote by B_V^k the operator corresponding to the variational quotient

$$\frac{\int_{U_k} V|w|^2\, dv}{\int_{U_k} (|\Delta w|^2 + |w|^2)\, dv}, \quad w \in H^{2,0}(U_k). \tag{3.10}$$

As usual, $H^{2,0}(\cdot)$ stands for the closed subspace in $H^2(\cdot)$, in which the set $C_0^\infty(\cdot)$ is dense.

Keeping in mind to apply Proposition 2.2, we take $B_1 = B_V$ and $B_2 = \sum_{k=1}^\infty \oplus B_V^k$ as the operators defined on $H^2(M)$ and on $\sum_{k=1}^\infty \oplus H^{2,0}(U_k)$

correspondingly. The mapping $S : u \mapsto \{\phi_k u\}_{1\le k<\infty}$ acts from $H^2(M)$ to the space $\sum_{k=1}^{\infty} \oplus H^{2,0}(U_k)$ and satisfies (2.9), therefore

$$n(\lambda, B_V) \le \sum_{k=1}^{\infty} n(L\lambda, B_V^k). \tag{3.11}$$

Thanks to (2.1), in harmonic coordinates the variational quotient (3.10) reduces to

$$\frac{\int_{B^e(4R)} V|u|^2 \sqrt{\mathbf{g}(y)}dy}{\int\limits_{B^e(4R)} \left(\left|g^{ij}\frac{\partial^2 u}{\partial y^i \partial y^j}\right|^2 + |u|^2\right) \sqrt{\mathbf{g}(y)}dy}, \quad u \in H^{2,0}(B^e(4R)). \tag{3.12}$$

Below we use the inequality for functions $u \in H^{2,0}(B^e(4R))$:

$$\int\limits_{B^e(4R)} \left(\left|g^{ij}\frac{\partial^2 u}{\partial y^i \partial y^j}\right|^2 + |u|^2\right) dy \ge c \int\limits_{B^e(4R)} \left(|\nabla_e^2 u|^2 + |u|^2\right) dy. \tag{3.13}$$

A constant c in (3.13) depends on the ellipticity constant of the operator $\sum_{i,j=1}^{d} g^{ij}\frac{\partial^2 u}{\partial y^i \partial y^j}$ and $\|g^{ij}\|_{C^\alpha}$. This follows from a much stronger classical result (see [LU], Lemma 11.1]). In our case the ellipticity constant is $Q = 4$ and, according to (2.5),

$$\|g^{ij}\|_{C^\alpha} \le C(d, K, i_0).$$

Taking into account (2.7), (3.13) and using Rosenblum's estimate (0.2), we find that

$$n(\lambda, B_V^k) = n(\lambda, (3.12)) \le C(d, K, i_0)\lambda^{-d/4} \int_{B^e(4R)} V^{d/4}\sqrt{\mathbf{g}(y)}dy.$$

Due to (3.11), we come to (1.7):

$$n(\lambda, B_V) \le C_1 \lambda^{-d/4} \sum_k \int_{U_k} V^{d/4}\, dv \le N_0\, C_1 \lambda^{-d/4} \int_M V^{d/4}\, dv.$$

Note that both N_0 and C_1 depend only on the structural constants d, K and i_0, and we are done. □

4. Proof of Theorem 1.3

The proof follows the same line as the proof of Theorem 5.1 of [LS]. At first we consider compactly supported V. Suppose that supp$V \subset X$, where $X \subset M$ is an open set with compact closure. Along with $H^2(M)$ and B_V, consider the

Sobolev space $H^{2,0}(X)$ with the metric $\int_X (|\Delta u|^2 + |u|^2)\, dv$ and the operator $B_{X,V}$ in $H^{2,0}(X)$, generated by the quadratic form $\int_X V|u|^2 dv$.

It follows from the variational principle that

$$n(\lambda, B_{X,V}) \le n(\lambda, B_V), \quad \lambda > 0. \tag{4.1}$$

To obtain for $n(\lambda, B_V)$ an appropriate estimate from above, choose a function $\phi \in C_0^\infty(M)$, such that $\phi = 1$ on supp$V$ and $\phi = 0$ outside X. Then

$$\int_M V|u|^2 dv = \int_X V|\phi u|^2 dv. \tag{4.2}$$

Fix also a number $\varepsilon \in (0, 1)$ and consider quadratic forms

$$\tilde{a}[u] = \varepsilon \int_M |\Delta u|^2 dv + \int_M |u|^2\, dv + (1-\varepsilon) \int_M \phi^2 |\Delta u|^2\, dv$$

and

$$\hat{a}[u] = \varepsilon \int_M |\Delta u|^2 dv + \int_M |u|^2\, dv + (1-\varepsilon) \int_M |\Delta(\phi u)|^2\, dv.$$

We have

$$\int_M (|\Delta u|^2 + |u|^2) dv \ge \tilde{a}[u] = \hat{a}[u] + P[u], \tag{4.3}$$

where

$$P[u] = (1-\varepsilon) \int_M \left(\phi^2 |\Delta u|^2 - |\Delta(\phi u)|^2\right)\, dv$$

is a quadratic form with the smooth and compactly supported coefficients. Since all the products of second order derivatives of u cancel, P is compact in $H^2(M)$. Consider now Hilbert spaces $\tilde{H}^2(M)$ and $\hat{H}^2(M)$, which coincide with $H^2(M)$ algebraically and topologically, but are endowed with different metric forms, namely $\tilde{a}[u]$ and $\hat{a}[u]$ respectively. Let $\tilde{B}_V$ and $\hat{B}_V$ be the operators, generated in $\tilde{H}^2(M)$ and in $\hat{H}^2(M)$ by the same quadratic form $\int_M V|u|^2 dv$. The passage from $\tilde{B}_V$ to $\hat{B}_V$ corresponds to a compact perturbation of the metric of Hilbert space, and by Proposition 2.3 we have

$$\limsup_{\lambda \to 0} \lambda^{d/4} n(\lambda, \tilde{B}_V) = \limsup_{\lambda \to 0} \lambda^{d/4} n(\lambda, \hat{B}_V).$$

Further, by (4.3) we obtain

$$n(\lambda, B_V) \le n(\lambda, \tilde{B}_V).$$

We observe that

$$\hat{a}[u] \ge (1-\varepsilon) \left(\int_M |\Delta(u\phi)|^2\, dv + \int_M |u\phi|^2\, dv \right). \tag{4.4}$$

The mapping $u \mapsto \phi u$ acts from $H^2(M)$ into $H^{2,0}(X)$. The assumptions of Proposition 2.2 are satisfied because of (4.2) and (4.4), and we conclude that

$$n(\lambda, \hat{B}_V) \leq n(\lambda(1-\varepsilon), B_{X,V}).$$

Using the variational principle, we obtain that

$$\limsup_{\lambda\to 0} \lambda^{d/4} n(\lambda, B_V) \leq \lim_{\lambda\to 0} \lambda^{d/4} n(\lambda(1-\varepsilon), B_{X,V}).$$

The result of [BS2] gives

$$\lim_{\lambda\to 0} \lambda^{d/4} n(\lambda, B_{X,V}) = c(d) \int_X V^{d/4}\, dv.$$

Since V is a compactly supported function, we have

$$\limsup_{\lambda\to 0} \lambda^{d/4} n(\lambda, B_V) \leq (1-\varepsilon)^{-d/4} c(d) \int_M V^{d/4}\, dv.$$

It follows (as $\varepsilon \to 0$) that

$$\limsup_{\lambda\to 0} \lambda^{d/4} n(\lambda, B_V) \leq c(d) \int_M V^{d/4}\, dv.$$

The similar inequality is valid for $\liminf$. Together with (4.1), this proves (1.8) for V with compact support. The passage to any $0 \leq V \in L_{d/2}$ is straightforward, in view of continuity of the asymptotic coefficients (see [BS1, Lemma 1.19]). □

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