Mathematical Research Letters **4**, 915–929 (1997)

PRODUCT FORMULAS ALONG T^3 FOR SEIBERG-WITTEN INVARIANTS

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1. Introduction

Suppose that X is a smooth closed oriented 4-manifold, and that X contains a smoothly embedded 2-torus $T^2 \hookrightarrow X$ with trivial self-intersection number. Similarly to Dehn-surgery on knots in 3-manifolds, a *generalized logarithmic transformation* of X along T^2 is defined by deleting a tubular neighborhood of T^2 from X and gluing it back via a diffeomorphism $\phi : \partial(D^2 \times T^2) \to \partial(X \setminus nd(T^2))$.

The aim of this paper is to study the effect of this smooth operation on the Seiberg-Witten invariants of the underlying 4-manifolds. Our first result gives a relation between the Seiberg-Witten invariants of the various manifolds obtained in this way.

Theorem 1.1. *Let M be a smooth compact oriented 4-manifold with $b_2^+(M) \geq 1$, and suppose that ∂M is diffeomorphic to T^3. Let us fix a basis $a, b, c \in H_1(\partial M, \mathbb{Z})$ and for each indivisible element $\gamma = pa + qb + rc$ let us fix an orientation reversing diffeomorphism $\phi_\gamma : \partial(D^2 \times T^2) \to \partial M$ with $(\phi_\gamma)_*([\partial(D^2 \times pt)]) = \gamma$. Let*

$$M_\gamma = M(\phi_\gamma) = M \cup_{\phi_\gamma} D^2 \times T^2.$$

(Since every ϕ_γ is isotopic to a linear map, it follows that the diffeomorphism type of M_γ depends only on γ not on the choice of ϕ_γ.) If the spin^c structure $\mathcal{L}' \to M_\gamma$ restricts non-trivially to $D^2 \times T^2$, then $SW_{M_\gamma} = 0$. For each spin^c structure $\mathcal{L}_0 \to M$ that restricts trivially to ∂M let $V_\gamma(\mathcal{L}_0)$ denote the set of isomorphism classes of spin^c structures over M_γ whose restriction to M is equal to $\mathcal{L}_0$. Then we have

$$\sum_{\mathcal{L} \in V_\gamma(\mathcal{L}_0)} SW_{M_\gamma}(\mathcal{L}) =$$

$$p\left(\sum_{\mathcal{L} \in V_a(\mathcal{L}_0)} SW_{M_a}(\mathcal{L})\right) + q\left(\sum_{\mathcal{L} \in V_b(\mathcal{L}_0)} SW_{M_b}(\mathcal{L})\right) + r\left(\sum_{\mathcal{L} \in V_c(\mathcal{L}_0)} SW_{M_c}(\mathcal{L})\right),$$

where it is understood that in the $b_2^+ = 1$ case the invariants in the above formula are computed in corresponding chambers.

Received March 18, 1997.

Remark 1.2. • We choose an orientation of $\det(H^2_{\geq 0}(M,\partial M,\mathbb{R}))\otimes(\det(H^1(M,\partial M,\mathbb{R})))^{-1}$ called a *homology orientation* of $(M,\partial M)$. Standard Mayer-Vietoris arguments show that a homology orientation for $(M,\partial M)$ induces a homology orientation for M_γ, i.e. an orientation of $\det H^2_+(M_\gamma,\mathbb{R})\otimes(\det H^1(M_\gamma,\mathbb{R}))^{-1}$, and hence orients the moduli spaces for M_γ, allowing us to define the function SW_{M_γ}. In the statement of the theorem all the functions SW_{M_γ}, etc. are computed using homology orientations derived from a common homology orientation of $(M,\partial M)$.

• If $b_2^+(M)\geq 2$, then $b_2^+(M_\gamma)\geq 2$ for all γ and there are no chamber structures for the invariants. If $b_2^+(M)=1$, then $b_2^+(M_\gamma)$ equal to either 1 or 2. If $b_2^+(M_\gamma)=1$, then the inclusion $M\hookrightarrow M_\gamma$ induces an isomorphism $H^2(M,\partial M,\mathbb{R})/\{x|x\cdot y=0\ \forall y\}\to H^2(M_\gamma,\mathbb{R})$. This identification of cohomology induces an identification of positive cones with their chamber structures. Since $c_1(\det\mathcal{L}_0)\in H^2(M)$ can be lifted to $H^2(M,\partial M)$, the *spin*^{c}-structure $\mathcal{L}_0$ determines a wall in $H^2(M,\partial M,\mathbb{R})/\{x|x\cdot y=0\ \forall y\}$. If $b_2^+(M_\gamma)=1$, then the set of *spin*^{c}-structures over which we sum is exactly the set whose wall agrees with that of $\mathcal{L}_0$. It is in this sense, that we compute the invariants in corresponding chambers.

Note, that if $i_*(\gamma)$ is indivisible, where $i_*:H_1(\partial M,\mathbb{Z})\to H_1(M,\mathbb{Z})$, then the sum in the left hand side of the above formula consists only of one term, and consequently we get a formula computing SW_{M_γ} in terms of SW_{M_a}, SW_{M_b} and SW_{M_c}. So in this case Theorem 1.1 provides a useful tool to compute Seiberg-Witten invariants. For applications of Theorem 1.1 along these lines, see [FS3], [Sz1] and [Sz2].

By using cylindrical end Seiberg-Witten moduli spaces over M we define in Theorem 3.1 relative Seiberg-Witten invariants $SW_M:\mathcal{C}_{M,\partial M}\to\mathbb{Z}$, where $\mathcal{C}_{M,\partial M}$ is defined as follows:

Let $\mathcal{K}\to S^1$ denote the product spin- structure over S^1, i.e. $\mathcal{K}$ represents the non-trivial element in the 1-dimensional *spin* cobordism group. Then the spin-structure $(\mathcal{K})^3\to T^3$ is invariant under self-diffeomorphisms of T^3, and as such gives a well-defined spin^c-structure $\mathcal{K}'\to\partial M$. Now let $\mathcal{C}_{M,\partial M}$ denote the isomorphism classes of pairs consisting of *spin*^{c}-structure $\mathcal{L}\to M$ and a spin^c isomorphism from $\mathcal{L}|_{\partial M}$ to $\mathcal{K}'$.

Note that similarly to the closed case we have a map $c:\mathcal{C}_{M,\partial M}\to\mathbb{H}^2(M,\partial M,\mathbb{Z})$ given by $\mathcal{L}\mapsto c_1(\det\mathcal{L})$. If $H_1(M,\partial M,\mathbb{Z})$ has no 2-torsion, then

$$c:\mathcal{C}_{M,\partial M}\to\{x\in H^2(M,\partial M,\mathbb{Z})|x\equiv w_2(M,\partial M)\pmod 2\}$$

is a bijection, where $w_2(M,\partial M)$ is the relative second Stiefel-Whitney class $w_2(M)$ determined by the spin-structure $(\mathcal{K})^3\to\partial M$. We say that $K\in H^2(M,\partial M,\mathbb{Z})$ is a relative basic class if there is a $\mathcal{L}\in\mathcal{C}_{M,\partial M}$ with $c(\mathcal{L})=K$ and $SW_M(\mathcal{L})\neq 0$. We define $SW_M(K)=\sum_{\mathcal{L}\in c^{-1}(K)}SW_M(\mathcal{L})$. Similarly to the closed case, the set of basic classes is finite.

Our second result, presented in Theorem 4.1, gives a relation between SW_M and $SW_{M(\phi)}$. As a corollary we prove the following.

Theorem 1.3. *Let M, γ, ϕ_γ and M_γ be as above. Suppose that $b_2^+(M) > 1$, and that $\gamma \in \operatorname{Ker} i_*$, where $i_* : H_1(\partial M, \mathbb{R}) \to H_1(M, \mathbb{R})$. Let $\mathcal{B}_M$, $\mathcal{B}_{M_\gamma}$ denote the set of basic classes of M and M_γ. We define formal series on $H_2(M_\gamma, \mathbb{R})$ by*

$$SW^*_{M_\gamma} = \sum_{K \in \mathcal{B}_{M_\gamma}} SW_{M_\gamma}(K)(\exp K).$$

Then we have

$$SW^*_{M_\gamma} = \left(\sum_{L \in \mathcal{B}_M} SW_M(L)(\exp(\rho(L))) \right) (\exp(T_\gamma) - \exp(-T_\gamma))^{-1},$$

where $\rho : H^2(M, \partial M, \mathbb{Z}) \to H^2(M_\gamma, \mathbb{Z})$ is given by the Mayer-Vietoris sequence, and where T_γ is the Poincare dual of the core $\{pt\} \times T^2 \hookrightarrow M_\gamma$.

Let $\mathrm{Diff}^+(M)$ denote the group of orientation preserving self-diffeomorphisms of M. In case $\mathrm{Diff}^+(M)$ is large enough, we also have a stonger result on how SW_{M_γ} depends on γ:

Corollary 1.4. *Let M be as in Theorem 1.1, and suppose that $b_2^+(M) > 1$. Suppose that there is a basis $a, b, c \in H_1(\partial M, \mathbb{Z})$ with the following properties:*

(i) *There is a 2-dimensional linear subspace $V \subset H_2(M, \partial M, \mathbb{Z})$ such that $j : H_2(M, \partial M, \mathbb{Z}) \to H_1(\partial M, \mathbb{Z})$ gives an isomorphism between V and the subspace of $V' \subset H_1(\partial M, \mathbb{Z})$ spanned by b, c.*

(ii) *For all $A \in SL(V')$, there is an $f \in \mathrm{Diff}^+(M)$, such that $f_*(V) = V$, and $(f_*)|_{V'} = A$.*

(iii) *$a \in \operatorname{Ker} i_*$, where $i_* : H_1(\partial M, \mathbb{R}) \to H_1(M, \mathbb{R})$.*

Then for all $\gamma = pa + qb + rc \in H_1(\partial M, \mathbb{Z})$ indivisible element we have

- *If $p = 0$, then $SW_{M_\gamma} = 0$.*
- *If $p \neq 0$, then*

$$SW^*_{M_\gamma} = SW^*_{M_a} \cdot \frac{\sinh(T_a)}{\sinh(T_\gamma)} = SW^*_{M_a} \cdot \frac{\sinh(T_a)}{\sinh(T_a/p)},$$

where T_a, T_γ denote the Poincare dual of the cores in M_a and M_γ.

Remark 1.5. • If $p \neq 0$, then there is a natural isomorphism between $T_\gamma^\perp \subset H^2(M_\gamma, \mathbb{Q})$ and $T_a^\perp \subset H^2(M_a, \mathbb{Q})$ defined by lifting any $x \in T_\gamma^\perp$ to a class in $\tilde{x} \in H^2(M, \partial M, \mathbb{Q})$ such that $\langle \tilde{x}, V \rangle = 0$ and extending by zero to define a class in $T_a^\perp$. Note that under this identification $T_\gamma = T_a/p$.

• Note that if the core $(pt \times T^2) \to M_a$ lies in a cusp neighborhood, cf. [FS1], [GM], and $[pt \times T^2]$ is not a torsion class, then M satisfies the conditions in Corollory 1.4. In this special case the formula of Corollary 1.4 was proved previously in [FS2] by using rational blowdowns.

As another corollary we get the following result.

Corollary 1.6. *Let X be a smooth closed oriented 4-manifold with $b_2^+(X) > 1$. Suppose that X contains a smoothly embedded 2-torus $T^2 \hookrightarrow X$, with self-intersection 0, such that $[T^2] \in H_2(X, \mathbb{Z})$ is not a torsion class. Then X has Seiberg-Witten simple type, i.e. if $\mathcal{L} \in \mathcal{C}_X$ satisfies*

$$d(\mathcal{L}) = \frac{1}{4}(c_1(\det \mathcal{L})^2 - 2e(X) - 3\,\mathrm{sign}(X)) > 0,$$

then $SW_X(\mathcal{L}) = 0$.

These results are proved by using cylindrical end Seiberg-Witten moduli spaces over M. The technical background is presented in Section 2. In Section 3 we define relative Seiberg-Witten invariants and prove some basic properties. The proof of Theorem 1.1, Theorem 1.3, Corollary 1.4 and Corollary 1.6 are given in Section 4.

Other product formulas along T^3 for the generalized fiber sum operation will be presented in [MMSzT] by using different techniques. Similar results for Donaldson invariants were given in [MSz].

For analogous results in the 3-dimensional case and applications, see [MT], [L], [C].

2. Cylindrical end moduli spaces

Suppose that M is a smooth oriented 4-manifold, and that the end of M is diffeomorphic to $T^3 \times [0, \infty)$. Fix a flat riemannian metric h on T^3, and a riemannian metric g on M such that g is equal to $h + dt^2$ near the end of M. Let us briefly recall the setup for Seiberg-Witten invariants. For more details see [KM2], [M], [Wi].

For any compactly supported real smooth self-dual two-form $\mu \in \Omega^2_+(M, \mathbb{R})$ and *spin*^{c} structure $\mathcal{L} \to M$ we define the Seiberg-Witten moduli space in the following way. Let $S^{\pm}(\mathcal{L})$ denote the plus and minus spin bundles associated to $\mathcal{L}$, and let $L = \det(\mathcal{L})$. Let $\mathcal{A}(L)$ denote the space of $L^2_{2,loc}$ connections on L, and $\Gamma(S^+(\mathcal{L}))$ the space of $L^2_{2,loc}$ sections in $S^+(\mathcal{L})$. The perturbed Seiberg-Witten equations are

- $F_A^+ = q(\phi) + i\mu$,
- $D_A(\phi) = 0$,

where $A \in \mathcal{A}(L)$, $\phi \in \Gamma(S^+(\mathcal{L}))$, $q : \Gamma(S^+(\mathcal{L})) \to \Omega^2_+(M, i\mathbb{R})$ is a quadratic bundle map defined by $q(\phi) = \phi \otimes \phi^* - (|\phi|^2/2)\,\mathrm{Id}$, and $D_A : \Gamma(S^+(\mathcal{L})) \to \Gamma(S^-(\mathcal{L}))$ is the Dirac operator associated to A. The *energy* of a solution of (A, ϕ) is defined by $\int_M |F_A|^2 dvol$. The moduli space $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is defined by dividing the space of finite energy solutions by the action of the gauge group $\mathcal{G}(M)$, where $\mathcal{G}(M)$ is the group of $L^2_{3,loc}$ maps $u : M \to U(1)$.

Our goal in this section is to study the qualitative nature of this moduli space. The first step is to understand solutions on T^3 and $T^3 \times \mathbb{R}$.

Notice that every *spin*^{c} structure $\mathcal{L} \to T^3 \times \mathbb{R}$ pulls back from T^3, and Clifford multiplication by dt gives an identification between $S^+(\mathcal{L})$ and $S^-(\mathcal{L})$. Let $S(\mathcal{L})$ denote the restriction of $S^+(\mathcal{L})$ to T^3, and take $L_0 = \det(\mathcal{L})$. Let us fix the riemannian metric $h + dt^2$, where h is a flat metric on T^3. Using the product structure over the cylinder we can write the connections in temporal gauge, and the Seiberg-Witten equations can be written as the gradient flow equation of the functional, see [KM2], [MSzT],

$$(1) \quad f(A,\phi) = \int_{T^3} a \wedge F_{A_0} + \frac{1}{2}\int_{T^3} a \wedge da + \int_{T^3} \langle \phi, D_A\phi \rangle dvol,$$

where $A \in \mathcal{A}(L_0)$, $\phi \in \Gamma(S(\mathcal{L}))$, A_0 is a fixed background connection and $a = A - A_0$.

The critical points of $f(A,\phi)$ are the solutions of the three dimensional analogue of the Seiberg-Witten equations:

- $F_A = q(\phi)$,
- $D_A(\phi) = 0$,

where A and ϕ are in $L^2_{3/2}$. After dividing out by the action of the gauge group $\mathcal{G}(T^3)$, i.e the group of $L^2_{5/2}$ maps $u : T^3 \to U(1)$, we get the moduli space $\mathcal{M}_{T^3}(\mathcal{L}, h)$.

Lemma 2.1. *Suppose that h is a flat metric on T^3, and let $\mathcal{L}_0$ denote the trivial spin^c structure over T^3. Then $\mathcal{M}_{T^3}(\mathcal{L}_0,h) = \mathcal{F}/\mathcal{G}(T^3)$, where $(A,\phi) \in \mathcal{F}$ if A is a flat S^1 connection over T^3, and $\phi \equiv 0$. In particular $\mathcal{M}_{T^3}(\mathcal{L}_0,h)$ can be identified with T^3. Furthermore if $\mathcal{L} \neq \mathcal{L}_0$ then $\mathcal{M}_{T^3}(\mathcal{L},h)$ is empty.*

Proof. Since h is a flat metric, the Weitzenbock formula for Dirac operator, see [KM2] or [M], implies that if $(A,\phi) \in \mathcal{M}_{T^3}(\mathcal{L},h)$ then $\phi \equiv 0$. This implies that $F_A = 0$ and $\mathcal{L}$ trivial. □

The component group of $\mathcal{G}(T^3) = \mathcal{G}(\partial M)$ is identified with $H^1(T^3,\mathbb{Z})$. Since $T^3 = \partial M$, we get a distinguished subgroup $\mathcal{G}_0(T^3) \subset \mathcal{G}(T^3)$: $u \in \mathcal{G}_0(T^3)$ if u can be extended to a $u' \in \mathcal{G}(M)$. Another interpretation of $\mathcal{G}_0(T^3)$ is as follows. Let $\rho \in H^1(U(1),\mathbb{Z})$ be the fundamental cohomology class of $U(1)$. Then $u : T^3 \to U(1)$ is in $\mathcal{G}_0(T^3)$ if $u^*(\rho)$ is in the image of $i^* : H^1(M,\mathbb{Z}) \to H^1(T^3,\mathbb{Z})$. (Although $\mathcal{G}_0(T^3)$ depends on M, we will suppress this in order to keep the notation simple.)

From now on we use the notations $\chi(T^3) = \mathcal{M}_{T^3}(\mathcal{L}_0,h) = \mathcal{F}/\mathcal{G}(T^3)$ and $\chi_0(T^3) = \mathcal{F}/\mathcal{G}_0(T^3)$. Notice that dividing by $\mathcal{G}(T^3)$ gives a covering $p : \chi_0(T^3) \to \chi(T^3)$. Note also that since the gauge group $(S^1)^{T^3}$ acts on the connections on $\det(\mathcal{L}_0)$ by the square of the usual action $\chi(T^3)$ is naturally a $(\mathbb{Z}/2)^3$ covering of $\mathrm{Hom}(\pi_1(T^3),U(1))$.

Standard techniques and Lemma 2.1 imply the following.

Theorem 2.2. *Let M be a smooth oriented cylindrical-end 4-manifold, such that the end of M is diffeomorphic to $T^3 \times [0,\infty)$. Let h be a flat metric on T^3,* *and g a riemannian metric on M, such that g is equal to $h+dt^2$ at the end of M. Let $\mu \in \Omega^2_+(M, \mathbb{R})$ be a compactly supported self-dual two-form. Let $\mathcal{L} \to M$ be a spin^c structure and $\mathcal{L}|_{T^3} \to T^3$ be its restriction to a slice. If $\mathcal{L}|_{T^3}$ is nontrivial then $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is empty. Furthermore if $\mathcal{L}|_{T^3}$ is trivial then $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is compact, and by taking limits at the non-compact end of the tube $T^3 \times [0 \times \infty)$ we have a continuous map $\partial_\infty : \mathcal{M}_M(\mathcal{L}, g, \mu) \to \chi_0(T^3)$. The homology orientation of $(M, \partial M)$ induces an orientation of $\mathcal{M}_M(\mathcal{L}, g, \mu)$.*

Proof. Suppose that $(A, \phi) \in \mathcal{M}_M(\mathcal{L}, g, \mu)$. Since (A, ϕ) has finite energy, it follows from [KM2, Proposition 8], that $\mathcal{M}_{T^3}(\mathcal{L}|_{T^3}, h)$ is not empty. Now Lemma 2.1 implies that $\mathcal{L}|_{T^3}$ is trivial. If $\mathcal{L}|_{T^3}$ is trivial, then the function f given in Equation 1 descends to a real valued function on the configuration space modulo gauge equivalence. Now the connectedness of the critical set implies that every finite energy flowline over $T^3 \times \mathbb{R}$ is static, i.e pulls back from T^3. This, and the arguments in [KM2, Lemma 4] imply that $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is compact. The existence of the continuous map ∂_∞ follows from the arguments of [MMR, page 63-70]: Just as the Chern-Simons functional is real analytic, it is easy to see that f given in Equation 1 is real analytic as well. So it is easy to see, that Simon's result [Si] applies in this context as well, showing that a finite energy path in the space of configurations has finite length. From that it is straightforward to deduce the existence of the continuous map ∂_∞. The statement about the orientation of the moduli space is completely analogous to the closed case. □

From now on we study the local properties of the map $\partial_\infty : \mathcal{M}_M(\mathcal{L}, g, \mu) \to \chi_0(T^3)$. Since this is equivalent to studying $\overline{\partial}_\infty = p \circ \partial_\infty$ we work over $\chi(T^3)$.

We shall see that $\chi(T^3)$ has a singular point. This creates singularities in $\mathcal{M}_M(\mathcal{L}_0, g, \mu)$. It turns out that, at least when the dimension of the moduli space $\mathcal{M}_M(\mathcal{L}_0, g, \mu)$ is less or equal to 4, the moduli space is a compact manifold with boundary, the boundary mapping to the singular point.

Let $(A, 0) \in \chi(T^3)$. The linearization of the 3-dimensional Seiberg-Witten equations around $(A, 0)$ gives

$$(2) \quad L^2_{5/2}(X, i\mathbb{R}) \xrightarrow{d_1} L^2_{3/2}(T^*(T^3) \otimes i\mathbb{R} \oplus S(\mathcal{L})) \xrightarrow{d_2} L^2_{1/2}(T^*(T^3) \otimes i\mathbb{R} \oplus S(\mathcal{L})),$$

where $d_1(f) = (2df, 0)$, and $d_2(a, \phi) = (*da, D_A(\phi))$.

Lemma 2.3. *For every $(A, 0) \in \chi(T^3)$ let $\mathcal{H}^1_{(A,0)}$ denote the vector space $\operatorname{Ker} d_1^* \cap \operatorname{Ker} d_2$, where $d_1^* : L^2_{3/2}(T^*(T^3) \otimes i\mathbb{R} \oplus S(\mathcal{L})) \longrightarrow L^2_{1/2}(T^3, i\mathbb{R})$ is the L^2-adjoint of d_1. Then $\mathcal{H}^1_{(A,0)} = \mathcal{H}^1(T^3, i\mathbb{R}) \oplus \operatorname{Ker}(D_A)$, where $\mathcal{H}^1(T^3, i\mathbb{R})$ is the space of purely imaginary harmonic 1-forms on T^3. There is a unique point $\theta = (\theta_0, 0)$ such that $\operatorname{Ker}(D_{\theta_0}) \neq 0$. Furthermore $\operatorname{Ker} D_{\theta_0} = \mathbb{C}^2$ is the space of contant sections of the trivialized spin-bundle $S^+(\mathcal{L}_0)$.*

Proof. It follows from (2) that $\mathcal{H}^1_{(A,0)} = \mathcal{H}^1(T^3, i\mathbb{R}) \oplus \operatorname{Ker} D_A$. Since h is a flat metric, it follows from the Weitzenbock formula that for $u \in \operatorname{Ker} D_A$ we have

$\nabla_A(u) = 0$. Since $S(\mathcal{L}) \to T^3$ is a trivial bundle, $u \neq 0$ implies that A is the trivial connection, and u is a constant section. □

Theorem 2.4. *With M, h, g, $\mathcal{L}$ as above, suppose that $b_2^+(M) > 0$ and that $\mathcal{L}|_{T^3}$ is trivial. Fix a homology orientation of $(M, \partial M)$. Let $U \subset \chi(T^3)$ be a small neighborhood of θ. Then for all generic compactly supported 2-forms $\mu \in \Omega^2(M, \mathbb{R})$ the moduli space $\mathcal{M}_M(\mathcal{L}, g, \mu) \setminus ((\overline{\partial}_\infty)^{-1}(U))$ is either empty or is a smooth compact oriented manifold of dimension*

$$d = \frac{1}{4}(1 + (c_1(L)^2 - 2e(M) - 3\operatorname{sign}(M))$$

with boundary. Furthermore if g and μ are fixed, then there are only finitely many spin^c structures $\mathcal{L} \to M$, for which the moduli space $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is not empty.

Proof. Suppose that $x \in \chi(T^3)$ and let μ_x denote the smallest absolute value of a non-zero eigenvalue of the complex (2) based at x. Let $h^0(x) = \dim(\operatorname{Ker} d_1)$, $h^1(x) = \dim \mathcal{H}^1_x = \dim(\operatorname{Ker} d_1^* \cap \operatorname{Ker} d_2)$. By Lemma 2.3 $h^0(x)$, $h^1(x)$ are constants on $\chi(T^3) \setminus U$. Since $\chi(T^3) \setminus U$ is compact it follows that there is a $\delta > 0$ such that $\delta \leq \mu_x/2$ for all $x \in \chi(T^3) \setminus U$. It follows that all $A \in \mathcal{M}_M(\mathcal{L}, g, \mu) \setminus (\overline{\partial}_\infty)^{-1}(U)$ decays exponentially to $\partial_\infty(A)$ with exponent at least δ. The smoothness of $\mathcal{M}_M(\mathcal{L}, g, \mu)$ for generic μ follows from standard arguments. In order to compute the formal dimensions of $\partial_\infty^{-1}(x)$ and $\mathcal{M}_M(\mathcal{L}, g, \mu)$ we use the index formula of [APS], which gives

$$\dim(\ \partial_\infty^{-1}(x)) = \frac{1}{4}(c_1(L)^2 - 2e(M) - 3\operatorname{sign}(M)) - \frac{h^0(x) + h^1(x)}{2} + \frac{\rho(x)}{2},$$

where $\rho(x)$ is the ρ invariant of the boundary operator $(d_1^* + d_2)$. In this case $\rho(x)$ doesn't depend on the choice of flat metric h, and $\rho(x)$ is constant on $\chi(T^3) \setminus \theta$. Since T^3 admits an orientation reversing self diffeomorphism it follows that $\rho(x) = 0$. Now $h^1(x) = 3$, $h^0(x) = 1$ implies

$$\dim \mathcal{M}_M(\mathcal{L}, g, \mu) = h^1(x) + \dim(\ \partial_\infty^{-1}(x)) = 1 + \frac{1}{4}(c_1(L)^2 - 2e(M) - 3\operatorname{sign}(M)).$$

Since μ is compactly supported, and there are no non-trivial finite energy flowlines on the tube, it follows from arguments in[KM2, Corollary 3], that for all but finitely many $\mathcal{L}$ the moduli space $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is empty. □

We still have to understand the local structure of $\mathcal{M}_M(\mathcal{L}, g, \mu)$ around the singular points $(\overline{\partial}_\infty)^{-1}(\theta)$.

From now on we work in a neighborhood of $\theta = (\theta_0, 0) \in \mathcal{A}(L) \otimes \Gamma(S(\mathcal{L}))$ and use the identification between $L^2_{3/2}(T^*(T^3) \otimes i\mathbb{R} \oplus S(\mathcal{L}))$ and $\mathcal{A}(L) \otimes \Gamma(S(\mathcal{L}))$ given by $(a, \phi) \to (\theta_0 + a, \phi)$.

In order to study the gradient flow of the functional f defined in (1) we use the center manifold technique of [MMR, page 78-92]. A *center manifold* around θ is a C^2 manifold which contains a neighborhood of θ in $\chi(T^3)$, its tangent space at θ is $\mathcal{H}^1_\theta$ and it is invariant under the gradient flow. The restriction of the flow to the center manifold gives a finite dimensional approximation to the flow around θ and determines the local structure of the cylindrical-end moduli space around the singular boundary value θ. In this case the center manifold can be taken to be $\mathcal{H}^1_\theta$:

Lemma 2.5. *At every $(a,\phi) \in \mathcal{H}^1_\theta$ the gradient vector $\nabla_f(a,\phi)$ is tangent to $\mathcal{H}^1_\theta$. In particular $\mathcal{H}^1_\theta$ is a center manifold around θ.*

Proof. Let $(a,\phi) \in \mathcal{H}^1_\theta$, and $A = \theta_0 + a$. Then

$$\nabla_f(a,\phi) = (*(F_A - q(\phi)), D_A(\phi)).$$

Since A is flat and ϕ is constant we have $\nabla_f(a,\phi) = (-*(q(\phi)), \frac{1}{2}a\cdot\phi)$. Since h is a flat metric on T^3, the space of purely imaginary harmonic 1-forms can be characterized as the space of constant sections of $\bigwedge^1(T^3, i\mathbb{R})$. It follows that $*(q(\phi))$ is a harmonic 1-form and $a\cdot\phi$ is a constant section of $S(\mathcal{L})$. This finishes the proof of Lemma 2.5. □

Now we have to study the gradient flow on $\mathcal{H}^1_\theta$. Let us fix an isomorphism between $\mathcal{H}^1_\theta$ and $\mathrm{Im}\,\mathbb{H}\oplus\mathbb{H}$, such that $*q(\phi)$ and $a\cdot\phi$ corresponds to $\phi i\overline{\phi}$ and $a\phi i$, where here a is an imaginary quaternion, $\phi \in \mathbb{H}$ and we are using quaternion multiplication. Now $\nabla_f(a,\phi) = (-\frac{1}{2}\phi i\overline{\phi}, a\phi i)$, and the gradient flow equations are

- $a'(t) = -\frac{1}{2}\phi(t)i\overline{\phi}(t)$,
- $\phi'(t) = a(t)\phi(t)i$

Lemma 2.6. *The quantities $2|a|^2 - |\phi|^2$ and $a \times (\phi i\overline{\phi})$, where $a\times b = \mathrm{Im}\,(a\cdot b)$ is the cross product in $\mathrm{Im}\,\mathbb{H}$, are preserved under the gradient flow.*

Proof. Since $\overline{a} = -a$, and $\overline{(\alpha\beta)} = \overline{\beta}\,\overline{\alpha}$ for $\alpha,\beta\in\mathbb{H}$, we have $(\overline{\phi})' = -i\overline{\phi}\,\overline{a} = i\overline{\phi}a$. Now

$$(2a\overline{a} - \phi\overline{\phi})' = a\phi i\overline{\phi} + \phi i\overline{\phi}a - \phi i\overline{\phi}a - a\phi i\overline{\phi} = 0.$$

Similarly

$$(a\times\phi i\overline{\phi})' = (-\frac{1}{2}\phi i\overline{\phi})\times\phi i\overline{\phi} + a\times(-2|\phi|^2\cdot a) = 0.$$

This finishes the proof of Lemma 2.6. □

We say that $y \in \mathcal{H}^1_\theta$ is in the *stable set* $\mathcal{S}$ if the flowline starting at y converges to some $(a,0)$. If the flowline converges to $(0,0)$ then we say that y is in the stable set of the origin. The stable set of the origin is denoted by $\mathcal{S}_\theta$.

Lemma 2.7. *Let $(a,\phi)\in\mathcal{H}^1_\theta$. Then $(a,\phi)\in\mathcal{S}$ if and only if $2|a|^2 \geq |\phi|^2$, $a\times\phi i\overline{\phi} = 0$ and $\langle a, \phi i\overline{\phi}\rangle \geq 0$, where $\langle\ ,\ \rangle$ is the usual real inner product on $\mathrm{Im}\,\mathbb{H}$. Also $(a,\phi)\in\mathcal{S}_\theta$ if and only if $(a,\phi)\in\mathcal{S}$ and $2|a|^2 = |\phi|^2$. Furthermore $\mathcal{S}\setminus(0,0)$ is a 5-dimensional manifold with boundary, where $\partial(\mathcal{S}\setminus(0,0)) = \mathcal{S}_\theta\setminus(0,0)$.*

Proof. It follows from Lemma 2.6 that $2|a|^2 \geq \phi|^2$, $a \times \phi i\overline{\phi} = 0$ and $\langle a, \phi i\overline{\phi}\rangle \geq 0$ are necessary conditions for (a, ϕ) to be in $\mathcal{S}$, and it is easy to check that they are also sufficient. The rest of the lemma is obvious. □

Let $U' \subset \operatorname{Ker} d_1^*$ be a small neighborhood of θ. Since $\mathcal{M}_M(\mathcal{L}, g, \mu)$ is compact, we can fix $T_0 >> 0$ and a neighborhood U of θ in $\chi(T^3)$ satisfying the following: $U \subset U' \cap \mathcal{H}^1_\theta$, and for all $A \in \mathcal{M}(U) = (\overline{\partial}_\infty)^{-1}(U)$ and $t \geq T_0$, the restriction of A to the slice $T^3 \times t$ is in U'.

Let $\tilde{\mathcal{M}}(U) \to \mathcal{M}(U)$ denote the base fibration, let $\mathcal{S}(U)$ denote the stable set of U, and let μ_θ be defined as above. It follows from the arguments of [MMR], that if U and U' are sufficiently small then we have a well-defined continuous map

$$\hat{\partial}_\infty : \tilde{\mathcal{M}}(U) \to \mathcal{S}(U),$$

such that the restriction of $A \in \tilde{\mathcal{M}}(U)$ to the tube $T^3 \times [T_0, \infty)$ is exponentially close, with exponent at least $\mu_\theta/2$ to the flowline in the center manifold starting from $\hat{\partial}_\infty(A)$. Now as a corollary of Lemma 2.7 and the arguments of [MMR] we have the following structure theorem.

Theorem 2.8. *Fix a metric g as above. The following holds for generic μ. Let $\mathcal{K} \subset \mathcal{M}_M(\mathcal{L}, g, \mu)$ be defined as $\mathcal{K} = (p \circ \partial_\infty)^{-1}(\theta)$, and let $\mathcal{K}' \subset \mathcal{K}$ denote the part of $\mathcal{K}$ where the decay on the tube is exponential, with exponent at least $\mu_\theta/2$. Then generically $\mathcal{K}$ and $\mathcal{K}'$ are smoothly stratified spaces, satisfying* $\dim \mathcal{K}' = \dim(\mathcal{M}_M(\mathcal{L}, g, \mu)) - 5$, $\dim \mathcal{K} = \dim(\mathcal{M}_M(\mathcal{L}, g, \mu)) - 1$, *where in these equations* dim *means the formal dimension, i.e. minus the Euler characteristic of the appropriate elliptic complex. Furthermore $\mathcal{M}_M(\mathcal{L}, g, \mu)\backslash\mathcal{K}'$ is a smooth oriented manifold of dimension equal to its formal dimension, with boundary $\mathcal{K} \setminus \mathcal{K}'$.*

□

3. Relative invariants

Let M be a smooth oriented compact 4-manifold with boundary, and suppose that ∂M is diffeomorphic to T^3. Let $\mathcal{C}'_M$ denote the set of isomorphism classes of *spin^c* structures $\mathcal{L} \to M$ such that $\mathcal{L}|_{\partial M}$ is trivial, and let $\mathcal{C}_{M,\partial M}$ be defined as in Section 1. Let $r : \mathcal{C}_{M,\partial M} \to \mathcal{C}'_M$ denote the forgetful map.

Our aim in this section is to define relative Seiberg-Witten invariants

$$SW_M : \mathcal{C}_{M,\partial M} \to \mathbb{Z}$$

by using cylindrical end moduli spaces over M. So let $M' = M \cup_{T^3} T^3 \times [0, \infty)$, fix a flat metric h on T^3, and a corresponding cylindrical end metric g on M'. Fix $\mathcal{L}$ and let $\mathcal{L}_0 = r(\mathcal{L})$, and for a generic compactly supported $\mu \in \Omega^2_+(M', \mathbb{R})$ let $\mathcal{M}_M(\mathcal{L}_0, g, \mu)$ denote the cylindrical end moduli space over M'. Then by Theorem 2.2, we have continuous maps

$$\partial_\infty : \mathcal{M}_M(\mathcal{L}_0, g, \mu) \to \chi_0(T^3),$$

and $\overline{\partial}_\infty = p \circ \partial_\infty : \mathcal{M}_M(\mathcal{L}_0, g, \mu) \to \chi(T^3)$. Let $\theta \in \chi(T^3)$ be the trivial solution, and let $\Lambda(\mathcal{L}_0) = p^{-1}(\theta)$ denote the lattice of singular points in $\chi_0(T^3)$. It is easy to see, that there is a natural one-to-one correspondence between $\Lambda(\mathcal{L}_0)$ and $r^{-1}(\mathcal{L}_0)$. In particular there is a $\theta(\mathcal{L}) \in \Lambda(\mathcal{L}_0)$ that corresponds to $\mathcal{L}$. Now we take a geometric representative D of $\mu(pt)^{d/2}$, where

$$d = d(\mathcal{L}) = \frac{1}{4}(c_1(\det \mathcal{L}_0)^2 - 2e(M) - 3sign(M)),$$

and define

$$\mathcal{N}_M(\mathcal{L}, g, \mu, D) = \mathcal{M}_M(\mathcal{L}_0, g, \mu) \cap D \cap \partial_\infty^{-1}(\theta(\mathcal{L})).$$

Note. If $d \equiv 1$ (2), then we define the relative invariant SW_M to be zero. Of course it is clear that, $d = \dim \mathcal{M}_{M \cup (D^2 \times T^2)}$, so if d is odd, then all the invariants related in Theorem 1.1 and Theorem 1.2 are zero. From now on we assume $d \equiv 0$ (2). The usual cobordism arguments and Theorem 2.8 imply the following.

Theorem 3.1. *Let M, $\mathcal{L}$, h, g be as above, and suppose that $b_2^+(M) > 1$. Then for all generic μ and D, the moduli space $\mathcal{N}_M(\mathcal{L}, g, \mu, D)$ is a compact, oriented 0-dimensional manifold, and by counting its points with signs, we define*

$$SW_M(\mathcal{L}) = \#(\mathcal{N}_M(\mathcal{L}, g, \mu, D)).$$

$SW_M(\mathcal{L})$ is independent of h, g, μ and D. $SW_M(\mathcal{L})$ is zero for all but finitely many $\mathcal{L} \in \mathcal{C}_{M,\partial M}$. Furthermore for any orientation preserving self-diffeomorphism $f : M \to M$, and $\mathcal{L} \in \mathcal{C}_{M,\partial M}$ we have

$$SW_M(\mathcal{L}) = (-1)^\epsilon SW_M(f^*(\mathcal{L})),$$

where $\epsilon \in \mathbb{Z}_2$ is the sign of the action of f^* on the homology orientation of $(M, \partial M)$. □

Remark 3.2. *If $b_2^+(M) = 1$, then the same definition gives the invariant $SW_M(\mathcal{L}, g, \mu)$, which in general can depend on g and μ.*

An equivalent definition of SW_M, is the following. Take a small closed ball $B_\mathcal{L}$ around $\theta(\mathcal{L}) \in \chi_0(T^3)$, and let $S^2_\mathcal{L} = \partial(B_\mathcal{L})$. Now let us define

$$\mathcal{N}'_M(\mathcal{L}, g, \mu, D) = \mathcal{M}_M(\mathcal{L}_0, g, \mu) \cap D \cap \partial_\infty^{-1}(S^2_\mathcal{L}).$$

It follows from Theorem 2.8, that for all generic μ and D, the moduli space $\mathcal{M}_M(\mathcal{L}_0, g, \mu) \cap D \cap \partial_\infty^{-1}(B_\mathcal{L})$ is a compact 1-manifold with boundary. Its boundary is $-\mathcal{N}'_M(\mathcal{L}, g, \mu, D) \coprod \mathcal{N}_M(\mathcal{L}, g, \mu, D)$. This implies:

Lemma 3.3. *If $b_2^+(M) \geq 1$, then for all generic μ and D, the moduli space $\mathcal{N}'_M(\mathcal{L}, g, \mu, D)$ is a compact oriented 0-manifold, and we have*

$$SW_M(\mathcal{L}) = \#(\mathcal{N}'_M(\mathcal{L}, g, \mu, D)).$$

□

This description of $SW_M(\mathcal{L})$ leads to the following simple type result.

Theorem 3.4. *Suppose that $b_2^+(M) \geq 1$, and the kernel of $i_* : H_1(T^3, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is non-trivial. Then for each $\mathcal{L} \in \mathcal{C}_M$ with $d(\mathcal{L}) > 0$ we have*

$$SW_M(\mathcal{L}) = 0,$$

where in the $b_2^+(M) = 1$ case this invariant is computed in the chamber determined by (g, μ).

Proof. Let us take a geometric representative D' of $\mu(pt)^{\frac{d}{2}-1}$. Then

$$\mathcal{N} = \mathcal{M}_M(\mathcal{L}_0, g, \mu) \cap D' \cap \partial_\infty^{-1}(S^2_{\mathcal{L}})$$

is a smooth, closed oriented 2-manifold. Let $\tilde{\mathcal{N}} \to \mathcal{N}$ be the base fibration, and let $c_1(\tilde{\mathcal{N}})$ denote its first Chern class. Then it follows from Lemma 3.3, that

$$SW_M(\mathcal{L}) = \langle c_1(\tilde{\mathcal{N}}), \mathcal{N} \rangle.$$

We compute $c_1(\tilde{\mathcal{N}})$ by making use of the map

$$\hat{\partial}_\infty : \tilde{\mathcal{M}}_M(\mathcal{L}_0, g, \mu) \cap \partial_\infty^{-1}(B_{\mathcal{L}}) \to \mathcal{H}^1_\theta$$

defined in Section 2. Note that $\hat{\partial}_\infty$ maps $\tilde{\mathcal{N}}$ to the stable set $\mathcal{S}(S^2_{\mathcal{L}})$ of $S^2_{\mathcal{L}}$. It is easy to see, cf. Lemma 2.6 and Lemma 2.7, that $\mathcal{S}(S^2_{\mathcal{L}})$ is the total space of a 2-dimensional disc bundle over $S^2_{\mathcal{L}}$, with first Chern class 1, where the zero section corresponds to $S^2_{\mathcal{L}} \subset \mathcal{H}^1_\theta$. Since $(\hat{\partial}_\infty^{-1}(0,0)) \cap D'$ is empty by dimension counting, it follows from the continuity of $\hat{\partial}_\infty$, that if $B_{\mathcal{L}}$ is chosen to be small enough, then $\hat{\partial}_\infty(\tilde{\mathcal{N}}) \subset \mathcal{S}(S^2_{\mathcal{L}}) \setminus S^2_{\mathcal{L}}$. Since $\hat{\partial}_\infty$ is S^1-equivariant, it follows that $c_1(\tilde{\mathcal{N}})$ is equal to the degree of $\partial_\infty : \mathcal{N} \to S^2_{\mathcal{L}}$.

After removing small open balls around the singular points of $\chi_0(T^3)$ we get a connected manifold $Y \subset \chi_0(T^3)$. Since Y is connected, we have a well defined degree deg of $\partial_\infty : \mathcal{M}_M(\mathcal{L}_0, g, \mu) \cap D' \cap \partial_\infty^{-1}(Y) \to Y$. Of course deg is equal to the degree of $\partial_\infty : \mathcal{N} \to S^2_{\mathcal{L}}$. On the other hand, since $\operatorname{Ker} i_* \neq 0$, it follows, that $\chi_0(T^3)$ and Y are non-compact, and so the compactness of $\mathcal{M}_M(\mathcal{L}_0, g, \mu) \cap D'$ implies that $deg = 0$. This finishes the proof of Theorem 3.4 □

4. Applications

Let M be a smooth compact oriented 4-manifold with boundary, and suppose that $\partial M = T^3$. For any orientation reversing diffeomorphsim $\phi : \partial(D^2 \times T^2) \to \partial M$ we define a closed oriented 4-manifold $M(\phi) = M \cup_\phi (D^2 \times T^2)$.

In this section we establish relations between the relative Seiberg-Witten invariant of M and the Seiberg-Witten invariant of $M(\phi)$.

Let $\gamma \in \phi_*([\partial D^2 \times pt]) \in H_1(\partial M, \mathbb{Z})$, and let us assume that $\gamma \in \operatorname{Ker} i_*$, where $i_* : H_1(\partial M, \mathbb{R}) \to H_1(M, \mathbb{R})$. In this case we can fix a $b \in H_2(M, \partial M, \mathbb{R})$, such that $\partial b = \gamma$. Now for every *spin*^{c} structure $\mathcal{L} \in \mathcal{C}_{M(\phi)}$, such that $\mathcal{L}$ restricts trivially to $D^2 \times T^2$, we define $\Lambda^+(\mathcal{L}) \subset \mathcal{C}_M$ in the following way. Let $\mathcal{L}_0 \to M$ be the restriction of $\mathcal{L}$ to M, and let $\Lambda(\mathcal{L}_0) = r^{-1}(\mathcal{L}_0) \subset \mathcal{C}_{M,\partial M}$. By pairing with b we get a linear map $f : \Lambda(\mathcal{L}_0) \to \mathbb{R}$, where $f(\mathcal{L}') = \langle c_1(\det \mathcal{L}'), b\rangle$. We say that $\mathcal{L}' \in \Lambda^+(\mathcal{L})$, if $\mathcal{L}' \in \Lambda(\mathcal{L}_0)$ and

$$\langle c_1(\det \mathcal{L}), b + [D^2 \times pt]\rangle \; < f(\mathcal{L}').$$

Note that the definition of $\Lambda^+(\mathcal{L})$ doesn't depend on the choice of b. Now we can state our result.

Theorem 4.1. *Let M, ϕ and $M(\phi)$ be as above. Suppose that $b_2^+(M) \geq 1$ and that $\gamma \in \operatorname{Ker} i_*$. Then for all spin^c structures $\mathcal{L} \to M(\phi)$, that restrict trivially to $D^2 \times T^2$, we have*

$$SW_{M(\phi)}(\mathcal{L}) = \sum_{\mathcal{L}' \in \Lambda^+(\mathcal{L})} SW_M(\mathcal{L}').$$

Proof. Fix a cylindrical end metric g_1 on $D^2 \times T^2$, such that the scalar curvature of g_1 is non-negative, and positive at some points, and the restriction of g_1 to the end is of the form $h_1 + dt^2$, where h_1 is a flat metric on T^3. Then ϕ and h_1 induce a flat metric h on ∂M. Let g be a compatible cylindrical end metric on $M' = M \cup_{T^3} (T^3 \times [0, \infty))$. Now let $\mathcal{N} = \mathcal{N}_M(\mathcal{L}_0, g, \mu, D)$, where $\mathcal{L}_0$ is the restriction of $\mathcal{L}$ to M, cf. Section 3. Then it follows from Theorem 2.8, that for generic μ and D, the moduli space $\mathcal{N}$ is a smooth compact 1-manifold with boundary $\cup_{\mathcal{L}' \in \Lambda(\mathcal{L}_0)} \partial_\infty^{-1}(\theta(\mathcal{L}'))$.

On the $D^2 \times T^2$ side the non-negative scalar curvature implies that the cylindrical end moduli space $\mathcal{M}_{D^2 \times T^2}$ over $D^2 \times T^2$ is equal to the space of flat S^1 connections with trivial spinor fields modulo those gauge automorphisms that can be extended over $M(\phi)$. We have an inclusion $\partial_\infty : \mathcal{M}_{D^2 \times T^2} \to \chi_0(\partial M)$. Note that for all $A \in \mathcal{M}_{D^2 \times T^2}$ we have $\operatorname{Ker} D_A = 0$. Clearly ∂_∞ depends on ϕ. Let W_γ be the image of $\mathcal{M}_{D^2 \times T^2}$. Then $W_\gamma \subset \chi_0(\partial M)$ is the disjoint union of hypersurfaces, each being the image of an affine linear subspace in $\mathbb{R}^3 = \tilde{\chi}_0(\partial M)$. The different components correspond to the different extensions of $\mathcal{L}_0$ to $M(\phi)$. Note that W_γ avoids the singular points in $\chi_0(\partial M)$. To see that let us choose a trivialization of the *spin*^{c} structure over $D^2 \times T^2$ by fixing a spin structure over $D^2 \times T^2$. It is easy to see, that the restriction to the boundary is the product of a spin structure over T^2 and the twisted spin structure over S^1 with holonomy -1. It follows that for all flat connection A over $D^2 \times T^2$, we have $\operatorname{Ker}(D_{\partial_\infty(A)}) = 0$.

Let $W_\gamma(\mathcal{L})$ denote the component of W_γ that corresponds to $\mathcal{L}$. By choosing μ generic we can assume that $\partial_\infty : \mathcal{N} \to \chi_0(T^3)$ and W_γ are transverse. Since every point in the intersection of $\partial_\infty(\mathcal{N})$ and W_γ is a smooth point of $\chi_0(T^3)$, it follows that every point in $\partial_\infty^{-1}(W_\gamma)$ is represented by a solution which decays exponentially with a fixed exponent to the boundary value. Standard gluing results as in [T] or [MM] show that the moduli space for M_γ is identified with

$$\mathcal{N} \times_{\chi_0(T^3)} W_\gamma = \partial_\infty^{-1}(W_\gamma).$$

It follows that that $SW_{M(\phi)}(\mathcal{L})$ is equal to the oriented intersection of $\partial_\infty(\mathcal{N})$ and $W_\gamma(\mathcal{L})$.

To compute this intersection, let us define $g : \mathcal{N} \to \mathbb{R}$ by

$$g(A, \phi) = \frac{i}{2\pi} \int_b F_A,$$

where $b \in H_2(M, \partial M, \mathbb{R})$ with $\partial b = \gamma$. Note that this integral is well-defined because of the decay condition on A. It follows from the definitions of f and g, that if $\partial_\infty(A, \phi) = \theta(\mathcal{L}')$, then $g(A, \phi) = f(\mathcal{L}')$. Let

$$\mathcal{N}(\mathcal{L}) = \{(A, \phi) \in \mathcal{N} |\; g(A, \phi) = \langle c_1(\det \mathcal{L}), b + [D^2 \times pt] \rangle\}.$$

It is easy to see that $\mathcal{N}(\mathcal{L}) = \mathcal{N} \cap \partial_\infty^{-1}(W_\gamma(\mathcal{L}))$. It follows that $\mathcal{N}(\mathcal{L})$ is a transverse level set in $\mathcal{N}$, and for all $(A, \phi) \in \mathcal{N}(\mathcal{L})$ the boundary value $\partial_\infty(A, \phi)$ is nonsingular. Now we have

$$SW_{M(\phi)}(\mathcal{L}) = \#(\mathcal{N}(\mathcal{L})).$$

On the other hand $\#(\mathcal{N}(\mathcal{L}))$ is clearly equal to the number, counted with signs, of those endpoints of $\mathcal{N}$ for which $g(A, \phi) > \langle c_1(\det \mathcal{L}), b + [D^2 \times pt] \rangle$. It follows, that

$$\#\mathcal{N}(\mathcal{L}) = \sum_{\mathcal{L}' \in \Lambda^+(\mathcal{L})} \#(\partial_\infty^{-1}(\mathcal{L}')) = \sum_{\mathcal{L}' \in \Lambda^+(\mathcal{L})} SW_M(\mathcal{L}')$$

and this finishes the proof of Theorem 4.1. □

Proof of Corollary 1.6. Let X be as in the statement of the corollary. Suppose that $\mathcal{L} \in \mathcal{C}_X$, and $d(\mathcal{L}) > 0$. Let $M = X \setminus nd(T^2)$, where nd denotes the open tubular neighborhood. Then $b_2^+(M) = b_2^+(X) - 1 \geq 1$. Let $\mathcal{L}_0 \in \mathcal{C}_M^0$ denote the restriction of $\mathcal{L}$ to M. Then $d(\mathcal{L}_0) = d(\mathcal{L}) > 0$. Since $[T^2]$ is not a torsion class it follows from Theorem 3.4, that for all $\mathcal{L}' \in \Lambda(\mathcal{L}_0)$ we have $SW_M(\mathcal{L}') = 0$. Now we can apply the formula in Theorem 4.1, and that gives $SW_X(\mathcal{L}) = 0$. □

Proof of Theorem 1.3. It follows from Theorem 4.1, that for all $\mathcal{L}_0 \in \mathcal{C}'_M$ we have

$$\sum_{\mathcal{L}' \in \Lambda(\mathcal{L}_0)} SW_M(\mathcal{L}') \exp(\rho(c(\mathcal{L}'))) \cdot (t - t^{-1})^{-1} = \sum_{\mathcal{L} \in V_\gamma(\mathcal{L}_0)} SW_{M(\phi)} \exp(c_1(\det \mathcal{L})),$$

where $t = \exp(T)$. By summing over $\mathcal{L}_0 \in \mathcal{C}'_M$ we get the formula of Theorem 1.3. □

Proof of Corollary 1.4. Let $\mathrm{Diff}^+(M)$ denote the group of orientation preserving self-diffeomorphisms of M. Then $\mathrm{Diff}^+(M)$ acts naturally on $\mathcal{C}_{M,\partial M}$. We claim that if the orbit of $\mathcal{L} \in \mathcal{C}_{M,\partial M}$ is not finite under this action, then $SW_M(\mathcal{L}) = 0$. This follows from the naturality of SW_M and the fact that $SW_M(\mathcal{L})$ is zero for all but finitely many $\mathcal{L} \in \mathcal{C}_{M,\partial M}$.

We define the subset $W \subset \mathcal{C}_{M,\partial M}$ as follows: $\mathcal{L} \in W$ if $c_1(\det(\mathcal{L}))$ is trivial on V, where $V \subset H_2(M, \partial M, \mathbb{Z})$ is as in Corollary 1.4. Since each $x \neq 0 \in \mathbb{Z}^2$ has infinite order under the action of $\mathbb{SL}_2(\mathbb{Z})$, it follows from the definition of

V, that for each $\mathcal{L} \in \mathcal{C}_{M,\partial M} \setminus W$ the orbit of $\mathcal{L}$ under Diff^+ is infinte, and so it follows that $SW_M(\mathcal{L}) = 0$

Now let us define $W_1 \subset H^2(M, \partial M, \mathbb{R})$ by: $x \in W_1$ if x restricts trivially to V. Let $\rho_a : H^2(M, \partial M, \mathbb{R}) \to H^2(M_a, \mathbb{R})$, $\rho_\gamma : H^2(M, \partial M, \mathbb{R}) \to H^2(M_\gamma, \mathbb{R})$ be given by Mayer-Vietoris sequence. Then it is easy to see, that $\rho_a|_{W_1}$ and $\rho_\gamma|_{W_1}$ are one-to-one. On the other hand we proved above that all basic classes $K \in \mathcal{B}_M$ satisfy $K \in W_1$. Now applying Theorem 1.3 for M_γ and M_a shows

$$SW^*_{M_\gamma} \cdot \sinh(T_\gamma) = SW^*_{M_a} \cdot \sinh(T_a),$$

and this finishes the proof of Corollary 1.4. □

Proof of Theorem 1.1. Let us take $\mathcal{N} = \mathcal{N}_M(\mathcal{L}_0, g, \mu, D)$, where μ and D are generic. Let $W_\gamma \subset \chi_0(\partial M)$ be defined as in the proof of Theorem 4.1. Then W_γ is the disjoint union of hypersurfaces, all of which avoid the singular points in $\chi_0(T^3)$, where different components correspond to different extensions of $\mathcal{L}_0$ to M_γ. Note that W_γ depends on γ. For generic choices of μ and D, the intersection of $\partial_\infty(N)$ and W_γ is transverse, and clearly as above

$$\sum_{\mathcal{L} \in V_\gamma(\mathcal{L}_0)} SW_{M_\gamma}(\mathcal{L}) = \#(\mathcal{N} \cap \partial_\infty^{-1}(W_\gamma)).$$

It is easy to see that for any curve $\delta \subset \chi_0(T^3)$, that connects two singular points in $\chi_0(T^3)$ or any loop δ in $\chi_0(T^3)$ such that δ intersects W_a, W_b, W_c and W_γ transversally we have

$$\#(\delta \cap W_\gamma) = p \cdot \#(\delta \cap W_a) + q \cdot \#(\delta \cap W_b) + r \cdot \#(\delta \cap W_c)$$

and since $\partial_\infty(N)$ is the finite union of such curves and loops, this proves Theorem 1.1 □

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