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A PROOF OF FEIGIN'S CONJECTURE

Sergey Arkhipov

1. Introduction

The present paper is devoted to further development of semiinfinite cohomology of small quantum groups. The topic appeared first in [Ar1], where the very definition of semiinfinite cohomology $\mathrm{Ext}_A^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\cdot)$ was given. The setup for the definition of semiinfinite cohomology of an algebra A includes two subalgebras $B, N \subset A$ and the *triangular decomposition* of A, i. e. the vector space isomorphism $B \otimes N \widetilde{\longrightarrow} A$ provided by the multiplication in A. Fix root data $(Y, X, \dots)$ of the finite type $(I, \cdot)$ and a positive integer number ℓ. The *small quantum group* $\mathfrak{u}_\ell$ with the standard triangular decomposition turns out to be a very interesting object for the investigation of semiinfinite cohomology. The explanation for this lies in the following fact proved by Ginzburg and Kumar in [GK]. Consider the set of nilpotent elements $\mathcal{N}$ in the simple Lie algebra $\mathfrak{g}$ corresponding to $(Y, X, \dots)$.

Theorem. $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \mathcal{F}un(\mathcal{N})$ *as an associative algebra. The grading on the right hand side is provided by the action of the group* $\mathbb{C}^*$ *on the affine variety* $\mathcal{N}$. □

On the other hand it is proved in [Ar2] that the algebra $\mathrm{Ext}^\bullet_A(\underline{\mathbb{C}},\underline{\mathbb{C}})$ acts naturally on the semi-infinite cohomology of A. Thus in particular for a $\mathfrak{u}_\ell$-module M one can concider $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, M)$ as a quasicoherent sheaf on $\mathcal{N}$. It is natural to look for the answer for semi-infinite cohoomlogy of $\mathfrak{u}_\ell$ in terms of geometry of $\mathcal{N}$. B. Feigin has proposed the following conjecture. Consider the standard positive nilpotent subalgebra $\mathfrak{n}^+ \subset \mathcal{N} \subset \mathfrak{g}$.

Conjecture A. *The quasicoherent sheaf on* $\mathcal{N}$ *provided by* $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ *is equal to the sheaf of algebraic distributions on* $\mathcal{N}$ *with support on* $\mathfrak{n}^+ \subset \mathcal{N}$. □

Moreover note that the simply connected Lie group G with the Lie algebra equal to $\mathfrak{g}$ acts naturally on $\mathcal{N}$. This action provides a structure of a $\mathfrak{n}^+$-integrable $\mathfrak{g}$-module on the described distributions' space. On the other hand it was shown in [Ar1] that there exists a natural $U(\mathfrak{g})$-module structure on $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. The $\mathfrak{g}$-module version of the Feigin conjecture states that the described $\mathfrak{g}$-modules are isomorphic. In [Ar1] and [Ar5] the conjecture was

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proved on the level of characters of the $\mathfrak{g}$-modules. In the present paper we give a full proof of the Feigin conjecture.

1.1. Let us describe briefly the structure of the paper. In the second section we recall nessesary facts concerning the usual cohomology of small quantum groups. The main facts here include the mentioned Ginzburg-Kumar theorem, the Kostant theorem describing the algebra structure on $\mathcal{F}un(\mathcal{N})$ (see 2.3.4.) and a homological description of a certain degeneration $\widetilde{\mathfrak{u}}_\ell$ of the algebra $\mathfrak{u}_\ell$ (see 2.4). It turns out that both the usual and the semi-infinite cohomology of the algebra $\widetilde{\mathfrak{u}}_\ell$ with trivial coefficients can be described as associated graded objects for the corresponding functors over $\mathfrak{u}_\ell$ with respect to certain filtrations.

In the third section we prove the $\mathcal{F}un(\mathcal{N})$-module version of the Feigin conjecture. The main arguments here are as follows. First we describe the $\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-module $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Next, using this description, we construct a morphism of $\mathcal{F}un(\mathcal{N})$-modules

$$\Phi:\ H^0(\mathcal{N}\setminus D_1\cup\ldots\cup D_{\sharp(R^+)},\mathcal{O}_\mathcal{N})\longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}),$$

where $D_1,\ldots,D_{\sharp(R^+)}$ denote the coordinate divisors such that $\cap_i D_i=\mathfrak{n}^+$. Note that the first $\mathcal{F}un(\mathcal{N})$-module contains the distributions module $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ as a certain quotient module.

Finally, using the connection between $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ and $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ we prove that the map Φ provides a $\mathcal{F}un(\mathcal{N})$-module isomorphism

$$H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})\,\widetilde{\longrightarrow}\,\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

In the fourth section we prove the $\mathfrak{g}$-module version of the Feigin conjecture. The main steps of the proof are as follows. First we construct explicitly the $\mathfrak{n}^+$-module isomorphism. Next we recall from [Ar1] that there exists a nondegenerate $\mathfrak{g}$-equivariant contragradient pairing on $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. We construct its geometric analogue on $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$. On the other hand it is easy to verify that $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is free over the algebra $U(\mathfrak{n}^-)$. It follows that both $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ and $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ are co-free over $U(\mathfrak{n}^+)$. Using the constructed contragradient pairung on the $\mathfrak{g}$-module of semi-infinite cohomology we see that the latter module is $U(\mathfrak{n}^-)$-free. It follows that both $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ and $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ are *tilting* $\mathfrak{g}$-modules. Finally a beautiful result of Andersen and Paradowski states that a tilting $\mathfrak{g}$-module is completely determined up to an isomorphism by its character (see [AP]).

2. Small quantum groups

2.1. Root data. Fix a *Cartan datum* $(I,\cdot)$ of the finite type and a *simply connected root datum* $(Y,X,\ldots)$ of the type $(I,\cdot)$. Thus we have $Y=\mathbb{Z}[I]$,

$X = \mathrm{Hom}(Y, \mathbb{Z})$, and the pairing $\langle\ ,\ \rangle : Y \times X \longrightarrow \mathbb{Z}$ coincides with the natural one (see [L3], I 1.1, I 2.2). In particular the data contain canonical embeddings $I \hookrightarrow Y, i \mapsto i$ and $I \hookrightarrow X, i \mapsto i' : \langle i', j\rangle := 2i \cdot j/i \cdot i$. The latter map is naturally extended to an embedding $Y \subset X$. Denote by ht the linear function on X defined on elements $i', i \in I$, by $\mathrm{ht}(i') = 1$ and extended to the whole X by linearity. The root system (resp. the positive root system) corresponding to the data $(Y, X, \dots)$ is denoted by R (resp. by R^+), below W denotes the Weyl group of R.

2.2. Quantum groups at roots of 1. Given the root datum $(Y, X, \dots)$ Drinfeld and Jimbo constructed an associative algebra U over the field $\mathbb{Q}(v)$ of rational functions in v with generators $E_i, F_i, K_i^{\pm 1}, i \in I$, and relations being the quantum analogues of the classical Serre relations in the universal enveloping algebra of the corresponding simple Lie algebra $\mathfrak{g}$. The explicit form of the relations can be found e. g. in [L1], I 3.1. We call the algebras U the *quantum groups*.

Lusztig (see [L1], V 31.1) defined a $\mathbb{Q}[v, v^{-1}]$-subalgebra $U_{\mathbb{Q}}$ in U being the quantum analogue of the integral form for the universal enveloping algebra of $\mathfrak{g}$ due to Kostant. In particular the elements $E_i, F_i, K_i^{\pm 1}, i \in I$, belong to $U_{\mathbb{Q}}$. Let ℓ be an odd number satisfying the conditions from [GK]. Fix a primitive ℓ-th root of unity ζ. Define a $\mathbb{C}$-algebra $\widetilde{U}_\ell := U_{\mathbb{Q}} \otimes_{\mathbb{Q}[v,v^{-1}]} \mathbb{C}$, where v acts on $\mathbb{C}$ by multiplication by ζ. It is known that the elements $K_i^\ell, i \in I$, are central in $\widetilde{U}_\ell$. Set $U_\ell := \widetilde{U}_\ell/(K_i^\ell - 1, i \in I)$. The algebra U_ℓ is generated by the elements E_i, $E_i^{(\ell)}$, F_i, $F_i^{(\ell)}$, $K_i^{\pm 1}$, $i \in I$. Here $E_i^{(\ell)}$ (resp. $F_i^{(\ell)}$) denotes the *ℓ-th quantum divided power* of the element E_i (resp. F_i) specialized at the root of unity ζ.

Following Lusztig we define the *small quantum group* $\mathfrak{u}_\ell$ at the root of unity ζ as the subalgebra in U_ℓ generated by all $E_i, F_i, K_i^{\pm 1}, i \in I$. Denote the subalgebra in $\mathfrak{u}_\ell$ generated by $E_i, i \in I$ (resp. $F_i, i \in I$, resp. $K_i, i \in I$), by $\mathfrak{u}_\ell^+$ (resp. $\mathfrak{u}_\ell^-$, resp. $\mathfrak{u}_\ell^0$). Note that the algebra $\mathfrak{u}_\ell$ is graded naturally by the abelian group X. Using the function ht we obtain a $\mathbb{Z}$-grading on $\mathfrak{u}_\ell$ from this X-grading. In particular the subalgebra $\mathfrak{u}_\ell^+$ (resp. $\mathfrak{u}_\ell^-$) is graded by $\mathbb{Z}_{\geq 0}$ (resp. by $\mathbb{Z}_{\leq 0}$).

Below we present several well known facts about the algebra $\mathfrak{u}_\ell$ to be used later. Recall that an augmented subalgebra $B \subset A$ with the augmentation ideal $\overline{B} \subset B$ is called *normal* if $A\overline{B} = \overline{B}A$. If so, the space $A/A\overline{B}$ becomes an algebra. It is denoted by $A//B$. Fix an augmentation on $\mathfrak{u}_\ell$ as follows: $E_i \mapsto 0, F_i \mapsto 0, K_i \mapsto 1$ for every $i \in I$. Set $\underline{\mathbb{C}} := \mathfrak{u}_\ell/\overline{\mathfrak{u}}_\ell$.

Lemma 2.2.1. ([AJS] 1.3, [L2] Theorem 8.10)

(i) *The multiplication in $\mathfrak{u}_\ell$ provides a vector space isomorphism $\mathfrak{u}_\ell = \mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^0 \otimes \mathfrak{u}_\ell^+; \dim \mathfrak{u}_\ell^\pm = \ell^{\sharp(R^+)}$; the subalgebra $\mathfrak{u}_\ell^0$ is isomorphic to the group algebra of the group $(\mathbb{Z}/\ell\mathbb{Z})^{\sharp(I)}$.*

(ii) *The subalgebra $\mathfrak{u}_\ell \subset U_\ell$ is normal and we have $U_\ell//\mathfrak{u}_\ell = U(\mathfrak{g})$.*

□

The subalgebra $\mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^0$ (resp. $\mathfrak{u}_\ell^0 \otimes \mathfrak{u}_\ell^+$) in $\mathfrak{u}_\ell$ is denoted by $\mathfrak{b}_\ell^-$ (resp. by $\mathfrak{b}_\ell^+$). Recall that a finite dimensional algebra A is called *Frobenius* if the left A-modules A and $A^* := \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ are isomorphic.

Lemma 2.2.2. ([Ar1], Lemma 2.4.5) *The algebras $\mathfrak{u}_\ell^+$ and $\mathfrak{u}_\ell^-$ are Frobenius.* □

Consider the filtration on the algebra $\mathfrak{u}_\ell$ as follows. Let the filtration component $F^{\leq d}(\mathfrak{u}_\ell)$ be linearly generated by X-homogeneous monomials $u = u^- \otimes u^0 \otimes u^+$ such that $|\,\mathrm{ht}(\deg u^-)| + |\,\mathrm{ht}(\deg u^+)| \leq d$. By definition set $\widetilde{\mathfrak{u}}_\ell := \mathrm{gr}^F \mathfrak{u}_\ell$. Evidently we have $\mathrm{gr}^F \mathfrak{b}_\ell^+ = \mathfrak{b}_\ell^+$, $\mathrm{gr}^F \mathfrak{b}_\ell^- = \mathfrak{b}_\ell^-$, $\mathrm{gr}^F \mathfrak{u}_\ell^0 = \mathfrak{u}_\ell^0$.

Lemma 2.2.3. *Elements of the subalgebra $\mathfrak{u}_\ell^- \subset \widetilde{\mathfrak{u}}_\ell$ commute with elements of the subalgebra $\mathfrak{u}_\ell^+ \subset \widetilde{\mathfrak{u}}_\ell$.* □

Both the X- and the $\mathbb{Z}$-grading as well as the augmentation on $\widetilde{\mathfrak{u}}_\ell$ are induced by the ones on $\mathfrak{u}_\ell$. Denote the category of X-graded finite dimensional left $\mathfrak{u}_\ell$-modules (resp. $\widetilde{\mathfrak{u}}_\ell$-modules) $M = \bigoplus_{\lambda \in X} M_\lambda$ such that K_i acts on M_λ by multiplication by the scalar $\zeta^{\langle i, \lambda \rangle}$ and $E_i : M_\lambda \longrightarrow M_{\lambda + i'}$, $F_i : M_\lambda \longrightarrow M_{\lambda - i'}$ for all $i \in I$, with morphisms preserving X-gradings, by $\mathfrak{u}_\ell$-mod (resp. by $\widetilde{\mathfrak{u}}_\ell$-mod). For $M, N \in \mathfrak{u}_\ell$-mod and $\lambda \in \ell \cdot X$ we define the shifted module $M\langle\lambda\rangle \in \mathfrak{u}_\ell$-mod : $M\langle\lambda\rangle_\mu := M_{\lambda+\mu}$ and set $\mathrm{Hom}_{\mathfrak{u}_\ell}(M, N) := \bigoplus_{\lambda \in \ell \cdot X} \mathrm{Hom}_{\mathfrak{u}_\ell\text{-mod}}(M\langle\lambda\rangle, N)$. The functor $\mathrm{Hom}_{\widetilde{\mathfrak{u}}_\ell}$ is defined in a similar way. Evidently the spaces $\mathrm{Hom}_{\mathfrak{u}_\ell}(\cdot, \cdot)$ and $\mathrm{Hom}_{\widetilde{\mathfrak{u}}_\ell}(\cdot, \cdot)$ posess natural $\ell \cdot X$-gradings.

2.3. Cohomology of small quantum groups. Consider the $\ell \cdot X \times \mathbb{Z}$-graded algebra $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. Note that by Shapiro lemma and Lemma 2.2.1. (ii) the Lie algebra $\mathfrak{g}$ acts naturally on the Ext algebra and the multiplication in the algebra satisfies Lebnitz rule with respect to the $\mathfrak{g}$-action. In [GK] Ginzburg and Kumar obtained a nice description of the multiplication structure as well as the $\mathfrak{g}$-module structure on $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ as follows.

2.3.1. Functions on the nilpotent cone. Let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g}$ by adjunction. The action preserves the set of nilpotent elements $\mathcal{N} \subset \mathfrak{g}$ called the *nilpotent cone* of $\mathfrak{g}$. The action is algebraic, thus it provides a morphism of $\mathfrak{g}$ into the Lie algebra of algebraic vector fields on the nilpotent cone $\mathcal{V}ect(\mathcal{N})$. The latter algebra acts on the algebraic functions $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$. The action is G-integrable. Note also that the natural action of the group $\mathbb{C}^*$ provides a grading on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ preserved by the G-action.

Theorem 2.3.2. (**[GK]**) *The algebra and $\mathfrak{g}$-module structures on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ and on $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ coincide. The homological grading on the latter algebra corresponds to the grading on the former one provided by the $\mathbb{C}^*$-action. The X-grading on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ provided by the weight decomposition with respect to the* *action of the Cartan subalgebra in* $\mathfrak{g}$ *corresponds to the natural* $\ell \cdot X$*-grading on the space* $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. $\square$

We will need a more detailed description of the algebra $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$. First the folowing result of Kostant shows the size of the algebra.

Proposition 2.3.3. ([H])

$$\mathrm{ch}\left(H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}}), t\right) = \frac{\sum\limits_{w\in W} t^{l(w)}}{\prod\limits_{\alpha\in R^+}(1-e^{-\alpha}t)(1-e^{\alpha}t)}.$$

Here the indeterminate t *stands for the homogeneous* $\mathbb{Z}$*-grading, for a weight* $\alpha = \sum a_i i'$ *the symbol* e^α *denotes the monomial* $\prod_i (e^{i'})^{a_i}$*, and* $l(w)$ *denotes the length of the element of the Weyl group.* $\square$

In particular the space

$$\begin{aligned}\left(H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})\right)_2 &= \bigoplus_{\alpha\in R^+} \mathbb{C}e^*_\alpha \oplus \bigoplus_{w\in W, \ell t(w)=1} \mathbb{C}\xi_w \oplus \bigoplus_{\alpha\in R^+} \mathbb{C}f^*_\alpha \\ &= \mathfrak{n}^{+*} \oplus \bigoplus_{w\in W, l(w)=1} \mathbb{C}\xi_w \oplus \mathfrak{n}^{-*},\end{aligned}$$

where the X-grading of the element e^*_α (resp. f^*_α, resp. ξ_w) is equal to $-\alpha$ (resp. to α, resp. to 0). Consider the subalgebra H_- (resp. H_+) in $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ generated by the space $\mathfrak{n}^{+*}$ (resp. $\mathfrak{n}^{-*}$). The detailed description of the algebra structure on $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ is given by the following statement.

Proposition 2.3.4. ([Ko], Theorem 1.5) *The algebra* H_- (*resp.* H_+) *is equal to the free commutative algebra* $S^\bullet(\mathfrak{n}^{+*})$ (*resp.* $S^\bullet(\mathfrak{n}^{-*})$). $\square$

Note that the inclusion $S^\bullet(\mathfrak{n}^{-*}) \hookrightarrow H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ (resp. $S^\bullet(\mathfrak{n}^{+*}) \hookrightarrow H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$) corresponds to the coordinate projection $\mathcal{N} \hookrightarrow \mathfrak{g} \longrightarrow \mathfrak{n}^-$ (resp. $\mathcal{N} \hookrightarrow \mathfrak{g} \longrightarrow \mathfrak{n}^+$). In particular we obtain a description of the algebra $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$.

Corollary 2.3.5. $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = S^\bullet(\mathfrak{n}^{-*}) \otimes H_0 \otimes S^\bullet(\mathfrak{n}^{+*})$ *as a module over* $S^\bullet(\mathfrak{n}^{-*}) \otimes S^\bullet(\mathfrak{n}^{+*})$. *Here* $H_0 := \bigoplus_{w\in W} \mathbb{C}\xi_w$. $\square$

Ginzburg and Kumar also proved the following statements (see [GK]).

(i) $\mathrm{Ext}^\bullet_{\mathfrak{u}^-_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = \bigoplus\limits_{w\in W} \mathbb{C}\nu^+_w \otimes S^\bullet(\mathfrak{n}^{-*})$ as a $X \times \mathbb{Z}$-graded vector space. Here the homological grading of the element ν^+_w, $w \in W$, is equal to $l(w)$, and its X-grading equals $\rho - w(\rho)$. The homological grading (resp. the X-grading) of the element $f^*_\alpha \in \mathfrak{n}^{-*}$ equals 2 (resp. $\ell\alpha'$). A similar statement holds for the Ext algebra of $\mathfrak{u}^+_\ell$.

(ii) $\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ (resp. $\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$) is equal to $S^\bullet(\mathfrak{n}^{-*})$ (resp. $S^\bullet(\mathfrak{n}^{+*})$) both as an associative algebra and as a $\mathfrak{n}^-$- (resp. $\mathfrak{n}^+$-) module.

(iii) The natural map $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) \longrightarrow \mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ (resp. $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) \longrightarrow \mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$) given by the restriction functor coincides with the morphism of the algebras of functions provided by the inclusion of the affine manifolds $\mathfrak{n}^- \hookrightarrow \mathcal{N}$ (resp. $\mathfrak{n}^+ \hookrightarrow \mathcal{N}$).

We set $A_+ := \bigoplus_{w\in W} \mathbb{C}\nu^+_w \subset \mathrm{Ext}^\bullet_{\mathfrak{u}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ and $A_- := \bigoplus_{w\in W} \mathbb{C}\nu^-_w \subset \mathrm{Ext}^\bullet_{\mathfrak{u}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Note that the restriction functor $\mathrm{Res}^{\mathfrak{b}^+_\ell}_{\mathfrak{u}^+_\ell}$ (resp. $\mathrm{Res}^{\mathfrak{b}^-_\ell}_{\mathfrak{u}^-_\ell}$) provides the inclusion of algebras

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = S^\bullet(\mathfrak{n}^{+*}) \longrightarrow \mathrm{Ext}^\bullet_{\mathfrak{u}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) \text{ (resp. } &\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) \\ = S^\bullet(\mathfrak{n}^{-*}) \longrightarrow \mathrm{Ext}^\bullet_{\mathfrak{u}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})).&\end{aligned}$$

Lemma 2.3.6.

(i) *The subspace* A_- (*resp.* A_+) *is a subalgebra in* $\mathrm{Ext}^\bullet_{\mathfrak{u}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ (*resp. in* $\mathrm{Ext}^\bullet_{\mathfrak{u}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$).

(ii) *The multiplication maps provide the vector space isomorphisms*

$$\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})\otimes A_+ \widetilde{\longrightarrow} \mathrm{Ext}^\bullet_{\mathfrak{u}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) \textit{ and } \mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})\otimes A_- \widetilde{\longrightarrow} \mathrm{Ext}^\bullet_{\mathfrak{u}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

□

Note that the algebra $\mathfrak{u}^0_\ell$ acts trivially on the first factors in these decompositions.

Proposition 2.3.7. *The algebras* A_- *and* A_+ *are Frobenius.*

Proof. We consider only the case of A_+. Recall that when calculating cohomology of the algebra $\mathfrak{u}^-_\ell$ Ginzburg and Kumar used certain degenerations of the algebra defined by Kac, De Concini and Procesi in [DCKP] with the help of quantum PBW filtration on $\mathfrak{u}^-_\ell$. The algebra $\mathrm{gr}^{PBW}(\mathfrak{u}^-_\ell)$ appears to be a certain quotient algebra of the *quantum symmetric algebra* with the set of generators enumerated by the standard basis in the space $\mathfrak{n}^-$. In particular it was proved in [GK] that $\mathrm{Ext}^\bullet_{\mathrm{gr}^{PBW}(\mathfrak{u}^-_\ell)}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \Lambda^\bullet_\ell(\mathfrak{n}^{-*})\otimes S^\bullet(\mathfrak{n}^{-*})$ as an associative algebra. Here $\Lambda^\bullet_\ell(\mathfrak{n}^{-*})$ denotes the *quantum exterior algebra* on generators enumerated by elements of the standard basis in $\mathfrak{n}^{-*}$. In particular this algebra is Frobenius with the trace map given by the projection on the top homological grading component (that is one dimensional).

Consider now the spectral sequence with the term E_2 equal to $\mathrm{Ext}^\bullet_{\mathrm{gr}^{PBW}(\mathfrak{u}^-_\ell)}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ that converges to $\mathrm{Ext}^\bullet_{\mathfrak{u}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Note that for ℓ big enough the subalgebra $\Lambda^\bullet_\ell(\mathfrak{n}^{-*}) \subset \mathrm{Ext}^\bullet_{\mathrm{gr}^{PBW}(\mathfrak{u}^-_\ell)}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is a DG-subalgebra. Moreover its images on the next terms of the spectral sequence are DG-subalgebras too. Now note that the subalgebra $\mathrm{gr}\, A_+$ in the term E_∞ of the spectral sequence comes from the described DG-subalgebra in the term E_2. In particular the one dimensional top homological grading component of $\Lambda_\ell^\bullet(\mathfrak{n}^{-*})$ has its nonzero representative on the level E_∞. Thus it has its nonzero representatives on all the levels E_m, $m \geq 2$.

Consider the algebra $\mathrm{gr}^F A_+$ appearing in the E_∞ term of the spectral sequence. Put the linear functional generating the trace map on $\mathrm{gr}^F A_+$ equal to the projection on the one dimensional top homological grading component. We claim that the corresponding trace pairing on the algebra is nondegenerate. Indeed, it is so on the algebra $\Lambda_\ell^\bullet(\mathfrak{n}^{-*})$ appearing in the E_2 term of the spectral sequence. On the other hand note that the X-grading components of the space $\Lambda_\ell^\bullet(\mathfrak{n}^{-*})$ with the weights $\rho - w(\rho)$, $w \in W$, are one dimensional. Moreover the subspace $\bigoplus_{w\in W} \left(\Lambda_\ell^\bullet(\mathfrak{n}^{-*})\right)_{\rho - w(\rho)}$ is self-dual with respect to the trace pairing on $\Lambda_\ell^\bullet(\mathfrak{n}^{-*})$. Thus the term E_∞ of the spectral sequence becomes an orthogonal direct summand in the term E_2. We have proved that the algebra $\mathrm{gr}^F A_+$ is Frobenius. Finally by Lemma 2.4.4 from [Ar1] tha algebra A_+ is itself Frobenius. $\square$

2.4. Cohomology of the algebra $\widetilde{\mathfrak{u}}_\ell$. Note that the algebra $\widetilde{\mathfrak{u}}_\ell$ contains the algebra $\mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^+$ *as a subalgebra*, moreover, this subalgebra is *normal* in $\widetilde{\mathfrak{u}}_\ell$ with the quotient algebra $\mathfrak{u}_\ell^0$. On the other hand, by standard arguments, we have

$$\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^+}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^-}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \otimes \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^+}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$$

as an associative algebra. Now recall that the algebra $\mathfrak{u}_\ell^0$ is semisimple being a group algebra of a finite group. We have proved the following statement.

Lemma 2.4.1. $\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = \left(\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^-}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \otimes \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^+}(\underline{\mathbb{C}}, \underline{\mathbb{C}})\right)^{\mathfrak{u}_\ell^0}$ *as a* $\ell \cdot X \times \mathbb{Z}$-*graded vector space. Here* $(\cdot)^{\mathfrak{u}_\ell^0}$ *denotes taking* $\mathfrak{u}_\ell^0$-*invariants. Moreover, the restriction functor* $\mathrm{Res}^{\widetilde{\mathfrak{u}}_\ell}_{\mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^+}$ *provides an isomorfism of associative algebras* $\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \cong \left(\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^+}(\underline{\mathbb{C}}, \underline{\mathbb{C}})\right)^{\mathfrak{u}_\ell^0}$. $\square$

The following proposition is proved similarly to Theorem 3.1 from [Ar5].

Lemma 2.4.2.

$$\mathrm{ch}\left(\left(\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^-}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \otimes \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^+}(\underline{\mathbb{C}}, \underline{\mathbb{C}})\right)^{\mathfrak{u}_\ell^0}, t\right) = \frac{\sum\limits_{w\in W} t^{2l(w)}}{\prod\limits_{\alpha\in R^+} (1 - e^{-\ell\alpha} t^2)(1 - e^{\ell\alpha} t^2)}.$$

Here the indeterminate t *stands for the homogeneous* $\mathbb{Z}$-*grading, for a weight* $\alpha = \sum a_i i'$ *the symbol* e^α *denotes the monomial* $\prod_i (e^{i'})^{a_i}$. $\square$

Let us denote the filtration on the space $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ that corresponds to the filtration F on the algebra $\mathfrak{u}_\ell$ by the same letter.

Corollary 2.4.3. $\mathrm{gr}^F \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = \mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ *as a graded associative algebra.* $\square$

Summing up the previous considerations we obtain the following statement.

Proposition 2.4.4. $\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = S^\bullet(\mathfrak{n}^{-*}) \otimes \mathrm{gr}^F H_0 \otimes S^\bullet(\mathfrak{n}^{+*})$ *as an associative algebra. Here the imbedding* $S(\mathfrak{n}^{-*}) \hookrightarrow \mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ *resp. the imbedding* $S(\mathfrak{n}^{+*}) \hookrightarrow \mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}))$ *is provided by the projection of algebras* $\widetilde{\mathfrak{u}}_\ell \longrightarrow \mathfrak{b}^-_\ell$*:* $\mathfrak{u}^+_\ell \longrightarrow 0$ *(resp.* $\widetilde{\mathfrak{u}}_\ell \longrightarrow \mathfrak{b}^+_\ell$*:* $\mathfrak{u}^-_\ell \longrightarrow 0$*).* $\square$

Comparing grading components in the previous equality with gradings far smaller than ℓ we obtain the following statement.

Corollary 2.4.5. *The associative algebra* $\mathrm{gr}^F H_0 = (A_- \otimes A_+)^{\mathfrak{u}^0_\ell}$. $\square$

3. $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ as a $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$-module

First we recall briefly the general setup for semi-infinite cohomology of the small quantum group. The definition of semi-infinite cohomology presented below is a specialization of the general one in the case of a finite dimensional graded algebra $\mathfrak{u} = \mathfrak{b}^- \otimes \mathfrak{u}^+$ such that $\mathfrak{b}^-$ is nonpositively graded, and $\mathfrak{u}^+$ is a positively graded *Frobenius* algebra with $\mathfrak{u}^+_0 = \mathbb{C}$.

3.1. Definition of semi-infinite cohomology. Consider first the *semiregular* $\mathfrak{u}$-bimodule $S^{\mathfrak{u}^+}_{\mathfrak{u}} = \mathfrak{u} \otimes_{\mathfrak{u}^+} \mathfrak{u}^{+*}$, with the right $\mathfrak{u}$-module structure provided by the isomorphism of left $\mathfrak{u}^+$-modules $\mathfrak{u}^+ \cong \mathfrak{u}^{+*}$. Note that $S^{\mathfrak{u}^+}_{\mathfrak{u}}$ is free over the algebra $\mathfrak{u}$ both as a right and as a left module.

For a complex of graded $\mathfrak{u}$-modules $M^\bullet = \bigoplus_{p,q \in \mathbb{Z}} M^q_p$, $d: M^q_p \longrightarrow M^{q+1}_p$ we define the support of $M^\bullet$ by $\mathrm{supp}\, M^\bullet := \{(p, q) \in \mathbb{Z}^2 | M^q_p \neq 0\}$. We say that a complex $M^\bullet$ is *concave* (resp. *convex*) if there exist $s_1, s_2 \in \mathbb{N}, t_1, t_2 \in \mathbb{Z}$ such that $\mathrm{supp}\, M^\bullet \subset \{(p, q) \in \mathbb{Z}^2 | s_1 q + p \leq t_1, s_2 q - p \leq t_2\}$ (resp. $\mathrm{supp}\, M^\bullet \subset \{(p, q) \in \mathbb{Z}^2 | s_1 q + p \geq t_1, s_2 q - p \geq t_2\}$).

Let $M^\bullet, N^\bullet \in \mathcal{C}om(\mathfrak{u}\text{-mod})$. Suppose that $M^\bullet$ is convex and $N^\bullet$ is concave. Choose a convex (resp. concave) complex $R^\bullet_\uparrow(M^\bullet)$ (resp. $R^\bullet_\downarrow(N^\bullet)$) in $\mathcal{C}om(\mathfrak{u}\text{-mod})$ quasiisomorphic to $M^\bullet$ (resp. $N^\bullet$) and consisting of $\mathfrak{u}^+$-free (resp. $\mathfrak{u}^-$-free) modules.

Definition 3.1.1. We set

$$\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}}(M^\bullet, N^\bullet) := H^\bullet(\mathrm{Hom}^\bullet_{\mathfrak{u}}(R^\bullet_\uparrow(M^\bullet), S^{\mathfrak{u}^+}_{\mathfrak{u}} \otimes_{\mathfrak{u}} R^\bullet_\downarrow(N^\bullet))).$$

Lemma 3.1.2. ([Ar1] Lemma 3.4.2, Theorem 5)
The spaces $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}}(M^\bullet, N^\bullet)$ *do not depend on the choice of resolutions and*

define functors

$$\mathrm{Ext}_{\mathfrak{u}}^{\frac{\infty}{2}+k}(\cdot,\cdot):\ \mathfrak{u}\text{-mod}\times\mathfrak{u}\text{-mod}\longrightarrow \mathcal{V}ect,\ k\in\mathbb{Z}.$$

□

Below we consider semi-infinite cohomology of algebras $\mathfrak{u}_\ell$, $\widetilde{\mathfrak{u}}_\ell$, $\mathfrak{b}_\ell^+$, $\mathfrak{b}_\ell^-$ etc. with coefficients in X-graded modules. The $\mathbb{Z}$-grading on such a module is obtained from the X-grading using the function $\mathrm{ht}:\ X\longrightarrow\mathbb{Z}$.

Evidently the spaces $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(M^\bullet,N^\bullet)$ (resp. $\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\widetilde{M}^\bullet,\widetilde{N}^\bullet)$) posess natural $\ell\cdot X$-gradings. The following statement is a direct consequence of Lemma 2.2.1. (ii).

Lemma 3.1.3. *Let $M,N\in\mathfrak{u}_\ell$-mod be restrictions of some U_ℓ-modules. Then the spaces $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(M,N)$ have natural structures of $\mathfrak{g}$-modules, and the $\ell\cdot X$-gradings on them coincide with the X-gradings provided by the weight decompositions of the modules with respect to the standard Cartan subalgebra in $\mathfrak{g}$.* □

In particular we consider the character of this $\mathfrak{g}$-module.

3.2. Semiinfinite cohomology of the trivial $\widetilde{\mathfrak{u}}_\ell$-module. The following statement sums up the main results from [Ar5].

Theorem.

(i) $\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})=\left(\mathrm{Ext}_{\mathfrak{u}_\ell^-\otimes\mathfrak{u}_\ell^+}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})\right)^{\mathfrak{u}_\ell^0}.$

(ii) $\mathrm{ch}\left(\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}),t\right)=t^{-\sharp(R^+)}e^{-2\ell\rho}\dfrac{\sum\limits_{w\in W}t^{2l(w)}}{\prod\limits_{\alpha\in R^+}(1-e^{-\ell\alpha}t^{-2})(1-e^{-\ell\alpha}t^{2})}.$

The right hand side of the equality is considered as an element in $\mathbb{C}[t,t^{-1}][[e^{-i'},i\in I]]$.

(iii) *Let M be a filtered module over the filtered algebra $\mathfrak{u}_\ell$ with the filtration F. Then there exists a spectral sequence with the term $E_1=\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\mathrm{gr}\,M)$, converging to $E_\infty=\mathrm{gr}^F\,\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},M)$.*

(iv) *For the trivial $\mathfrak{u}_\ell$-module the described spectral sequence degenerates in the term E_1. In particular as a $\ell\cdot X\times\mathbb{Z}$-graded vector space $\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})=\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$.*

□

Let us compare semi-infinite cohomology of the trivial $\mathfrak{b}_\ell^+$-module with the usual cohomology of $\mathfrak{b}_\ell^+$ with coefficients in this module.

Lemma 3.2.1. $\mathrm{Ext}_{\mathfrak{b}_\ell^+}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})=\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell}(\underline{\mathbb{C}},\mathbb{C}((-\ell+1)2\rho))$.

Proof. The triangular decomposition of the algebra $\mathfrak{b}_\ell^+$ is as follows: $\mathfrak{b}_\ell^+ = \mathfrak{u}_\ell^+ \otimes \mathfrak{u}_\ell^0$. Then by definition of semi-infinite cohomology, using the fact that $\mathfrak{u}_\ell^0$ semisimple, we see that

$$\mathrm{Ext}_{\mathfrak{b}_\ell^+}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = H^\bullet\left(\mathrm{Hom}^\bullet_{\mathfrak{b}_\ell^+}(R^\bullet_\uparrow(\underline{\mathbb{C}}), S_{\mathfrak{b}_\ell^+}^{\mathfrak{u}_\ell^+}\otimes_{\mathfrak{u}_\ell}\underline{\mathbb{C}})\right),$$

where $R^\bullet_\uparrow(\underline{\mathbb{C}})$ denotes a concave $\mathfrak{u}_\ell^+$-free resolution of the trivial $\mathfrak{b}_\ell^+$-module. Next note that the $\mathfrak{b}_\ell^+$-module $S_{\mathfrak{b}_\ell^+}^{\mathfrak{u}_\ell^+}\otimes_{\mathfrak{b}_\ell^+}\underline{\mathbb{C}} = \mathfrak{b}_\ell^{+*}\otimes_{\mathfrak{b}_\ell^+}\underline{\mathbb{C}} = \mathbb{C}(-(\ell-1)2\rho)$. Thus we have

$$\begin{aligned}\mathrm{Ext}_{\mathfrak{b}_\ell^+}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) &= H^\bullet\left(\mathrm{Hom}^\bullet_{\mathfrak{b}_\ell^+}(R^\bullet_\uparrow(\underline{\mathbb{C}}),\mathbb{C}(-(\ell-1)2\rho))\right)\\ &= \left(H^\bullet(\mathrm{Hom}^\bullet_{\mathfrak{u}_\ell^+}(R^\bullet_\uparrow(\underline{\mathbb{C}}),\mathbb{C}(-(\ell-1)2\rho)))\right)^{\mathfrak{u}_\ell^0} = \mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\mathbb{C}(-(\ell-1)2\rho)).\end{aligned}$$

□

Next we investigate the structure of the $\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-module $\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$.

Proposition 3.2.2. *Up to the homological grading shift by $\sharp(R^+)$ we have*

$$\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \mathrm{Coind}_{\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}})}^{\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}\left(\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\mathbb{C}(-(\ell-1)2\rho))\right)$$

as a $\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$*-module.*

Proof. By Theorem 3.2, Lemma 3.2.1. and Corollary 2.4.3. we have

$$\begin{aligned}\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) &= \left(\mathrm{Ext}_{\mathfrak{u}_\ell^-\otimes\mathfrak{u}_\ell^+}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})\right)^{\mathfrak{u}_\ell^0}\\ &= \left(\mathrm{Tor}_\bullet^{\mathfrak{u}_\ell^-}(\underline{\mathbb{C}},\underline{\mathbb{C}})\otimes\mathbb{C}((1-\ell)2\rho)\otimes\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}})\right)^{\mathfrak{u}_\ell^0}\\ &= \left(S^\bullet(\mathfrak{n}^-)\otimes A_+^*\otimes S^\bullet(\mathfrak{n}^{+*})\otimes A_-\otimes\mathbb{C}((1-\ell)2\rho)\right)^{\mathfrak{u}_\ell^0}\\ &= S^\bullet(\mathfrak{n}^-)\otimes S^\bullet(\mathfrak{n}^{+*})\otimes\left(A_+^*\otimes\mathbb{C}((1-\ell)2\rho)\otimes A_-\right)^{\mathfrak{u}_\ell^0}.\end{aligned}$$

Now recall that the algebra A_+ is Frobenius. Making the isomorphism $A_+ = A_+^*$ compatible with the $\mathfrak{u}_\ell^0$-action, we see that $A_+ = A_+^*\otimes\mathbb{C}(-2\rho)$. Thus we obtain

$$\begin{aligned}\mathrm{Ext}_{\widetilde{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) &= S^\bullet(\mathfrak{n}^-)\otimes S^\bullet(\mathfrak{n}^{+*})\otimes\left(A_+\otimes\mathbb{C}(-2\rho)\otimes\mathbb{C}((1-\ell)2\rho)\otimes A_-\right)^{\mathfrak{u}_\ell^0}\\ &= S^\bullet(\mathfrak{n}^-)\otimes\mathrm{gr}^F H_0\otimes S^\bullet(\mathfrak{n}^{+*})\otimes\mathbb{C}(-2\ell\rho).\end{aligned}$$

Note that the algebra $\mathrm{gr}\, H_0$ is Frobenius itself, thus the latter $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-module can be considered both as

$$\mathrm{Coind}^{\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}_{\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}\left(\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\mathbb{C}(-2\ell\rho))\right) \text{ and as}$$
$$\mathrm{Ind}^{\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}_{\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}\left(\mathrm{Tor}^{\mathfrak{b}^-_\ell}_\bullet(\underline{\mathbb{C}},\mathbb{C}(-2\ell\rho))\right).$$

Finally precise calculation of homological gradings shows that the grading on the first module should be shifted by $-\sharp(R^+)$, and the grading on the second one should be shifted by $\sharp(R^+)$. Thus we have

$$\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \mathrm{Coind}^{\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}_{\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}\left(\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\mathbb{C}(-(\ell-1)2\rho))\right)$$
$$= \mathrm{Ind}^{\mathrm{Ext}^\bullet_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}_{\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})}\left(\mathrm{Tor}^{\mathfrak{b}^-_\ell}_\bullet(\underline{\mathbb{C}},\mathbb{C}(-(\ell-1)2\rho))\right).$$

□

Corollary 3.2.3. $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ *is both free over the algebra* $\mathrm{Ext}^\bullet_{\mathfrak{b}^+_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ *and co-free over the algebra* $\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. □

3.3. Functions on $\mathcal{N}$ with singularities along coordinate hyperplanes. Consider the divisors on the nilpotent cone of the form $D_\alpha := \{n \in \mathcal{N} | f^*_\alpha(n) = 0\}$, where f^*_α denote the standard basic linear functions on $\mathfrak{n}^-$, $\alpha \in R^+$. Note that we have $\mathcal{N} \supset \mathfrak{n}^+ = \bigcap_{\alpha\in R^+} D_\alpha$.

Consider the open subset in the nilpotent cone of the form $\mathcal{N}_{(f^*_{\alpha_1},\dots,f^*_{\alpha_{\sharp(R^+)}})} := \mathcal{N} \setminus \bigcup_{\alpha\in R^+} D_\alpha$ and the $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module $H^0(\mathcal{N}_{(f^*_{\alpha_1},\dots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N})$.

It is well known how to obtain this $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module from the free module $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$ via an inductive limit construction.

For $\alpha \in R^+$ consider the $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module map $\mu_\alpha : H^0(\mathcal{N},\mathcal{O}_\mathcal{N}) \longrightarrow H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$ given by multiplicaton by the element $f^*_\alpha \in \mathfrak{n}^{-*}$.

Next we construct an inductive system of $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-modules $\mathcal{I}_\bullet$ enumerated by the partially ordered set $\mathbb{Z}^{R^+}_+$ with $\mathcal{I}_v = H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$ for any $v \in \mathbb{Z}^{R^+}_+$ and with morphisms

$$\mathcal{I}_{(\dots,v_\alpha,\dots)} \longrightarrow \mathcal{I}_{(\dots,v_\alpha+1,\dots)}$$

equal to μ_α.

Lemma 3.3.1. *The* $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$*-module* $H^0(\mathcal{N}_{(f^*_{\alpha_1},\dots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N})$ *is equal to* $\varinjlim \mathcal{I}_\bullet$. □

Note that the $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module $H^{\sharp R^+}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ can be realised naturally as a bottom subquotient module in $H^0(\mathcal{N}_{(f^*_{\alpha_1},\dots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N})$ under the filtration on the latter module by singularities of functions. More precisely, recall that $H^0(\mathcal{N},\mathcal{O}_\mathcal{N}) = S^\bullet(\mathfrak{n}^{-*})\otimes H_0\otimes S^\bullet(\mathfrak{n}^{+*})$as a module over $S^\bullet(\mathfrak{n}^{+*})\otimes S^\bullet(\mathfrak{n}^{-*})$. Fix the set of standard generators in the algebra $S(\mathfrak{n}^{-*})$: $S(\mathfrak{n}^{-*}) = \mathbb{C}[f^*_\alpha,\ \alpha\in R^+]$. Then by the previous Lemma $H^0(\mathcal{N}_{(f^*_{\alpha_1},\dots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N})$ is equal to $\mathbb{C}[f^{*\pm 1}_\alpha,\alpha\in R^+]\otimes H_0\otimes S^\bullet(\mathfrak{n}^{+*})$. On the other hand note that $H^{\sharp R^+}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is isomorphic to $\bigotimes_{\alpha\in R^+}\mathbb{C}[f^{*\pm1}_\alpha]/\mathbb{C}[f^*_\alpha]\otimes H_0\otimes S^\bullet(\mathfrak{n}^{+*})$.

Our main goal here is to construct an isomorphism of $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-modules

$$H^{\sharp R^+}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})\widetilde{\longrightarrow}\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

We start with constructing a system of $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module morphisms $\varphi_\bullet$: $\mathcal{I}_\bullet\longrightarrow\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ that would provide a morphism from the direct limit of the inductive system to the semi-infinite cohomology module.

3.4. Construction of the morphisms $\varphi_\bullet$. Note that since each module $\mathcal{I}_v$, $v\in\mathbb{Z}^{R^+}_+$, is free over the algebra $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$ with one generating element, to construct the system of morphisms from the described inductive system to $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ one has to specify the images of $1\in H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$ under the morphisms φ_v, $v\in\mathbb{Z}^{R^+}_+$. Then $\varphi_v(a) = a\cdot\varphi_v(1)$, where $a\in H^0(\mathcal{N},\mathcal{O}_\mathcal{N}) = \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ (the latter algebra acts naturally on $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$). Suppose that

$$\varphi_{(\dots,v_\alpha,\dots)}:\ 1\mapsto m_{(\dots,v_\alpha,\dots)}\in \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}});\ \varphi_{(\dots,v_\alpha+1,\dots)}:\ 1\mapsto m_{(\dots,v_\alpha+1,\dots)}\in\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

Then evidently the only condition on the sequence of elements $\{m_v\}_{v\in\mathbb{Z}^{R^+}_+}$ is that

$$f^*_\alpha\cdot m_{(\dots,v_\alpha+1,\dots)} = m_{(\dots,v_\alpha,\dots)}.$$

Consider the element

$$\nu^+_e\otimes\nu^-_e\in(A_+\otimes A_-)^{\mathfrak{u}^0}\otimes\mathbb{C}(-2\ell\rho) = \mathrm{gr}^F H_0\otimes\mathbb{C}(-2\ell\rho)\subset\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\tilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

Here we preserve notanion from 2.3 for the base elements in the algebras A_+ and A_-, and $e\in W$ denotes the unity element.

Note that the corresponding $\ell\cdot X\times\mathbb{Z}$-grading component in the space $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\tilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is one dimensional. Since the filtration F on the space $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is well defined with respect to the $\ell\cdot X\times\mathbb{Z}$-grading, we can consider the described vector as an element of $\mathrm{Ext}^{\frac{\infty}{2}-\sharp(R^+)}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$.

We are starting to construct the system of morphisms φ_v, $v\in\mathbb{Z}^{R^+}_+$. Put

$$\varphi_{(0,\dots,0)}:\ \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})[\sharp(R^+)]\longrightarrow\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}),\ \varphi_{(0,\dots,0)}(1) = \nu^-_e\otimes\nu^+_e.$$

Theorem 3.4.1. $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ *is co-free over the algebra* $S^\bullet(\mathfrak{n}^{-*}) = \mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^-}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ *with the space of co-generators equal to* $H_0\otimes\mathbb{C}(-2\ell\rho)\otimes S^\bullet(\mathfrak{n}^{+*})$.

Proof. First recall that by 2.3.5. the $\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-module $\mathrm{Ext}_{\widetilde{\mathfrak{u}_\ell}}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is free with the space of generators equal to

$$S^\bullet(\mathfrak{n}^-)\otimes H_0\otimes\mathbb{C}(-2\ell\rho).$$

On the other hand $\mathrm{Ext}_{\widetilde{\mathfrak{u}_\ell}}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \mathrm{gr}^F\,\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$, and this equality is an isomorphism of modules over $\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \mathrm{gr}^F\,\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Fix the set of linear independent elements $\{\overline{b}_p|p\in\mathbb{Z}_+^{R^+}\times W\}$ in the generators space of $\mathrm{Ext}_{\widetilde{\mathfrak{u}_\ell}}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ and choose the set of representatives of the elements $\{b_p|p\in\mathbb{Z}_+^{R^+}\times W\}\subset\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. It is easy to verify that the $\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-submodule in $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ generated by this set is free. Moreover its character coincides with the one of $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$, thus the module of semi-infinite cohomology is $\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^+}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-free itself. Now to complete the proof of the Theorem recall the following statement from [Ar1].

Denote by $\hat{\mathfrak{u}}_\ell$ the finite quantum group defined in the same way as $\mathfrak{u}_\ell$, but with ζ replaced by ζ^{-1} in the defining relations.

Lemma 3.4.2. ([Ar1], Proposition 5.4.3) *There exists a nondegenerate contragradient pairing*

$$\langle\ ,\ \rangle:\ \mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})\times\mathrm{Ext}_{\hat{\mathfrak{u}}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})\longrightarrow\mathbb{C}$$

well defined with respect to the actions of the algebra $H^0(\mathcal{N},\mathcal{O}_\mathcal{N}) = \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \mathrm{Ext}^\bullet_{\hat{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ *and of the Lie algebra* $\mathfrak{g}$. □

Using this pairing we obtain the set of co-generators $\{c_p|p\in\mathbb{Z}_+^{R^+}\}$ for the co-free $\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^-}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-module $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ enumerated by the base vectors of the space $H_0\otimes\mathbb{C}(-2\ell\rho)\otimes S^\bullet(\mathfrak{n}^{+*})$. The Theorem is proved. □

Note that we can choose the cogenerators c_p constructed above to be $X\times\mathbb{Z}$-homogeneous. In particular there is a unique choice of the homogeneous base vector in the space

$$\left(\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}-\sharp(R^+)}(\underline{\mathbb{C}},\underline{\mathbb{C}})\right)_{-2\ell\rho}$$

with the top filtration factor equal to $\nu_e^-\otimes\nu_e^+\in\mathrm{Ext}_{\widetilde{\mathfrak{u}_\ell}}^{\frac{\infty}{2}-\sharp(R^+)}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Now consider the direct sum decomposition of the $\mathrm{Ext}^\bullet_{\mathfrak{b}_\ell^-}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-module

$$\mathrm{Ext}_{\widetilde{\mathfrak{u}_\ell}}^{\frac{\infty}{2}-\sharp(R^+)+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}}) = \bigoplus_{p\in\mathbb{Z}_+^{R^+}\times W} S^\bullet(\mathfrak{n}^-)\otimes c_p$$

and the direct summand $S^\bullet(\mathfrak{n}^-)\otimes(\widetilde{\nu_e^+\otimes\nu_e^-})$ where $\widetilde{\nu_e^+\otimes\nu_e^-}$ denotes the element representing $\nu_e^+\otimes\nu_e^-$. Fixing standard generators f_α, $\alpha\in R^+$ in the space $\mathfrak{n}^-$ we obtain a linear base in $S^\bullet(\mathfrak{n}^-)\otimes(\nu_e^+\otimes\nu_e^-)$ of the form

$$m_v := f_{\alpha_1}^{v_1}\cdot\ldots\cdot f_{\alpha_{\sharp(R^+)}}^{v_{\sharp(R^+)}}\otimes(\widetilde{\nu_e^+\otimes\nu_e^-}).$$

Corollary 3.4.3. *We have* $f_\alpha^*\cdot m_{(\ldots,v_\alpha+1,\ldots)} = m_{(\ldots,v_\alpha,\ldots)}$. □

Thus we obtain a sequence of morphisms of $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$-modules

$$\varphi_v:\ \mathcal{I}_v = \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})[\sharp R^+ + 2|v|] \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}),\ \varphi_v(1)=m_v,\ v\in\mathbb{Z}_+^{R^+}.$$

Here $|v| = v_1+\ldots+v_{\sharp(R^+)}$. We have constructed a morphism

$$H^0(\mathcal{N}_{(f^*_{\alpha_1},\ldots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N}) = \varinjlim \mathcal{I}_\bullet \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

3.5. Construction of the isomorphism. $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})\widetilde{\longrightarrow}\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ Our nearest goal is to show that the morphism $\varprojlim(\varphi_\bullet)$ provides a map of $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-modules $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N}) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$.

Lemma 3.5.1. *For any* $w\in H^0(\mathcal{N}_{(f^*_{\alpha_1},\ldots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N})$ *there exists* $v^0 = (v^0_{\alpha_1},\ldots,v^0_{\alpha_{\sharp(R^+)}})$ *such that for any* $v=v^0+v^1$, $v^1\in\mathbb{Z}_+^{R^+}$, *the element* $\varprojlim(\varphi_\bullet)(f^{*v_1}_{\alpha_1}\cdot\ldots\cdot f^{*v_{\sharp(R^+)}}_{\alpha_{\sharp(R^+)}}w) = 0$.

Proof. The statement follows immediately from the restrictions on $\ell\cdot X\times\mathbb{Z}$-gradings of the space $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. □

Recall that $H_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is the largest quotient module of $H^0(\mathcal{N}_{(f^*_{\alpha_1},\ldots,f^*_{\alpha_{\sharp(R^+)}})},\mathcal{O}_\mathcal{N})$ supported on $\mathfrak{n}^+\subset\mathcal{N}$.

Corollary 3.5.2. *The map* $\varprojlim(\varphi_\bullet)$ *defines a morphism* $\Phi:\ H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_\mathcal{N}) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. □

In fact it follows from the previous considerations that for any finite dimensional graded $\mathfrak{u}_\ell$-module M the quasicoherent sheaf on $\mathcal{N}$ corresponding to the $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},M)$ is supported on $\mathfrak{n}^+\subset\mathcal{N}$.

Note that the maps μ_α, φ_v and $\varprojlim(\varphi_\bullet)$ are well defined with respect to the filtration F. The maps $\mathrm{gr}^F\mu_\alpha:\mathrm{Ext}^\bullet_{\tilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})\longrightarrow\mathrm{Ext}^\bullet_{\tilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})[2]$ are provided by multiplication by the elements $f^*_\alpha\in\mathrm{Ext}^\bullet_{\mathfrak{b}^-_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})\subset\mathrm{Ext}^\bullet_{\tilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Moreover we have

$$\mathrm{gr}^F\left(\varinjlim\mathcal{I}_\bullet\right) = \varinjlim\left(\mathrm{gr}^F\mathcal{I}_\bullet\right) = \bigotimes_{\alpha\in R^+}\mathbb{C}[f^{*\pm1}_\alpha]\otimes\mathrm{gr}^F H_0\otimes S^\bullet(\mathfrak{n}^{+*}).$$

Note also that the morphisms

$$\mathrm{gr}^F \varphi_v : \ \mathrm{Ext}^{\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})[\sharp(R^+)+2|v|] \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$$

are defined by

$$\mathrm{gr}^F \varphi_v(1) = \overline{m}_v = f_{\alpha_1}^{v_1}\dots f_{\alpha_{\sharp(R^+)}}^{v_{\sharp(R^+)}}(\nu_e^-\otimes\nu_e^+).$$

It follows that the map $\mathrm{gr}^F \lim\limits_{\leftarrow}(\varphi_\bullet)$ coincides with the natural one

$$\bigotimes_{\alpha\in R^+}\mathbb{C}[f_\alpha^{*\pm1}]\otimes \mathrm{gr}^F H_0\otimes S^\bullet(\mathfrak{n}^{+*}) \longrightarrow \bigotimes_{\alpha\in R^+}\left(\mathbb{C}[f_\alpha^{*\pm1}]/\mathbb{C}[f_\alpha^*]\right)\otimes \mathrm{gr}^F H_0\otimes S^\bullet(\mathfrak{n}^{+*}).$$

In particular the map $\mathrm{gr}^F \lim\limits_{\leftarrow}(\varphi_\bullet)$ is surjective.

Lemma 3.5.3. *The map $\lim\limits_{\leftarrow}(\varphi_\bullet)$ is surjective itself.*

Proof. Note that the morphisms $\lim\limits_{\leftarrow}(\varphi_\bullet)$ and $\mathrm{gr}^F \lim\limits_{\leftarrow}(\varphi_\bullet)$ preserve $\ell\cdot X\times\mathbb{Z}$-grading components. The $\ell\cdot X\times\mathbb{Z}$-grading components of the spaces $\lim\limits_{\rightarrow}\mathcal{I}_\bullet$, $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$, $\lim\limits_{\rightarrow}\left(\mathrm{gr}^F\mathcal{I}_\bullet\right)$ and $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\widetilde{\mathfrak{u}}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ are finite dimensional.

Thus we can check injectivity of the map dual to $\lim\limits_{\leftarrow}(\varphi_\bullet)$ instead of surjectivity of $\lim\limits_{\leftarrow}(\varphi_\bullet)$. But the injectivity of a filtered map follows from the injectivity of the associated graded map. □

Thus we obtain a surjective map $\Phi:\ H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}}) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$. Next recall the following statement from [Ar1].

Proposition 3.5.4. ([Ar1], Theorem A.2.2)

$$\mathrm{ch}\left(H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}}),t^2\right) = \mathrm{ch}\left(\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}),t\right).$$

Here the indeterminate t in the left hand side of the equality stands for the grading by the natural action of $\mathbb{C}^*$, in the right hand side of the equality it stands for the homological grading. □

We have proved the Feigin conjectire on the level of $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-modules.

Theorem 3.5.5. *The morphism Φ constructed above provides an isomorphism of the $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-modules*

$$\Phi:\ H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}}) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}}).$$

□

Remark 3.5.6. In fact we have proved a slightly more general statement as follows. Let M be a $X\times\mathbb{Z}$-graded $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-module with the character equal to the one of $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$. Suppose that M is generated over $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ by a $S^\bullet(\mathfrak{n}^{-*})$-cofree $S^\bullet(\mathfrak{n}^{-*})$-submodule co-generated by one element. Then the $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-module M is isomorphic to $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$.

4. Isomorphism of $\mathfrak{g}$-modules

Recall that both sides of the $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-module isomorphism Φ carry natural structures of $\mathfrak{g}$-modules.

The one on $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is a consequence of the fact that the algebra $\mathfrak{u}_\ell$ is a normal subalgebra in U_ℓ with the quotient algebra $U_\ell//\mathfrak{u}_\ell$ equal to $U(\mathfrak{g})$.

The $\mathfrak{g}$-module structure on $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ comes from the natural Lie algebra inclusion $\mathfrak{g}\hookrightarrow \mathcal{V}ect(\mathcal{N})$ and the standard fact due to Kempf (see [K]) that the Lie algebra of algebraic vector fields acts on local cohomology spaces. Below we prove that the two described $\mathfrak{g}$-module structures are isomorphic.

4.1. Isomorphism of the $\mathfrak{n}^+$-modules. Consider the standard positive nilpotent Lie subalgebra $\mathfrak{n}^+\subset\mathfrak{g}$. We prove first that $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ and $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ are isomorphic as $\mathfrak{n}^+$-modules.

Note first that the $\mathfrak{n}^+$- and $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-actions on both $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ and $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ satisfy the equation

$$n\cdot(a\cdot c)=a\cdot(n\cdot c)+[n,a]\cdot c,$$

where $n\in\mathfrak{n}^+, a,[n,a]\in H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$, $c\in H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ (resp. $c\in \mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$) and $[*,*]$ denotes the natural $\mathfrak{g}$-action on $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$.

Consider the $S^\bullet(\mathfrak{n}^{-*})$-module direct sum decomposition

$$\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}-\sharp(R^+)+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})=\bigoplus_{p\in\mathbb{Z}_+^{R^+}\times W} S^\bullet(\mathfrak{n}^-)\otimes c_p$$

with $c_0=\widetilde{\nu_e^+\otimes\nu_e^-}\in \mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}-\sharp(R^+)}(\underline{\mathbb{C}},\underline{\mathbb{C}})$.

Lemma 4.1.1. *$S^\bullet(\mathfrak{n}^-)\otimes c_0\subset \mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}-\sharp(R^+)}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is a $\mathfrak{n}^+$-submodule.*

Proof. We prove the statement of the Lemma by induction by homological grading. First $(S^\bullet(\mathfrak{n}^-)\otimes c_0)^{-\sharp(R+)}=\mathbb{C}c_0$, and the statement follows from the X-grading restrictions on the $\mathfrak{n}^+$-action.

Suppose the statement is proved for $S^k(\mathfrak{n}^-)\otimes c_0$, $k<k_0$. We prove that $S^{k_0}(\mathfrak{n}^-)\otimes c_0$ is a $\mathfrak{n}^+$-submodule in $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}-\sharp(R^+)-2k_0}(\underline{\mathbb{C}},\underline{\mathbb{C}})$.

Let $b\in S^{k_0}(\mathfrak{n}^-)$ and $n\in\mathfrak{n}^+$. Consider the element $n\cdot b\in \mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}-\sharp(R^+)-2k_0}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ and its decomposition $n\cdot b=(n\cdot b)^{\mathrm{yes}}+(n\cdot b)^{\mathrm{no}}$, where $(n\cdot b)^{\mathrm{yes}}\in S^\bullet(\mathfrak{n}^-)\otimes c_0$ and $(n\cdot b)^{\mathrm{no}}\in \bigoplus_{p\in\mathbb{Z}_+^{R^+}\times W, p\neq 0} S^\bullet(\mathfrak{n}^-)\otimes c_p$. Then for any $\alpha\in R^+$ the element

$f_\alpha^* \cdot (n \cdot b) = n \cdot (f_\alpha^* \cdot b) + [n, f_\alpha^*] \cdot b$ belongs to $S^\bullet(\mathfrak{n}^-)\otimes c_0$, since by induction hypothesis both summands lie there. Thus for any $\alpha \in R^+$ we have $f_\alpha^* \cdot (n \cdot b)^{\mathrm{no}} = 0$, i. e. the element belongs to the $S^\bullet(\mathfrak{n}^{-*})$-invariants subspace of $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. But by Theorem 3.4.1. the invariants space is equal to $H_0 \otimes S^\bullet(\mathfrak{n}^{+*})\otimes\mathbb{C}(-2\ell\rho)$. The latter subspace is situated in homological gradings greater than or equal to $-\sharp(R^+)$. On the other hand the $\mathfrak{n}^+$-submodule in question is situated in homological gradings less than or equal to $-\sharp(R^+)$. It follows that $(n \cdot b)^{\mathrm{no}} = 0$. $\square$

Corollary 4.1.2. *The action of $\mathfrak{n}^+$ on $S^\bullet(\mathfrak{n}^-)\otimes c_0$ coincides with the one on $S^\bullet(\mathfrak{g}/\mathfrak{b}^+)$.*

Proof. Identify $S^\bullet(\mathfrak{n}^-)\otimes c_0$ with $\mathrm{Hom}_{\mathbb{C}}(S^\bullet(\mathfrak{n}^{-*}), \mathbb{C})$. For such a linear function b note that $b(a) = (a \cdot b)(1)$. Note also that $S^\bullet(\mathfrak{n}^{-*}) = S^\bullet((\mathfrak{g}/\mathfrak{b}^+)^*)$ is a $\mathfrak{n}^+$-submodule in $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$. Thus for $n \in \mathfrak{n}^+$, $b \in \mathrm{Hom}_{\mathbb{C}}(S^\bullet(\mathfrak{n}^{-*}), \mathbb{C})$ we have

$$(n \cdot b)(a) = a \cdot (n \cdot b)(1) = n \cdot (a \cdot b)(1) - ([n, a] \cdot b)(1) = -b([n, a]).$$

$\square$

Recall that the $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-module $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ is generated by the subspace $S^\bullet(\mathfrak{n}^-)\otimes c_0$ and is in fact isomorphic to $\mathrm{Ind}_{S^\bullet(\mathfrak{n}^{-*})}^{H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})} S^\bullet(\mathfrak{n}^-)\otimes c_0$. Summing up the previous considerations we note that we have proved the following statement.

Theorem 4.1.3. *Up to an isomorphism there exists only one $\mathfrak{n}^+$-module structure on the space*

$$\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N}, \mathcal{O}_\mathcal{N})$$

well defined with respect to the $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-structure. $\square$

Remark 4.1.4. The $\mathfrak{n}^+$-submodule $S^\bullet(\mathfrak{n}^-)\otimes c_0 \subset H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ can be described geometrically as follows. Consider the natural projection $\pi : \mathcal{N} \longrightarrow \mathfrak{n}^-$. Identify $\mathfrak{n}^-$ with $\mathfrak{g}/\mathfrak{b}^+$. Then the map π is equivariant with respect to the adjoint action of the Lie algebra $\mathfrak{n}^+$. Consider the natural morphism of the local cohomology spaces $H_0^{\sharp(R^+)}(\mathfrak{n}^-, \mathcal{O}_{\mathfrak{n}^-}) \longrightarrow H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ given by the inverse image (note that $\mathfrak{n}^+ = \pi^{-1}(0)$). One checks directly that the morphism is in fact an imbedding compatible with the natural $\mathfrak{n}^+$-actions and that its image coincides with $S^\bullet(\mathfrak{n}^-)\otimes c_0$.

4.2. Tilting modules. It remains to extend the described $\mathfrak{n}^+$-module isomorphism to a $\mathfrak{g}$-module isomorphism.

Recall that a finitely generated $\mathfrak{n}^+$-locally finite $\mathfrak{g}$-module M diagonazible over $\mathfrak{h}$ is called a *tilting module* if it is filtered both by Verma modules and by contragradient Verma modules. H. H. Andersen and J. Paradowski have proved the following statement.

Theorem 4.2.1. **[AP]** *A tilting $\mathfrak{g}$-module is uniquely determined up to an isomorphism by its character with respect to the root space decomposition for the Cartan action.* □

Our goal here is to prove that both $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ and $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ are tilting $\mathfrak{g}$-modules.

4.3. A nondegenerate pairing on $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$. The following construction was proposed by V. Ostrik. Instead of constructing a contragradient pairing on $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ we obtain a bilinear pairing

$$H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})\times H^{\sharp(R^+)}_{\mathfrak{n}^-}(\mathcal{N},\mathcal{O}_{\mathcal{N}})\longrightarrow\mathbb{C}$$

well defined with respect to the $\mathfrak{g}$-module structures. Consider the canonical map for the local cohomology spaces provided by the cup product

$$\langle\ ,\ \rangle:\ H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})\times H^{\sharp(R^+)}_{\mathfrak{n}^-}(\mathcal{N},\mathcal{O}_{\mathcal{N}})\longrightarrow H^{2\sharp(R^+)}_{\mathfrak{n}^-\cap\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}}).$$

Lemma 4.3.1.

(i) $H^{2\sharp(R^+)}_{\mathfrak{n}^-\cap\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})=H^{2\sharp(R^+)}_{0}(\mathcal{N},\mathcal{O}_{\mathcal{N}})=\mathbb{C}$. Here 0 denotes the vertex of the nilpotent cone.

(ii) The map $\langle\ ,\ \rangle$ provides a nondegenerate pairing.

□

4.3.2. Springer-Grothendieck resolution of the nilpotent cone. To prove that $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N},\mathcal{O}_{\mathcal{N}})$ is $\mathfrak{n}^-$-free recall that in [Ar1] we obtained another geometric realization of this $\mathfrak{g}$-module as follows.

Consider the simply connected Lie group G with the Lie algebra equal to $\mathfrak{g}$. Choose a maximal torus $H\subset G$ providing the root decomposition of $\mathfrak{g}$ and in particular its triangular decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$. Consider the Borel subgroup $B\subset G$ with the Lie algebra $\mathfrak{b}^+=\mathfrak{h}\oplus\mathfrak{n}^+$ and the flag variety G/B. The group G acts on G/B by left translations and the restriction of this action to B is known to have finitely many orbits. These orbits are isomorphic to affine spaces and called *the Schubert cells.* The Bruhat decomposition of G shows that the Schubert cells are enumerated by the Weyl group. Denote the orbit corresponding to the element $w\in W$ by S_w.

It is well known that the cotangent bundle $T^*(G/B)$ has a nice realization $T^*(G/B)=\{(B_x,n)|n\in\mathrm{Lie}(B_x)\}$, where B_x denotes some Borel subgroup in G and n is a nilpotent element in the Lie algebra $\mathrm{Lie}(B_x)$. The map

$$\mu:\ T^*(G/B)\longrightarrow\mathcal{N},\ (B_x,n)\mapsto n,$$

is known to be a resolution of singularities of $\mathcal{N}$ called *the Springer-Grothendieck resolution.*

Recall the following statement from [Ar1].

Proposition 4.3.3. (e.g. [CG], 3.1.36)

(i) $\mu^{-1}(\mathfrak{n}^+) = \bigsqcup_{w \in W} T^*_{S_w}(G/B)$, *where* $T^*_{S_w}(G/B)$ *denotes the conormal bundle to* S_w *in* G/B.

(ii) $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_{\mathcal{N}}) \widetilde{\longrightarrow} H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(T^*(G/B), \mathcal{O}_{T^*(G/B)})$ *as a* $\mathfrak{g}$*-module.*

□

Corollary 4.3.4. *The* $\mathfrak{g}$*-module* $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ *is free over the algebra* $U(\mathfrak{n}^-)$. □

In particular it is filtered by Verma modules. Using the fact that the $\mathfrak{g}$-module is self-dual with respect to the described contragradient pairing we construct a filtration by contragradient Verma modules on $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$. We have proved the following statement.

Proposition 4.3.5. *The* $\mathfrak{g}$*-module* $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ *is a tilting module.*

4.4. On the other hand recall that in Lemma 3.4.2. we obtained a nondegenerate $\mathfrak{g}$-equivariant contragradient pairing

$$\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \times \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\hat{\mathfrak{u}}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \longrightarrow \mathbb{C}.$$

It follows from Proposition 4.3.5. and Theorem 4.1.3. that the $\mathfrak{g}$-module $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ is cofree over the algebra $U(\mathfrak{n}^+)$. Thus it is filtered by contragradient Verma modules. Again using the contragradient pairing on the $\mathfrak{g}$-module we produce on it a filtration with subquotients equal to direct sums of Verma modules.

We have proved the following statement.

Proposition 4.4.1. *The* $\mathfrak{g}$*-module* $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ *is a tilting module.* □

Finally we come to the main statement of the present paper.

Theorem 4.4.2. *The* $\mathfrak{g}$*-modules* $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ *and* $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ *are isomorphic.*

Proof. The statement of the Theorem follows from Theorem 4.2.1. and Propositions 4.3.5. and 4.4.1. □

The Feigin conjecture is proved.

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Independent Univ. of Moscow, Pervomajskaya st. 16-18, Moscow 105037, RUSSIA
E-mail address: hippie@ium.ips.ras.ru