

## A METHOD OF CONSTRUCTION OF FINITE-DIMENSIONAL TRIANGULAR SEMISIMPLE HOPF ALGEBRAS

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### Introduction

The goal of this paper is to give a new method of constructing finite-dimensional semisimple triangular Hopf algebras, including minimal ones which are non-trivial (i.e. not group algebras). The paper shows that such Hopf algebras are quite abundant. It also discovers an unexpected connection of such Hopf algebras with bijective 1-cocycles on finite groups and set-theoretical solutions of the quantum Yang-Baxter equation defined by Drinfeld [Dr1].

Finite-dimensional triangular Hopf algebras were studied by several authors (see e.g. [CWZ,EG,G,Ma]). In [EG] the authors prove that any finite-dimensional semisimple triangular Hopf algebra over an algebraically closed field of characteristic 0 (say  $\mathbb{C}$ ) is obtained from a group algebra after twisting its comultiplication in the sense of Drinfeld [Dr2]. Twists are easy to construct for abelian groups  $A$ , they are just 2-cocycles for  $A^*$  with values in  $\mathbb{C}^*$ . A general simple construction of triangular semisimple Hopf algebras which are non-trivial, is the following: take a non-abelian group  $G$ , an abelian subgroup of it  $A$ , and a twist  $J \in \mathbb{C}[A] \otimes \mathbb{C}[A]$  which does not commute with  $g \otimes g$  for all  $g \in G$ , and twist  $\mathbb{C}[G]$  by  $J$  to obtain  $\mathbb{C}[G]^J$  (see e.g. [EV,Mo,N]). Examples of such Hopf algebras were constructed by the second author in [G]. They are Hopf algebras of dimension  $pq^2$  where  $p$  and  $q$  are any prime numbers so that  $q$  divides  $p-1$ . It was also proved in [G] that the dual of the Drinfeld double of these Hopf algebras is triangular. Nevertheless Hopf algebras which are constructed in this way are not minimal, and their minimal Hopf subalgebras are trivial. In fact, as far as we know, in the literature there are no non-trivial semisimple minimal triangular Hopf algebras. A natural question thus arose: Are there finite-dimensional non-trivial minimal semisimple triangular Hopf algebras? In this paper we describe a method for constructing such Hopf algebras.

The paper is organized as follows. First, we show how to construct twists for certain solvable non-abelian groups by iterating twists of their abelian subgroups, and thus obtain non-trivial semisimple triangular Hopf algebras. Second, we show that in some cases this construction gives non-trivial semisimple *minimal* triangular Hopf algebras. Finally, we show how any non-abelian group which admits a bijective 1-cocycle gives rise to a non-trivial semisimple *minimal*

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triangular Hopf algebra. Such non-abelian groups exist in abundance and were constructed in [ESS] in connection with set-theoretical solutions to the quantum Yang-Baxter equation.

We shall work over the field of complex numbers  $\mathbb{C}$ , although it can be replaced by any algebraically closed field of characteristic 0.

## 1. Twists

Recall Drinfeld's notion of a twist for Hopf algebras.

**Definition 1.1. [Dr2]** *Let  $\mathcal{A}$  be a Hopf algebra over a field  $k$ . A twist for  $\mathcal{A}$  is an invertible element  $J \in \mathcal{A} \otimes \mathcal{A}$  which satisfies*

$$(1.1) \quad (\Delta \otimes I)(J)J_{12} = (I \otimes \Delta)(J)J_{23} \text{ and } (\varepsilon \otimes I)(J) = (I \otimes \varepsilon)(J) = 1$$

where  $I : \mathcal{A} \rightarrow \mathcal{A}$  is the identity map.

Given a twist  $J$  for  $\mathcal{A}$ , we can construct a new Hopf algebra  $\mathcal{A}^J$ , which is the same as  $\mathcal{A}$  as an algebra, with coproduct  $\Delta^J$  given by

$$(1.2) \quad \Delta^J(x) = J^{-1}\Delta(x)J, \quad x \in \mathcal{A}.$$

If  $(\mathcal{A}, R)$  is quasitriangular then so is  $\mathcal{A}^J$  with the R-matrix

$$(1.3) \quad R^J = J_{21}^{-1}RJ,$$

and if  $(\mathcal{A}, R)$  is triangular then so is  $(\mathcal{A}^J, R^J)$ . Note that twists can be composed; that is, if  $J$  is a twist for  $\mathcal{A}$  and  $J'$  is a twist for  $\mathcal{A}^J$  then  $JJ'$  is a twist for  $\mathcal{A}$ , and  $\mathcal{A}^{JJ'} = (\mathcal{A}^J)^{J'}$ .

Now let  $(\mathcal{A}, R)$  be a finite-dimensional semisimple triangular Hopf algebra over  $\mathbb{C}$ . Assume that the Drinfeld element  $u$  equals 1 (this can always be attained by a simple modification of  $R$ , see [EG]). It was shown in [EG, Theorem 2.1] that in this case there exists a finite group  $G$  such that  $\mathcal{A} = \mathbb{C}[G]^J$  as a triangular Hopf algebra, where  $J$  is a suitable twist. Thus, construction of triangular semisimple finite-dimensional Hopf algebras reduces to construction of twists for group algebras.

*Remark 1.2.* In [CWZ], the authors generalize a number of results on commutative algebras to the case of quantum commutative algebras in the tensor category of  $\mathcal{A}$ -modules, where  $\mathcal{A}$  is a finite-dimensional semisimple triangular Hopf algebra. Let us point out that [EG, Theorem 2.1] is very useful for proving such generalizations. For example, the main result of [CWZ] is Theorem 4.7, which in particular states that if the characteristic of the ground field is 0, then any quantum commutative algebra  $M$  in  $\mathcal{A}\text{-mod}$  is integral over its subalgebra of invariants  $M^{\mathcal{A}}$ . Let us show how [EG, Theorem 2.1] enables to give an easy proof of this result.

Let  $\alpha \in M$  and pick a finite-dimensional  $\mathcal{A}$ -submodule  $V$  of  $M$  so that  $\alpha \in V$ . We wish to show that the images of  $K_i = \bigoplus_{j=0}^i M^{\mathcal{A}} \otimes V^{\otimes j}$  under multiplication  $K_i \rightarrow M$  stabilize. This will imply integrality of  $M$  over  $M^{\mathcal{A}}$ , as then the algebra  $M^{\mathcal{A}}[V]$  is a faithful module over  $M^{\mathcal{A}}[\alpha]$  which is finitely generated over  $M^{\mathcal{A}}$  (see [L, p. 356]).

Let  $G$  be a finite group and  $u \in G$  a central element such that  $u^2 = 1$ . Denote by  $\text{Rep}(G, u)$  the category of super-vector spaces with a  $G$ -action, such that  $u$  acts as 1 on even vectors and as  $-1$  on odd vectors. This is a symmetric tensor category.

Theorem 2.1 in [EG] states that there exists a fiber functor  $F : \mathcal{A}\text{-mod} \rightarrow \text{Rep}(G, u)$ , which is a tensor equivalence of  $\mathcal{A}\text{-mod}$  to the category  $\text{Rep}(G, u)$  for some  $(G, u)$  as above. Thus,  $M_0 = F(M)$  is a commutative algebra in  $\text{Rep}(G, u)$  (i.e., a supercommutative algebra with a  $G$  action, and  $u$  acts as 1 on even vectors and as  $-1$  on odd vectors. In particular  $F(M^{\mathcal{A}}) = M_0^G$ .

We claim that  $M_0$  is integral over  $M_0^G$ . It is clear if  $u = 1$  since any  $x \in M_0$  satisfies  $P(x) = 0$  where  $P(y) = \prod_{g \in G} (y - gx)$ . Suppose  $u \neq 1$ . Let  $x \in M_0$ ,  $x = x_0 + x_1$  where  $x_0$  is even,  $x_1$  is odd. Let  $G = \{g_1, ug_1, g_2, ug_2, \dots, g_n, ug_n\}$ , and let  $P(y) = \prod_{i=1}^n (y - g_i x)(y - ug_i x) = \prod_{i=1}^n (y - g_i x_0 - g_i x_1)(y - g_i x_0 + g_i x_1) = \prod_{i=1}^n (y - g_i x_0)^2$ . We have  $P(x) = 0$ , and the coefficients of  $P$  are in  $M_0^G$ .

This implies that if  $U$  is any finite-dimensional subobject (i.e. a  $G$ -submodule) of  $M_0$ , then the images of  $K_i = \bigoplus_{j=0}^i M_0^G \otimes U^{\otimes j}$  under the multiplication map  $K_i \rightarrow M_0$ , stabilize (after  $i = |G|$ ). In particular, it is true for  $U = F(V)$ . But, this property is categorical, i.e. preserved by tensor functors. Therefore, it remains valid if we replace  $M_0$  with  $M$ ,  $M_0^G$  with  $M^{\mathcal{A}}$ , and  $U$  with  $V$ , which implies the claim.  $\square$

The following definition is due to Radford [R].

**Definition 1.3.** Let  $(\mathcal{A}, R)$  be a quasitriangular Hopf algebra, and write  $R = \sum_i x_i \otimes y_i$  in the shortest possible way. Then  $(\mathcal{A}, R)$  is called *minimal* if  $\mathcal{A}$  is generated by its Hopf subalgebras  $sp\{x_i\}$  and  $sp\{y_i\}$ .

It is easy to check that if  $(\mathcal{A}, R)$  is triangular, then it is minimal if and only if  $R$  defines a nondegenerate bilinear form on  $\mathcal{A}^*$ . A twist  $J$  for  $\mathbb{C}[G]$  for which  $(\mathbb{C}[G]^J, J_{21}^{-1}J)$  is minimal triangular is said to be *minimal*. By [EG], for any finite-dimensional semisimple triangular Hopf algebra  $(\mathcal{A}, R)$  there exist finite groups  $H \subset G$  and a minimal twist  $J$  for  $\mathbb{C}[H]$  such that  $\mathcal{A} = \mathbb{C}[G]^J$ . Therefore, construction of finite-dimensional semisimple triangular Hopf algebras reduces to construction of minimal ones.

For a general group  $G$ , minimal twists for  $\mathbb{C}[G]$  are quite difficult to construct. However, if  $G$  is abelian, they can be constructed easily (if they exist). Namely, a twist for  $\mathbb{C}[G]$ , when regarded as a  $\mathbb{C}^*$ -valued function on  $G^* \times G^*$ , is the same thing as a 2-cocycle of the group  $G^*$  with coefficients in  $\mathbb{C}^*$ . In many cases, such a twist is minimal.

Our goal here is to use this simple construction to construct twists in the case

of a non-abelian  $G$ .

## 2. Construction of twists for non-abelian groups

In this section we give a method of constructing twists for non-abelian groups.

**Definition 2.1.** Let  $G, A$  be groups and  $\rho : G \rightarrow \text{Aut}(A)$  a homomorphism. By a 1-cocycle of  $G$  with coefficients in  $A$  we mean a map  $\pi : G \rightarrow A$  which satisfies the equation

$$(2.1) \quad \pi(gg') = \pi(g)(g\rho(g')), \quad g, g' \in G,$$

where  $\rho(g)(x) = gx$  for  $g \in G, x \in A$ .

We will be interested in the case when  $\pi$  is a *bijection*, because of the following proposition.

**Proposition 2.2.** Let  $G, A$  be groups,  $\pi : G \rightarrow A$  a bijective 1-cocycle, and  $J$  a twist for  $\mathbb{C}[A]$  which is  $G$ -invariant. Then  $\bar{J} := (\pi^{-1} \otimes \pi^{-1})(J)$  satisfies (1.1). Thus, if  $\bar{J}$  is invertible then it is a twist for  $\mathbb{C}[G]$ .

*Proof.* It is obvious that the second equation of (1.1) is satisfied for  $\bar{J}$ . So we only have to prove the first equation of (1.1) for  $\bar{J}$ . Let  $J = \sum a_{xy} x \otimes y$ . Then

$$(2.2) \quad \begin{aligned} & (\pi \otimes \pi \otimes \pi)((\Delta \otimes I)(\bar{J})\bar{J}_{12}) \\ &= \sum_{x,y,z,t \in A} a_{xy} a_{zt} \pi(\pi^{-1}(x)\pi^{-1}(z)) \otimes \pi(\pi^{-1}(x)\pi^{-1}(t)) \otimes \pi(\pi^{-1}(y)) \\ &= \sum_{x,y,z,t \in A} a_{xy} a_{zt} x(\pi^{-1}(x)z) \otimes x(\pi^{-1}(x)t) \otimes y. \end{aligned}$$

Using the  $G$ -invariance of  $J$ , we can remove the  $\pi^{-1}(x)$  in the last expression and get

$$(2.3) \quad (\pi \otimes \pi \otimes \pi)((\Delta \otimes I)(\bar{J})\bar{J}_{12}) = (\Delta \otimes I)(J)J_{12}.$$

Similarly,

$$(2.4) \quad (\pi \otimes \pi \otimes \pi)((I \otimes \Delta)(\bar{J})\bar{J}_{23}) = (I \otimes \Delta)(J)J_{23}.$$

But  $J$  is a twist, so the right hand sides of (2.3) and (2.4) are equal. Since  $\pi$  is bijective, this implies equation (1.1) for  $\bar{J}$ .  $\square$

Let  $G_1, \dots, G_n = G$ ,  $H_1, \dots, H_n$  be finite groups, and  $\rho_{i-1} : G_{i-1} \rightarrow \text{Aut}(H_i)$ ,  $i = 2, \dots, n$ , be homomorphisms, such that  $G_1 = H_1$ , and  $G_i = G_{i-1} \ltimes H_i$ , where the semidirect product is made using the action  $\rho_{i-1}$ . We will call such data a *hierarchy of length  $n$* . Note that  $G$  canonically contains  $G_1, \dots, G_n$  and  $H_1, \dots, H_n$  as subgroups. Let  $A = H_1 \times \dots \times H_n$ , and  $\rho : G \rightarrow \text{Aut}(A)$  be defined by  $\rho(x_n \cdots x_1)(y_1, \dots, y_n) = (y_1, \rho_1(x_1)y_2, \dots, \rho_{n-1}(x_{n-1} \cdots x_1)y_n)$ .

**Proposition 2.3.** *The map  $\pi : G \rightarrow A$  given by  $\pi(x_n \cdots x_1) = (x_1, \dots, x_n)$ ,  $x \in H_i$  is a bijective 1-cocycle.*

*Proof.* Straightforward.  $\square$

In the situation of Proposition 2.3 suppose further that  $J_i$  is a twist for  $\mathbb{C}[H_i]$ ,  $i = 1, \dots, n$ , and that  $J_i$  is invariant under  $\rho_{i-1}(G_{i-1})$ ,  $i = 2, \dots, n$ . We will call such data a *twist hierarchy of length  $n$* . In this situation, the element  $J = J_1 \otimes \cdots \otimes J_n$  is a twist for  $\mathbb{C}[A]$ . Therefore by Proposition 2.2,  $\bar{J} = (\pi^{-1} \otimes \pi^{-1})(J)$  satisfies (1.1).

**Proposition 2.4.**  $\bar{J} = J_n \cdots J_1$ , and is a twist for  $\mathbb{C}[G]$ .

*Proof.* It is clear that  $\bar{J}$  is invertible and  $\bar{J}^{-1} = J_1^{-1} \cdots J_n^{-1}$ . Therefore,  $\bar{J}$  is a twist for  $\mathbb{C}[G]$ .  $\square$

### 3. Minimal triangular Hopf algebras arising from symplectic hierarchies

In this section we show that the twist constructed in Proposition 2.4 is often minimal.

Let  $H$  be a finite abelian group, and  $(, ) : H \times H^* \rightarrow 2\pi i(\mathbb{R}/\mathbb{Z})$  be the standard pairing. By a *symplectic structure on  $H$*  we mean an isomorphism  $B : H \rightarrow H^*$  such that  $B^* = -B$ . If  $B$  is a symplectic structure then the bilinear form  $(x, By)$  on  $H$  will be denoted by  $\langle x, y \rangle$ . A symplectic structure defines an element

$$(3.1) \quad J_B = |H|^{-1} \sum_{x, y \in H} e^{\langle x, y \rangle} x \otimes y \in \mathbb{C}[H] \otimes \mathbb{C}[H].$$

This element is invertible with  $J_B^{-1} = |H|^{-1} \sum_{x, y \in H} e^{-\langle x, y \rangle} x \otimes y$ , so it is a twist for  $\mathbb{C}[H]$ .

It is clear that an automorphism of  $H$  fixes  $J_B$  if and only if it fixes  $B$ .

If  $H$  is an abelian group of odd order, then taking the square is an isomorphism  $H \rightarrow H$ . For  $x \in H$ , we will denote the preimage of  $x$  under this isomorphism by  $x^{1/2}$ .

If  $|H|$  is odd then it is straightforward to check that the R-matrix of  $\mathbb{C}[H]^{J_B}$  is  $R_B = (J_B)_{21}^{-1} J_B = |H|^{-1} \sum e^{\langle x, y^{1/2} \rangle} x \otimes y$ . The matrix  $c_{xy} = |H|^{-1} e^{\langle x, y^{1/2} \rangle}$  is nondegenerate; its inverse is  $e^{-\langle x, y^{1/2} \rangle}$  by the inversion formula for Fourier transform. So if  $|H|$  is odd then  $J_B$  is minimal.

Consider now a twist hierarchy of length  $n$ , of the following form:  $H_i$  are abelian of odd order, and  $J_i = J_{B_i}$ , where  $B_i$  is a symplectic structures on  $H_i$  for all  $i$ . Such a hierarchy will be called *symplectic*.

**Theorem 3.1.** *For a symplectic hierarchy, the twist  $\bar{J} = J_n \cdots J_1 \in \mathbb{C}[G] \otimes \mathbb{C}[G]$  is minimal, i.e. the triangular Hopf algebra  $(\mathbb{C}[G]^{\bar{J}}, \bar{J}_{21}^{-1} \bar{J})$  is minimal. Its*

universal  $R$ -matrix has the form

$$(3.2) \quad R = |G|^{-1} \sum_{x_i, y_i \in H_i} e^{\sum_{j=1}^n \langle \rho_{j-1}(x_{j-1}^{1/2} \cdots x_1^{1/2})^{-1} x_j, \rho_{j-1}(y_{j-1}^{1/2} \cdots y_1^{1/2})^{-1} y_j^{1/2} \rangle} x_n \cdots x_1 \otimes y_n \cdots y_1.$$

Before we prove Theorem 3.1 we need the following lemma.

**Lemma 3.2.** *Let  $K$  be an abelian group of odd order with a symplectic form, acting symplectically on another abelian group  $H$  of odd order with a symplectic form via  $\rho : K \rightarrow \text{Aut}(H)$ . Let*

$$(3.3) \quad S_{K,H}(x, y) = \sum_{z, z', t, t' \in K} e^{\langle z, t \rangle + \langle z', t' \rangle + \langle \rho(z^{-1})x, \rho(t^{-1})y^{1/2} \rangle} z z' \otimes t t'.$$

Then,

$$(3.4) \quad S_{K,H}(x, y) = |K| \sum_{u, v \in K} e^{\langle u, v^{1/2} \rangle + \langle \rho(u^{-1/2})x, \rho(v^{-1/2})y^{1/2} \rangle} u \otimes v.$$

*Proof.* Set in (3.3)  $u = z z', v = t t'$ . Then (3.3) transforms to the form

$$(3.5) \quad S_{K,H}(x, y) = \sum_{z, u, t, v \in K} e^{2\langle z, t \rangle + \langle u, v \rangle - \langle u, t \rangle - \langle z, v \rangle + \langle \rho(z^{-1})x, \rho(t^{-1})y^{1/2} \rangle} u \otimes v.$$

Introducing a new variable  $b = z t^{-1}$ , we can sum over  $b$  instead of  $z$  in (3.5). This yields

$$(3.6) \quad \begin{aligned} S_{K,H}(x, y) &= \sum_{b, u, t, v \in K} e^{2\langle b t, t \rangle + \langle u, v \rangle - \langle u, t \rangle - \langle b t, v \rangle + \langle \rho(t^{-1})\rho(b^{-1})x, \rho(t^{-1})y^{1/2} \rangle} u \otimes v. \end{aligned}$$

Using the skew symmetry of  $\langle, \rangle$  (which implies  $\langle t, t \rangle = 0$  because of odd order) and the invariance of the symplectic form on  $H$  under  $K$ , we get

$$(3.7) \quad S_{K,H}(x, y) = \sum_{b, u, t, v \in K} e^{\langle u, v \rangle + \langle b^2 u^{-1} v, t \rangle - \langle b, v \rangle + \langle \rho(b^{-1})x, y^{1/2} \rangle} u \otimes v.$$

Now we can sum over  $t$ , using the identity  $\sum_t e^{\langle a, t \rangle} = |K| \delta_{a1}$ . This yields

$$(3.8) \quad S_{K,H}(x, y) = |K| \sum_{u, v \in K} e^{\langle u^{1/2}, v \rangle + \langle \rho(u^{1/2} v^{-1/2})x, y^{1/2} \rangle} u \otimes v,$$

and we are done.  $\square$

*Proof of Theorem 3.1.* The first statement clearly follows from the second one. Indeed, let  $R = \sum_{g,h \in G} a_{gh} g \otimes h$ . It follows from (3.2) that by permutation of rows and columns the matrix  $a = (a_{gh})$  can be transformed into  $b = (b_{gh})$ , where

$$(3.9) \quad b_{x_n \dots x_1, y_n \dots y_1} = |G|^{-1} e^{\sum \langle x_i, y_i \rangle}.$$

This matrix is a tensor product  $b^n \otimes \dots \otimes b^1$ , where  $b_{x,y}^i = |H_i|^{-1} e^{\langle x,y \rangle}$ ,  $x, y \in H_i$ . As we mentioned, the matrices  $b_i$  are nondegenerate, therefore so are  $b$  and  $a$ .

We now calculate  $R$ . It is straightforward to check that the twist  $J_i$  satisfies  $J_i^{21} = J_i^{-1}$  for all  $i$ . Therefore,  $R = J_1 \dots J_{n-1} J_n^2 J_{n-1} \dots J_1$ . It is also straightforward to check that  $J_B^2 = J_{B'}$  for any  $B$ , where  $B'x = Bx^{1/2}$ . Therefore, we have:

$$(3.10) \quad R = \frac{|H_n|}{|G|^2} \sum_{z_i, t_i, x'_n, y'_n, z'_i, t'_i} e^{\sum_i \langle z_i, t_i \rangle + \sum_i \langle z'_i, t'_i \rangle + \langle x'_n, (y'_n)^{1/2} \rangle} \\ \times z_1 \dots z_{n-1} x'_n z'_{n-1} \dots z'_1 \otimes t_1 \dots t_{n-1} y'_n t'_{n-1} \dots t'_1.$$

Moving  $x'_n$  and  $y'_n$  to the left, we get from (3.10):

$$(3.11) \quad R = \frac{|H_n|}{|G|^2} \sum_{z_i, t_i, x'_n, y'_n, z'_i, t'_i} e^{\sum_i \langle z_i, t_i \rangle + \sum_i \langle z'_i, t'_i \rangle + \langle x'_n, (y'_n)^{1/2} \rangle} \times \\ x_n z_1 \dots z_{n-1} z'_{n-1} \dots z'_1 \otimes y_n t_1 \dots t_{n-1} t'_{n-1} \dots t'_1,$$

where  $x_n = \rho_{n-1}(z_1 \dots z_{n-1})x'_n$  and  $y_n = \rho_{n-1}(t_1 \dots t_{n-1})y'_n$ . Now we can replace summation over  $x'_n, y'_n$  with summation over  $x_n, y_n$  and get

$$(3.12) \quad R = \frac{|H_n|}{|G|^2} \sum_{z_i, t_i, x_n, y_n, z'_i, t'_i} e^{\sum_i \langle z_i, t_i \rangle + \sum_i \langle z'_i, t'_i \rangle + \langle \rho_{n-1}(z_{n-1}^{-1} \dots z_1^{-1})x_n, \rho_{n-1}(t_{n-1}^{-1} \dots t_1^{-1})y_n^{1/2} \rangle} \\ \times x_n z_1 \dots z_{n-1} z'_{n-1} \dots z'_1 \otimes y_n t_1 \dots t_{n-1} t'_{n-1} \dots t'_1.$$

Thus, the statement of the theorem is equivalent to the identity

$$(3.13) \quad \frac{|H_n|}{|G|} \sum_{z_i, t_i, z'_i, t'_i} e^{\sum_i \langle z_i, t_i \rangle + \sum_i \langle z'_i, t'_i \rangle + \langle \rho_{n-1}(z_{n-1}^{-1} \dots z_1^{-1})x_n, \rho_{n-1}(t_{n-1}^{-1} \dots t_1^{-1})y_n^{1/2} \rangle} \\ \times z_1 \dots z_{n-1} z'_{n-1} \dots z'_1 \otimes t_1 \dots t_{n-1} t'_{n-1} \dots t'_1 \\ = \sum_{x_i, y_i \in H_i, 1 \leq i \leq n-1} e^{\sum_{j=1}^n \langle \rho_{j-1}(x_{j-1}^{1/2} \dots x_1^{1/2})^{-1} x_j, \rho_{j-1}(y_{j-1}^{1/2} \dots y_1^{1/2})^{-1} y_j^{1/2} \rangle} \\ \times x_{n-1} \dots x_1 \otimes y_{n-1} \dots y_1.$$

So it remains to prove (3.13).

Denote the left hand side of (3.13) by  $T_{H_1, \dots, H_n}(x_n, y_n)$ . Using Lemma 3.2 for  $K = H_{n-1}$ ,  $H = H_n$ ,  $x = x_n$ ,  $y = y_n$ , we get

$$(3.14) \quad T_{H_1, \dots, H_n}(x_n, y_n) = \frac{|H_n||H_{n-1}|}{|G|} \sum_{z_i, t_i, z'_i, t'_i, x'_{n-1}, y'_{n-1}} e^{\sum_i \langle z_i, t_i \rangle + \sum_i \langle z'_i, t'_i \rangle} \\ \times e^{\langle x'_{n-1}, (y'_{n-1})^{\frac{1}{2}} \rangle + \langle \rho_{n-1}((x'_{n-1})^{-\frac{1}{2}}) \rho_{n-1}(z_{n-2}^{-1} \cdots z_1^{-1}) x_n, \rho_{n-1}((y'_{n-1})^{-\frac{1}{2}}) \rho_{n-1}(t_{n-2}^{-1} \cdots t_1^{-1}) y_n^{\frac{1}{2}} \rangle} \\ \times z_1 \cdots z_{n-2} x'_{n-1} z'_{n-2} \cdots z'_1 \otimes t_1 \cdots t_{n-2} y'_{n-1} t'_{n-2} \cdots t'_1.$$

(here  $i$  runs from 1 to  $n-2$ ). Let us now move  $x'_{n-1}$  and  $y'_{n-1}$  to the left. Then we get

$$(3.15) \quad T_{H_1, \dots, H_n}(x_n, y_n) = \frac{|H_n||H_{n-1}|}{|G|} \sum_{z_i, t_i, z'_i, t'_i, x_{n-1}, y_{n-1}} e^{\sum_i \langle z_i, t_i \rangle + \sum_i \langle z'_i, t'_i \rangle + \langle \rho_{n-2}(z_{n-2}^{-1} \cdots z_1^{-1}) x_{n-1}, \rho_{n-2}(t_{n-2}^{-1} \cdots t_1^{-1}) y_{n-1}^{1/2} \rangle} \\ \times e^{\langle \rho_{n-1}(z_{n-2}^{-1} \cdots z_1^{-1}) \rho_{n-1}(x_{n-1}^{-1/2}) x_n, \rho_{n-1}(t_{n-2}^{-1} \cdots t_1^{-1}) \rho_{n-1}(y_{n-1}^{-1/2}) y_n^{1/2} \rangle} \\ \times x_{n-1} z_1 \cdots z_{n-2} z'_{n-2} \cdots z'_1 \otimes y_{n-1} t_1 \cdots t_{n-2} t'_{n-2} \cdots t'_1.$$

Thus,

$$(3.16) \quad T_{H_1, \dots, H_{n-1}, H_n}(x_n, y_n) = \sum_{x_{n-1}, y_{n-1} \in H_{n-1}} (x_{n-1} \otimes y_{n-1}) \\ \times T_{H_1, \dots, H_{n-2}, H_{n-1} \times H_n}(x_{n-1} \rho_{n-1}(x_{n-1}^{-1/2}) x_n, y_{n-1} \rho_{n-1}(y_{n-1}^{-1/2}) y_n).$$

Now we can easily prove (3.13) by induction on  $n$ . The base of induction ( $n = 1$ ) is obvious, and the induction step follows directly from (3.16). This concludes the proof of the theorem.  $\square$

**Example 3.3.** Let  $H_1 = H_2 = \mathbb{Z}_p \times \mathbb{Z}_p$ , where  $p > 2$  is prime. Let  $J_1 = J_2$  be the twists for  $\mathbb{Z}_p \times \mathbb{Z}_p$  arising from the symplectic form  $\langle (j, k), (l, m) \rangle = \frac{2\pi i}{p}(jm - kl)$ . Let  $\mathbb{Z}_p \times \mathbb{Z}_p$  act on itself via  $(j, k) \cdot (l, m) = (l + jm, m)$ . Then by Theorem 3.1,  $((\mathbb{Z}_p \times \mathbb{Z}_p) \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p), \bar{J}_{21}^{-1} \bar{J})$ , where  $\bar{J} = J_2 J_1$ , is a  $p^4$ -dimensional semisimple minimal triangular Hopf algebra.

#### 4. The double of a bijective 1-cocycle and minimal triangular Hopf algebras

In fact, the method of Proposition 2.2 leads to more examples of triangular structures than described in Sections 2 and 3. Namely, given a quadruple  $(G, A, \rho, \pi)$  as in Section 2, such that  $A$  is abelian, define  $\tilde{G} = G \ltimes A^*$ ,  $\tilde{A} = A \times A^*$ ,  $\tilde{\rho} : \tilde{G} \rightarrow \text{Aut}(\tilde{A})$  by  $\tilde{\rho}(g) = \rho(g) \times \rho^*(g)^{-1}$ , and  $\tilde{\pi} : \tilde{G} \rightarrow \tilde{A}$  by  $\tilde{\pi}(a^*g) = \pi(g)a^*$ ,



$a^* \in A^*$ ,  $g \in G$ . It is straightforward to check that  $\tilde{\pi}$  is a bijective 1-cocycle. We call the quadruple  $(\tilde{G}, \tilde{A}, \tilde{\rho}, \tilde{\pi})$  the *double* of  $(G, A, \rho, \pi)$ .

Consider  $J \in \mathbb{C}[\tilde{A}] \otimes \mathbb{C}[\tilde{A}]$  given by

$$(4.1) \quad J = |A|^{-1} \sum_{x \in A, y^* \in A^*} e^{(x, y^*)} x \otimes y^*.$$

It is straightforward to check that  $J$  is a twist, and that it is  $G$ -invariant. This allows to construct the corresponding element

$$(4.2) \quad \bar{J} = |A|^{-1} \sum e^{(x, y^*)} \pi^{-1}(x) \otimes y^*.$$

**Proposition 4.1.**  $\bar{J}$  is invertible, and

$$(4.3) \quad \bar{J}^{-1} = |A|^{-1} \sum_{z \in A, t^* \in A^*} e^{-(z, t^*)} \pi^{-1}(T(z)) \otimes t^*,$$

where  $T : A \rightarrow A$  is a bijective map (not a homomorphism, in general) defined by  $\pi^{-1}(x^{-1})\pi^{-1}(T(x)) = 1$ .

*Proof.* Denote the right hand side of (4.3) by  $J'$ . We need to check that  $J' = \bar{J}^{-1}$ . It is enough to check it after evaluating any  $\alpha \in A$  on the second component of both sides. We have

$$(4.4) \quad (1 \otimes \alpha)(\bar{J}) = |A|^{-1} \sum_{x, y^*} e^{(x\alpha, y^*)} \pi^{-1}(x) = \pi^{-1}(\alpha^{-1}),$$

$$(1 \otimes \alpha)(J') = |A|^{-1} \sum_{x, y^*} e^{-(x\alpha^{-1}, y^*)} \pi^{-1}(T(x)) = \pi^{-1}(T(\alpha)).$$

This concludes the proof of the proposition.  $\square$

We can now prove:

**Theorem 4.2.** Let  $\bar{J}$  be as in (4.2). Then  $\bar{J}$  is a twist for  $\mathbb{C}[\tilde{G}]$ , and it gives rise to a minimal triangular Hopf algebra  $\mathbb{C}[\tilde{G}]^{\bar{J}}$ , with universal  $R$ -matrix

$$(4.5) \quad R = |A|^{-2} \sum_{x, y \in A, x^*, y^* \in A^*} e^{(x, y^*) - (y, x^*)} x^* \pi^{-1}(x) \otimes \pi^{-1}(T(y)) y^*.$$

*Proof.* The first two statements follow directly from Propositions 2.2 and 4.1. The minimality follows from the fact that  $\{x^* \pi^{-1}(x) | x^* \in A^*, x \in A\}$  and  $\{\pi^{-1}(T(y)) y^* | y \in A, y^* \in A^*\}$  are bases of  $\mathbb{C}[\tilde{G}]$ , and the fact that the matrix  $c_{xx^*, yy^*} = e^{(x, y^*) - (y, x^*)}$  is invertible (because it is proportional to the matrix of Fourier transform on  $A \times A^*$ ).  $\square$

Thus, every bijective 1-cocycle  $\pi : G \rightarrow A$  gives rise to a minimal triangular structure on  $\mathbb{C}[G \ltimes A^*]$ . So it remains to construct a supply of bijective 1-cocycles. This was done in [ESS]. The theory of bijective 1-cocycles was developed in [ESS], because it was found that they correspond to set-theoretical solutions of the quantum Yang-Baxter equation. In particular, many constructions of these 1-cocycles were found. We refer the reader to [ESS] for further detail.

**Example 4.3.** Let  $G = S_3$  be the permutation group of three letters, and  $A = \mathbb{Z}_2 \times \mathbb{Z}_3$ . Define an action of  $G$  on  $A$  by  $s(a, b) = (a, (-1)^{\text{sign}(s)}b)$  for  $s \in G$ ,  $a \in \mathbb{Z}_2$  and  $b \in \mathbb{Z}_3$ . Define a bijective 1-cocycle  $\pi = (\pi_1, \pi_2) : G \rightarrow A$  as follows:  $\pi_1(s) = 0$  if  $s$  is even and  $\pi_1(s) = 1$  if  $s$  is odd, and  $\pi_2(id) = 0$ ,  $\pi_2((123)) = 1$ ,  $\pi_2((132)) = 2$ ,  $\pi_2((12)) = 2$ ,  $\pi_2((13)) = 0$  and  $\pi_2((23)) = 1$ . Then by Theorem 4.2,  $\mathbb{C}[\tilde{G}]^{\bar{J}}$  is a non-trivial minimal triangular Hopf algebra of dimension 36. This is in fact a special case of the construction in Proposition 2.4.

*Remark 4.4.* Note that in [ESS], the 1-cocycle equation is  $\pi(gg') = \pi(g')(g')^{-1}\pi(g)$ , rather than (2.1). However, these equations are equivalent and related by the transformation  $g \rightarrow g^{-1}$ .

*Remark 4.5.* The Lie-theoretical analogue of the theory of bijective 1-cocycles for finite groups is the theory of left invariant affine space structures on Lie groups, which has been intensively discussed in the literature (see [Bu] and references therein). We note that many methods of constructing such structures on nilpotent Lie groups (which are based on Lie algebra theory and the Campbell-Hausdorff formula) work over a finite field and therefore apply to p-groups.

In conclusion let us formulate some questions which are inspired by the above results.

**Question 4.6.** 1. In the situation of Proposition 2.2, is  $\bar{J}$  always invertible? If yes and if  $J$  is minimal, is  $\bar{J}$  minimal?

2. In the situation of Proposition 2.4, assume that the twists  $J_1, \dots, J_n$  are minimal. Does this imply that  $\bar{J} = J_n \cdots J_1$  is minimal?

3. If (in the setting of Section 1) the Hopf algebra  $\mathbb{C}[G]^J$  is minimal triangular, does  $G$  have to be solvable (cf. Remark 4.7.)?

*Remark 4.7.* It was shown in [ESS] that if a bijective 1-cocycle on  $G$  exists then  $G$  has to be solvable. However, this theorem relies on a non-trivial theorem of P. Hall in finite group theory, saying that if  $G$  with  $|G| = \prod p_i^{n_i}$  has a subgroup of index  $p_i^{n_i}$  for any  $i$  then  $G$  is solvable.

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