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C^∞-REGULARITY OF THE INTERFACE OF THE EVOLUTION P-LAPLACIAN EQUATION

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1. Introduction

We consider, in this paper, the Cauchy problem for the *evolution p-Laplacian equation*

$$\begin{cases} u_t = \operatorname{div}(|Du|^{p-2}\, Du) & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ u(x,0) = u^0 & x \in \mathbb{R}^n, \end{cases}$$

in the range of exponents $p > 2$, with initial data u^0 non-negative, integrable and compactly supported. It is well known, [D], that the above problem admits a unique non-negative weak solution $u \in C([0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times [\tau,\infty))$, for any $\tau > 0$. Moreover, $u \in C^{1,\alpha}(\mathbb{R}^n)$, for some $\alpha > 0$, and u is smooth on the set where $|Du| \neq 0$. However, the problem becomes degenerate when $Du = 0$, therefore solutions are not always smooth. This can be demonstrated by explicit examples, such as the self-similar Barenblatt solution

$$u(x,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}},$$

defined for $t > 0$ and $p > 2$, with

$$\lambda = \frac{1}{n(p-2)+p},$$

and

$$\gamma_p = \left(\frac{1}{\lambda} \right)^{\frac{1}{p-1}} \frac{p-2}{p}.$$

In this example the solution becomes singular at the boundary of its support, where the gradient of B becomes zero. Because the equation is degenerate, the interface

$$\partial \overline{\{(x,t) : \ u(x,t) > 0\}},$$

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behaves like a free-boundary propagating with finite speed. Other examples with similar behavior at the free-boundary can be easily computed, such as the translating solutions

$$u(x,t) = \left[\frac{p-2}{p-1}(\beta + t - |x|)_+\right]^{\frac{p-1}{p-2}},$$

and

$$u(x,t) = \left[\frac{p-2}{p-1}(|x| - t - \beta)_+\right]^{\frac{p-1}{p-2}},$$

with β a positive constant.

Notice that in all of the above examples, if one substitutes

$$u = \frac{p-2}{p-1} f^{\frac{p-1}{p-2}},$$

then the function f becomes smooth up to the free-boundary $u = 0$, while u does not. Also, in the above examples $|Df| \neq 0$ at the interface. This indicates that to investigate the regularity of the interface, one may have to study the regularity of the function f instead of the regularity of u itself. The function f evolves as

$$f_t = f\,\Delta_p f + \frac{p-1}{p-2}\,|Df|^p,$$

an equation which becomes degenerate when $f = 0$ and not when $Df = 0$. However, one must assume that $|Df| \neq 0$ at time $t = 0$ at the interface, to guarantee that the free-boundary begins to move at every point at time $t > 0$. This is a necessary assumption in order to gain regularity.

Our goal is to show that, under certain assumptions on the initial data, the function f, defined as above in terms of u, is smooth up to the interface, for time $0 < t \leq T$, for some $T > 0$. As a consequence, the free-boundary is smooth.

Our approach is similar to the one we used in [DH] for the regularity of the interface of the porous medium equation

$$u_t = \Delta u^m, \qquad m > 1,$$

which becomes degenerate when $u = 0$ and the free-boundary $\partial\overline{\{u > 0\}}$ propagates with finite speed. We have shown in [DH] that, under certain assumptions in the initial data, the pressure $f = m\,u^{m-1}$ is smooth up to the interface and also that the interface is smooth. It should be mentioned that the C^∞-regularity and analyticity of the free boundary of the porous medium equation in dimension $n = 1$ has been shown in [AV] and [A] respectively. In dimensions $n \geq 2$, the C^α, Lipschitz and $C^{1,\alpha}$ regularity of the interface of the porous medium equation, has been shown in [CF], [CVW] and [CW] respectively. Finally, the Lipschitz and $C^{1,\alpha}$ regularity of the interface for the evolution p-Laplacian equation has been shown in [ZY] and [K] respectively.

To simplify the computations we will present the problem in spatial dimension $n = 2$. However, all the results in this paper can be generalized to any dimension $n \geq 3$.

We begin by determining an appropriate class of initial data. We assume that Ω_1 and Ω_2 are two domains in $\mathbb{R}^2$ such that

$$\Omega_1 \subset\subset \Omega_2,$$

so that the domain $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ is non-empty. We also assume that u^0 is a non-negative function on Ω such that $0 < u^0 < 1$ and $|Du^0| > 0$ in Ω with $u_0 = 0$ at $\partial\Omega_2$, and $u_0 = 1$ at $\partial\Omega_1$. The assumption that $|Du^0| > 0$ in the interior of the domain Ω is necessary for the equation to become degenerate only at the boundary of Ω. The corresponding condition for the porous medium equation was that $u^0 > 0$ in the interior of Ω. Notice that if $u^0 \equiv 0$ at $\partial\Omega$, then there must be at least a point in the interior of Ω where $|Du^0| = 0$. This explains our assumption that $u^0 = 1$ at $\partial\Omega_1$. However, if $|Du^0| = 0$ at $\partial\Omega_1$, then at time $t > 0$, the front $u = 1$ will behave like a free-boundary propagating with finite speed. We will, therefore, consider the free-boundary problem

$$\begin{cases} u_t = \operatorname{div}(|Du|^{p-2}\, Du) & x \in \Omega(t),\ 0 < t \leq T, \\ u(x,0) = u^0 & x \in \Omega, \end{cases} \tag{1.1}$$

in the range of exponents $p > 2$, where at each time $t \in (0, T]$, $\Omega(t)$ denotes the set

$$\Omega(t) = \Omega_2(t) \setminus \overline{\Omega_1(t)},$$

where $0 < u < 1$ and $|Du| > 0$, while

$$u(\cdot, t) = 0 \text{ at } \partial\Omega_2(t), \quad \text{and} \quad u(\cdot, t) = 1 \text{ on } \partial\Omega_1(t).$$

We denote by Δ_p the operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}\, Du).$$

Setting

$$u = \frac{p-2}{p-1}\, f^{\frac{p-1}{p-2}}$$

near the free boundary $u = 0$, we have already noticed, that the function f satisfies the equation

$$f_t = f\, \Delta_p f + \frac{p-1}{p-2}\, |Df|^p$$

and this equation becomes degenerate when $u = f = 0$. We will assume that

$$|Df| \neq 0, \qquad \text{at } \partial\Omega_2,$$

at time $t = 0$, condition which is necessary to guarantee that the free-boundary $f = 0$ will start to move at every point at $t > 0$. Near the free-boundary $u = 1$, we set

$$1 - u = \frac{p-2}{p-1}\, g^{\frac{p-1}{p-2}}.$$

Then g satisfies the equation

$$g_t = g\,\Delta_p g - \frac{p-1}{p-2}\,|Dg|^p.$$

We also assume that at time $t = 0$

$$|Dg| \neq 0, \qquad \text{at } \partial\Omega_1,$$

so that the free-boundary $u = 1$ ($g = 0$) will start to move at every point at time $t > 0$. We can combine both cases together, by setting

$$u = \alpha_p \int_0^f s^{\frac{1}{p-2}}\,(1-s)^{\frac{1}{p-2}}\,ds, \tag{1.2}$$

with

$$\alpha_p = \left(\int_0^1 s^{\frac{1}{p-2}}\,(1-s)^{\frac{1}{p-2}}\,ds\right)^{-1}.$$

Notice that $0 \leq f \leq 1$ with $f = 0$ iff $u = 0$ and $f = 1$ iff $u = 1$. The evolution of the function f defined by (1.2) is governed by the equation

$$f_t = \alpha_p^{p-2}\, f\,(1-f)\,\Delta_p f + \alpha_p^{p-2}\,\frac{p-1}{p-2}(1-2f)\,|Df|^p,$$

which becomes degenerate whenever $f = 0$ or $f = 1$. To simplify the notation we denote by β_p the constant

$$\beta_p = \alpha_p^{p-2}\,\frac{p-1}{p-2}.$$

Then, the problem (1.1) becomes equivalent to the free-boundary problem

$$\begin{cases} f_t = \alpha_p^{p-2} f\,(1-f)\,\Delta_p f + \beta_p(1-2f)\,|Df|^p & (x,t) \in Q_T, \\ f(x,0) = f^0 & x \in \Omega, \end{cases} \tag{1.3}$$

with initial data f^0 defined, in terms of u^0, by

$$u^0 = \alpha_p \int_0^{f^0} s^{\frac{1}{p-2}} (1-s)^{\frac{1}{p-2}} \, ds.$$

In this paper we denote by Q_T the domain

$$Q_T = \cup_{0<t\le T}(\Omega(t) \times \{t\}), \tag{1.4}$$

where $|Du| > 0$ and by Γ_T the interface

$$\Gamma_T = \cup_{0<t\le T}(\partial\Omega(t) \times \{t\}).$$

According to the conditions set above, $0 < f^0 < 1$ in Ω, $f^0 = 0$ on $\partial\Omega_2$, $f^0 = 1$ on $\partial\Omega_1$ and

$$|Df^0| > 0, \qquad \text{on } \overline{\Omega}$$

since $|Du^0| > 0$ in the interior of Ω and $|Df^0| > 0$ at $\partial\Omega$.

Under the above assumptions on f^0, and denoting by d the distance to the boundary of Ω, we will prove the following result:

Main Theorem (C^∞-regularity of the free-boundary). *Assume that the functions f^0, Df^0 and $d\,D^2 f^0$, restricted to the domain $\Omega = \Omega_2 \setminus \overline{\Omega}_1$, extend continuously up to the boundary of Ω, with extensions which are Hölder continuous of class $C^\alpha(\overline{\Omega})$, for some $\alpha > 0$. Assume also that $0 < f^0 < 1$ in Ω, with $f^0 = 0$ on $\partial\Omega_2$, $f^0 = 1$ on $\partial\Omega_1$ and $|Df^0| > 0$ on $\overline{\Omega}$. Then, there exists a number $T > 0$ for which the free-boundary problem* (1.3) *admits a solution f which is smooth up to the interface Γ_T. In particular, the free-boundary Γ_T is a smooth surface.*

Our proof is based on the idea of introducing an appropriate change of variables which transforms the free-boundary problem (1.3) to an *initial value problem* with fixed boundary of the form

$$\begin{cases} Mw = 0 & (u,v,t) \in \mathcal{D} \times [0,T]\,, \\ w(u,v,0) = w^0 & (u,v) \in \mathcal{D}, \end{cases} \tag{1.5}$$

on the parabolic cylinder $\mathcal{D} \times [0,T]$, where $\mathcal{D} = \{\, (x,y);\ 1 < x^2 + y^2 < 2 \,\}$ and

$$Mw = w_t - F(t,u,v,w,Dw,D^2 w),$$

is a quasilinear operator which becomes degenerate at $\partial\mathcal{D}$. To show that the problem (1.5) admits a solution, we apply the Inverse Function Theorem between, appropriately defined, Banach spaces. We will show in Section 3 that the linerization of the operator M at a point $\bar{w}$ sufficiently close to the initial data w^0, resembles the model degenerate equation

$$f_t = x\,(f_{xx} + f_{yy}) + \nu\, f_x, \tag{1.6}$$

with $\nu > 0$, on the half space $x \geq 0$ and no boundary conditions on f along the boundary $x = 0$. It is a simple observation that this equation admits solutions which behave like $x^{1-\nu}$ near the boundary $x = 0$. However, once you require that the initial data f^0 has f^0, Df^0 and $x\,D^2 f^0$ in C^α, for some $\alpha > 0$, solutions of (1.6) become smooth up to the boundary $x = 0$, for $t > 0$.

Equation (1.6) has been studied by the authors in [DH], Part I. The results from [DH] which will be used here are summarized, for the convenience of the reader, in the next section.

2. Preliminary results.

The diffusion in equation (1.6) is governed by the Riemannian metric ds where

$$ds^2 = \frac{dx^2 + dy^2}{2x}\ .$$

We call this the *cycloidal metric* because its geodesics are cycloid curves. It has been computed in [DH] that the distance between two points $\binom{x_1}{y_1}$ and $\binom{x_2}{y_2}$ in this metric is a function $s\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right]$ which is equivalent to the function

$$\overline{s}\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] = \frac{|x_1 - x_2| + |y_1 - y_2|}{\sqrt{x_1} + \sqrt{x_2} + \sqrt{|y_1 - y_2|}}$$

in the sense that $s \leq C\,\overline{s}$ and $\overline{s} \leq C\,s$, for some constant C. For the parabolic problem we use the parabolic distance

$$s\left[\begin{pmatrix} x_1 \\ y_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ t_2 \end{pmatrix}\right] = s\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] + \sqrt{|t_1 - t_2|}\ .$$

In terms of this distance we can define the Banach space $\mathcal{C}_s^\alpha(\mathcal{A})$ of Hölder continuous functions on $\mathcal{A}$ with respect to the metric s and the norm $||\cdot||_{C_s^\alpha(\mathcal{A})}$, as usual. Next, suppose the set $\mathcal{A}$ is the closure of its interior, and the function f on $\mathcal{A}$ has continuous derivatives $f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ in the interior of $\mathcal{A}$, and that f_t, f_x, f_y and $xf_{xx}, xf_{xy}, xf_{yy}$ extend continuously to the boundary, and the extensions are Hölder continuous on $\mathcal{A}$ of class $\mathcal{C}_s^\alpha(\mathcal{A})$ as before. Let $\mathcal{C}_s^{2+\alpha}(\mathcal{A})$ be the Banach space of all such functions with norm

$$\begin{aligned}\|f\|_{\mathcal{C}_s^{2+\alpha}(\mathcal{A})} &= \|f\|_{C_s^\alpha(\mathcal{A})} + \|f_x\|_{C_s^\alpha(\mathcal{A})} + \|f_y\|_{C_s^\alpha(\mathcal{A})} + \|f_t\|_{C_s^\alpha(\mathcal{A})} \\ &\quad + \|xf_{xx}\|_{C_s^\alpha(\mathcal{A})} + \|xf_{xy}\|_{C_s^\alpha(\mathcal{A})} + \|xf_{yy}\|_{C_s^\alpha(\mathcal{A})}\ .\end{aligned}$$

The operator

$$L_0 = f_t - x(f_{xx} + f_{yy}) - \nu f_x,$$

defines a continuous linear map $L_0 : C_s^{2+\alpha}(\mathcal{A}) \to C_s^{\alpha}(\mathcal{A})$. We can extend these definitions to spaces of higher order derivatives. Let k be a positive integer and let $\mathcal{A}$ be a subset of the half space $x \geq 0$ as above. We denote by $C_s^{k,\alpha}(\mathcal{A})$ and $C_s^{k,2+\alpha}(\mathcal{A})$ the spaces of all functions f whose k-th order derivatives $D_x^i D_y^j D_t^l f$, $i+j+l=k$ exist and belong to the space $C_s^{\alpha}(\mathcal{A})$ and $C_s^{2+\alpha}(\mathcal{A})$ respectively. Both spaces, equipped with the norms

$$||f||_{C_s^{k,\alpha}(\mathcal{A})} = \sum_{i+j+l\leq k} ||D_x^i D_y^j D_t^l g||_{C_s^{\alpha}(\mathcal{A})},$$

and

$$||f||_{C_s^{k,2+\alpha}(\mathcal{A})} = \sum_{i+j+l\leq k} ||D_x^i D_y^j D_t^l f||_{C_s^{2+\alpha}(\mathcal{A})},$$

are Banach spaces, and the operator $L_0 : C_s^{k,2+\alpha}(\mathcal{A}) \to C_s^{k,\alpha}(\mathcal{A})$ defines a continuous linear map.

Denoting by $\mathbb{R}^2_+$ the half space $x \geq 0$ in $\mathbb{R}^2$, we can now state the following Existence and Uniqueness result for equation (1.1) proven in [DH], Part I.

2.1. Theorem (Existence and Uniqueness) *Let k be a nonnegative integer and let α be a number in $0 < \alpha < 1$. Assume that $g \in \mathcal{C}_s^{k,\alpha}\left(\mathbb{R}^2_+ \times [0,\infty)\right)$ and $f^0 \in \mathcal{C}_s^{k,2+\alpha}(\mathbb{R}^2_+)$, both g and f^0 compactly supported in $\mathbb{R}^2_+ \times [0,\infty)$ and $\mathbb{R}^2_+$ respectively. Then, for any $\nu > 0$ and $T > 0$, the initial value problem*

$$\begin{cases} L_0 f = g & \text{in } \mathbb{R}^2_+ \times [0,T], \\ f(\cdot,0) = f^0 & \text{on } \mathbb{R}^2_+, \end{cases}$$

admits a unique solution $f \in \mathcal{C}_s^{k,2+\alpha}(\mathbb{R}^2_+ \times [0,T])$. Moreover

$$||f||_{\mathcal{C}_s^{k,2+\alpha}(\mathbb{R}^2_+\times[0,T])} \leq C(T)\left(||f^0||_{\mathcal{C}_s^{k,2+\alpha}(\mathbb{R}^2_+)} + ||g||_{\mathcal{C}_s^{k,\alpha}(\mathbb{R}^2_+\times[0,T])}\right),$$

for some constant $C(T)$ depending only α, k, ν and T.

Define the box of side r around a point $P = \begin{pmatrix} x_0 \\ y_0 \\ t_0 \end{pmatrix}$ to be

$$\mathcal{B}_r(P) = \{\, x \geq 0,\ |x-x_0| \leq r,\ |y-y_0| \leq r,\ t_0 - r \leq t \leq t_0 \,\}\ .$$

We let $\mathcal{B}_r$ be the box around the point $P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The basic tool in the proof of Theorem 2.1 is a Schauder type estimate, weighted according to the cycloidal metric s. It is proven in [DH], Part I, and it states as follows:

2.2. Theorem (Schauder Estimate) *Let k be a nonnegative integer and let $0<\alpha<1$ and $\nu>0$. Then, for any $r<1$ there exists a constant C depending on k, α, ν and r so that*

$$\|f\|_{C_s^{k,2+\alpha}(\mathcal{B}_r)} \le C\left(\|f\|_{C_s^{\circ}(\mathcal{B}_1)} + \|L_0 f\|_{C_s^{k,\alpha}(\mathcal{B}_1)}\right),$$

for all C^∞- smooth functions f on $\mathcal{B}_1$.

For our purposes, we need to generalize the above results on quasilinear operators of the form

$$Mw = w_t - \theta\, F^{ij}(t,x,y,w,Dw)\, w_{ij} - G(t,x,y,w,Dw),$$

defined on $R_T = \mathcal{D}\times[0,T]$, where $\mathcal{D}$ is a compact domain on $\mathbb{R}^2$, and with θ a smooth function such that

$$\theta \sim \mathrm{dist}(P,\partial\mathcal{D})$$

near the boundary of $\mathcal{D}$. This has been done in detail in [DH], Part II. To help the reader we will state again the main result. The *cycloidal distance* in $\mathcal{D}$ is equivalent to the function

$$\bar{s}(P_1,P_2) = \frac{|P_1-P_2|}{\sqrt{|P_1-P_2|}+\sqrt{d(P_1)}+\sqrt{d(P_2)}}$$

with $d=d(P)$ denoting the distance to the boundary of $\mathcal{D}$. For any $\mathcal{A}$ subset of $\mathcal{D}\times[0,T]$, the spaces $C_s^\alpha(\mathcal{A})$, $C_s^{2+\alpha}(\mathcal{A})$, $C_s^{k,\alpha}(\mathcal{A})$ and $C_s^{k,2+\alpha}(\mathcal{A})$ are defined similarly as above. Notice that this time, the norm in $C_s^{2+\alpha}(\mathcal{A})$ is defined by

$$||w||_{C_s^{2+\alpha}(\mathcal{A})} = ||w||_{C_s^\alpha(\mathcal{A})} + ||w_i||_{C_s^\alpha(\mathcal{A})} + ||d\, w_{ij}||_{C_s^\alpha(\mathcal{A})},$$

where d denotes the distance to the boundary and the summation convention is used. The linearization $DM(\bar{w})$ of the operator M at a point $\bar{w}$ will be assumed to be an operator of the form

$$Lw = w_t - (\theta a^{ij} w_{ij} + b^i\, w_i + c\, w),$$

defined on R_T, where θ is a smooth function on $\mathcal{D}$, strictly positive in its interior, with $||\theta||_{C^\infty(\mathcal{D})}\le 1$ and such that

$$\theta(P) = \mathrm{dist}(P,\partial\mathcal{D}), \qquad \text{if } \mathrm{dist}(P,\partial\mathcal{D})\le\sigma,$$

for some $\sigma>0$. The coefficients a^{ij}, b^i and c will be assumed to belong to the space $C_s^{k,\alpha}(\overline{R_T})$, for some $\alpha>0$ and k non-negative integer, and satisfy the ellipticity condition

$$a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2 > 0, \qquad \forall \xi\in\mathbb{R}^2\setminus\{0\},$$

and the bounds

$$||a^{ij}||_{C_s^{k,\alpha}(\overline{R_T})}, \qquad ||b^i||_{C_s^{k,\alpha}(\overline{R_T})}, \qquad ||c||_{C_s^{k,\alpha}(\overline{R_T})} \le 1/\lambda,$$

and

$$b^i n_i \ge \nu > 0 \quad \text{on} \quad \partial\mathcal{D} \times [0,T], \tag{2.1}$$

for some positive constants λ and ν, where $n = (n_1, n_2)$ denotes the interior normal to $\partial\mathcal{D}$. We will say that the operator L belongs to the class $\mathcal{F}(k, \alpha, \nu, \lambda, \sigma)$ if it satisfies all of the above hypotheses.

The following result from [DH], Part II, will be used for the proof of the Main Theorem in this paper.

2.3. Theorem. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^2$ with smooth boundary and let k be a nonnegative integer, and $0 < \alpha < 1$, $T > 0$ positive numbers. Also, let w^0 be a function in $C_s^{k,2+\alpha}(\overline{\mathcal{D}})$. Assume that the linearization $DM(\bar{w})$ of the quasilinear operator*

$$Mw = w_t - \theta\, F^{ij}(t,x,y,w,Dw)\, w_{ij} - G(t,x,y,w,Dw)$$

defined on $R_T = \mathcal{D} \times [0,T]$, belongs to the class $\mathcal{F}(k,\alpha,\nu,\lambda,\sigma)$, at all points $\bar{w} \in C_s^{k,2+\alpha}(\overline{R}_T)$, such that $||\bar{w} - w^0||_{C_s^{k,2+\alpha}(\overline{\mathcal{D}})} \le \mu$, $\mu > 0$. Then, there exists a number τ_0 in $0 < \tau_0 \le T$ depending on the constants α, k, λ, ν and μ, for which the initial value problem

$$\begin{cases} w_t = \theta\, F^{ij}(t,x,y,w,Dw)\, w_{ij} + G(t,x,y,w,Dw) & \text{in } \mathcal{D} \times [0,\tau_0], \\ w(\cdot,0) = w^0 & \text{on } \mathcal{D}, \end{cases}$$

admits a solution w in the space $C_s^{k,2+\alpha}(\overline{\mathcal{D}} \times [0,\tau_0])$. Moreover,

$$||w||_{C_s^{k,2+\alpha}(\overline{\mathcal{D}}\times[0,\tau_0])} \le C\, ||w^0||_{C_s^{k,2+\alpha}(\overline{\mathcal{D}})},$$

for some positive constant C which depends only on α, k, λ, ν and σ.

We conclude this Section with the generalization of the local Schauder estimate in Theorem 2.2 for variable coefficient equations. This result will be used in the proof of Theorem 4.3 in Section 4. It is sufficient to consider operators L which are defined on the half space $x \ge 0$ and have the form

$$Lw = w_t - (x\, a^{ij} w_{ij} + b^i\, w_i + c\, w).$$

As before, we denote by $\mathcal{B}_r$ the box of radius r around the point $P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

2.4. Theorem (Schauder Estimate) *Assume that the coefficients* a^{ij}, b^i *and* c *of the operator* L *belong to the space* $C_s^\alpha(\mathcal{B}_1)$, *for some number* α *in* $0<\alpha<1$ *and satisfy*

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0, \qquad \forall \xi \in \mathbb{R}^2 \setminus \{0\},$$

and

$$||a^{ij}||_{C_s^\alpha(\mathcal{B}_1)}, \qquad ||b^i||_{C_s^\alpha(\mathcal{B}_1)}, \qquad ||c||_{C_s^\alpha(\mathcal{B}_1)} \leq 1/\lambda,$$

and

$$b^1 \geq \nu > 0, \qquad \text{at} \;\; x=0,$$

for some positive constants λ *and* ν. *Then, for any* $r<1$, *there exists a constant* C *depending only on* α, λ, ν *and* r, *such that*

$$\|f\|_{C_s^{2+\alpha}(\mathcal{B}_r)} \leq C\left(\|f\|_{C_s^\circ(\mathcal{B}_1)} + \|Lf\|_{C_s^\alpha(\mathcal{B}_1)}\right),$$

for all functions $f \in C_s^{2+\alpha}(\mathcal{B}_1)$.

3. Local coordinate change.

To motivate the proof of the Main Theorem, we will compute in this Section the transformation of the equation

$$f_t = \alpha_p^{p-2}\, f(1-f)\,\Delta_p f + \beta_p(1-2f)\,|Df|^p, \tag{3.1}$$

when one exchanges dependent and independent variables near the free-boundary $z=0$ or $z=1$. Let $P_0=(x_0,y_0,t_0)$ be a point at the free boundary $z=0$ or $z=1$. We can assume (by rotating the coordinates) that, at the point P_0

$$f_x(P_0)>0, \quad \text{and} \quad f_y(P_0)=0.$$

Hence, we can solve around the point P_0 the equation $z=f(x,y,t)$ with respect to x, yielding to a map

$$x = h(z,y,t),$$

defined on a sufficiently small box

$$\mathcal{B}_\eta = \{0 \leq z \leq 1, \quad |z-z_0| \leq \eta \; |y-y_0| \leq \eta, \quad -\eta \leq t-t_0 \leq 0\},$$

around the point $Q_0=(z_0,y_0,t_0)$, where $z_0=0$ or $z_0=1$. Equation (3.1) can also expressed in the form

$$\begin{aligned} f_t = \alpha_p^{p-2}\, f(1-f)\,(f_x^2+f_y^2)^{\frac{p-4}{2}} \\ \left\{\left((p-1)f_x^2+f_y^2\right)\, f_{xx} + 2(p-2)\, f_x f_y f_{xy} + \left(f_x^2+(p-1)f_y^2\right)\, f_{yy}\right\} \\ + \beta_p(1-2f)\,(f_x^2+f_y^2)^{\frac{p}{2}}. \end{aligned}$$

To compute the evolution of the function $x = h(z, y, t)$ we use the identities

$$f_x = \frac{1}{h_z}, \qquad f_t = -\frac{h_t}{h_z}, \qquad f_y = -\frac{h_y}{h_z},$$

and

$$\begin{aligned} f_{xx} &= -\frac{1}{h_z^3}\, h_{zz}, \\ f_{xy} &= -\frac{1}{h_z^2}\, h_{zy} + \frac{h_y}{h_z^3}\, h_{zz}, \\ f_{yy} &= -\frac{1}{h_z}\left(\frac{h_y^2}{h_z^2}\, h_{zz} - 2\frac{h_y}{h_z}\, h_{zy} + h_{yy} \right). \end{aligned}$$

After several computations, we conclude that h satisfies the equation

$$M(h) = 0,$$

with M given by

$$\begin{aligned} M(h) = h_t - \Bigg\{ &\beta_p\, z\,(1-z)\, \frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}} \\ &\left(\frac{(1+h_y^2)^2}{h_z^2}\, h_{zz} - \frac{2(1+h_y^2)}{h_z}\, h_{zy} + \left((p-1)^{-1} + h_y^2\right)\, h_{yy} \right) \\ &+\beta_p\,(2z-1)\, \frac{(1+h_y^2)^{\frac{p}{2}}}{h_z^{p-1}} \Bigg\}, \end{aligned}$$

where we remind that

$$\beta_p = \alpha_p^{p-2}\, \frac{p-1}{p-2}.$$

The above operator is quasilinear parabolic and becomes degenerate at $z = 0$ or $z = 1$.

We wish to show that the operator M satisfies the hypotheses of Theorem 2.3. For this reason we linearize M around the point h. The reader can easily check, that if $h \in C_s^{2+\alpha}(\mathcal{B}_\eta)$, with η sufficiently small, then the linearization $DM(h)$ of M at h takes the form

$$\begin{aligned} DM(h)(\tilde h) = \tilde h_t - \Bigg\{ &\beta_p\, z\,(1-z)\, \frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}} \\ &\left(\frac{(1+h_y^2)^2}{h_z^2}\, \tilde h_{zz} - \frac{2(1+h_y^2)}{h_z}\, \tilde h_{zy} + ((p-1)^{-1} + h_y^2)\, \tilde h_{yy} \right) \\ &+b^1(z,y,h,Dh,D^2h)\, \tilde h_z + b^2(z,y,Dh,D^2h)\, \tilde h_y \Bigg\}, \end{aligned}$$

where all the coefficients belong to $C_s^{\alpha}(\mathcal{B}_\eta)$, since $h \in C_s^{2+\alpha}(\mathcal{B}_\eta)$. The matrix

$$\frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}} \begin{pmatrix} \frac{(1+h_y^2)^2}{h_z^2} & \frac{1+h_y^2}{h_z} \\ \frac{1+h_y^2}{h_z} & ((p-1)^{-1}+h_y^2) \end{pmatrix},$$

is strictly positive in the box $\mathcal{B}_\eta$, if η is sufficiently small, since at the point Q_0, $h_z > 0$ and $h_y = 0$. Moreover, the computations show that

$$b^1(z, y, h, Dh, D^2h) = (p-1)\beta_p\,(1-2z)\,\frac{(1+h_y^2)^{\frac{p}{2}}}{h_z^p} + \mathcal{O}(z(1-z)),$$

where the function $\mathcal{O}$, which depends also on $||h||_{C_s^{\alpha+2}(\mathcal{B}_\eta)}$, satisfies

$$\mathcal{O}(z(1-z)) = 0 \qquad \text{at } \ z=0 \ \text{ and } \ z=1.$$

Therefore, $b^1 > 0$ at $z = 0$ and $b^1 < 0$ at $z = 1$, conditions which are equivalent to assumption (2.1) of Theorem 2.3. It follows above discussion that the operator M satisfies the hypotheses of Theorem 2.3. However, the change of coordinates presented here is only local and can't be used directly for the proof of the Main Theorem stated in the Introduction. For this reason a more subtle, global change of coordinates will be introduced in Section 4. A similar coordinate change has been used in [DH]. The detailed computations will be omitted, since they can be easily adapted from the proofs of Section III.3 in [DH].

4. C^∞-regularity of the free-boundary.

Let $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ be a domain in $\mathbb{R}^2$ and f^0 a function in $C_s^{k,\alpha}(\overline{\Omega})$ which satisfies the assumptions of the Main Theorem, stated in the Introduction. Then, $0 < f^0 < 1$ in Ω, $f^0 = 0$ at $\partial\Omega_2$, $f^0 = 1$ at $\partial\Omega_1$ and

$$|D\, f^0(x,y)| \geq c > 0, \qquad \forall (x,y) \in \overline{\Omega}, \tag{4.1}$$

for some fixed positive number c. Denote, as before, by $\mathcal{D}$ the region

$$\mathcal{D} = \{(u,v) \in \mathbb{R}^2 : 1 < u^2+v^2 < 2\,\} = \mathcal{D}_2 \setminus \overline{\mathcal{D}}_1,$$

with $D_i = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < i\,\}$, $i = 1,2$. Pick a smooth surface $\mathcal{S}$, sufficiently close to the surface $z = f^0(x,y)$, such that the boundary of $\mathcal{S}$ lies either on the $z = 0$ or on the $z = 1$ plane, and choose $S : \mathcal{D} \to \mathbb{R}^3$ a smooth parametrization for the surface $\mathcal{S}$ which maps $\partial\mathcal{D}_2$ onto $\mathcal{S} \cap \{\, z = 0\,\}$ and $\partial\mathcal{D}_1$ onto $\mathcal{S} \cap \{\, z = 1\,\}$. Also, let

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \end{pmatrix},$$

be a smooth vector field, transverse to the surface $\mathcal{S}$. Since $|Df^0| \geq c$ along $\partial\Omega$ and the surface $\mathcal{S}$ is sufficiently close to the surface $z = f^0(x, y)$, we can choose $\mathcal{T}$ to be parallel to the $z = 0$ and $z = 1$ planes in a small neighborhood of $\partial\mathcal{D}$. In other words, we will assume that there exists a number $\delta > 0$, such that

$$\mathcal{T}_3 \begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad \text{on } \mathcal{D}^\delta,$$

with $\mathcal{D}^\delta = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \geq 1 + \delta \quad \text{or} \quad u^2 + v^2 \leq 2 - \delta\}$. For $\eta > 0$ sufficiently small, we define the change of spatial coordinates $\Phi : \mathcal{D} \times [-\eta, \eta] \to \mathbb{R}^3$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \\ w \end{pmatrix} = S \begin{pmatrix} u \\ v \end{pmatrix} + w\, \mathcal{T} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.2}$$

The map Φ defines x, y and z as functions of the new coordinates u, v and w. Assume that $z = f(x, y, t)$ satisfies the initial free-boundary problem (1.3), i.e. f solves

$$\begin{cases} f_t = \alpha_p^{p-2} f\,(1 - f)\,\Delta_p f + \beta_p(1 - 2f)\,|Df|^p & (x, t) \in Q_T, \\ f(x, 0) = f^0 & x \in \Omega, \end{cases}$$

where $Q_T = \cup_{0<t\leq T}(\Omega(t) \times \{t\})$ with $\Omega(t) = \Omega_2(t) \setminus \overline{\Omega_1(t)}$ and $f(\cdot, t) = 0$ in $\partial\Omega_2(t)$, $f(\cdot, t) = 1$ in $\partial\Omega_1(t)$ while $0 < f(\cdot, t) < 1$ in the interior of $\Omega(t)$. Under the coordinate change Φ, the initial data $f^0(x, y)$ transforms to a function $w^0(u, v)$ which can be made arbitrarily small, by choosing the smooth surface $\mathcal{S}$ sufficiently close to the surface $z = f^0(x, y)$. The reader can check that $w^0 \in C_s^{k,2+\alpha}(\overline{\mathcal{D}})$, since $f \in C_s^{k,2+\alpha}(\overline{\Omega})$.

When z evolves as a function of (x, y, t) then, through this coordinate change, w evolves as a function of (u, v, t) with $(u, v) \in \mathcal{D}$. Moreover, by the choice of the parametrization S, we have $(u, v) \in \partial\mathcal{D}_2$ iff $z = 0$, i.e. $(x, y) \in \partial\Omega_2(t)$, while $(u, v) \in \partial\mathcal{D}_1$ iff $z = 1$, i.e. $(x, y) \in \partial\Omega_1(t)$. In other words, through the coordinate change Φ the free-boundary Γ_T is mapped onto the fixed boundary $\partial\mathcal{D} \times (0, T]$. The evolution of w is described in the following Theorem, where to simplify the notation we use subscripts $i, j \in \{u, v\}$ to denote differentiation with respect to the variables u, v:

4.1. Theorem. *Let Ω and $f^0 \in C_s^{k,2+\alpha}(\overline{\Omega})$, be as above. Then, under the coordinate change Φ the initial free-boundary problem* (1.3) *converts into the initial value problem*

$$\begin{cases} Mw = 0 & (u, v, t) \in \mathcal{D} \times [0, T], \\ w(u, v, 0) = w^0 & (u, v) \in \mathcal{D}, \end{cases}$$

with

$$Mw = w_t - \theta\, F^{ij}(t,u,v,w,Dw)\, w_{ij} - G(t,u,v,w,Dw),$$

and $w^0 \in C_s^{k,2+\alpha}(\overline{\mathcal{D}})$*. Moreover, if* $T \le \tau_k$*, with* τ_k *sufficiently small depending on* c *and* k*, the operator* M *satisfies all the hypotheses of Theorem* 2.3.

Proof. One uses similar computations as in the proof of Theorem III.3.1 in [DH] to transform the free-boundary problem (1.3) into the initial value problem for the quasilinear operator M. Condition (4.1) implies that if the time T is sufficiently small, then the linearization $DM(\bar w)$ of the operator M at a point $\bar w \in C_s^{k,2+\alpha}(\overline{\mathcal{D}}\times[0,T])$ sufficiently close to w^0 satisfies the hypotheses of Theorem 2.3. This has been shown in Section 3 for the local change of coordinates. Here, the computations are much more involved, but following the same basic ideas. The proof will be omitted since it can be easily adapted from the proof of Theorem III.3.1 in [DH]. □

An immediate consequence of Theorems 4.1 and 2.3 is the following, important for our purposes, result:

4.2. Theorem. *Under the same hypotheses as in Theorem 4.1, there exists a number* τ_k*, for which the initial free-boundary problem* (1.3) *admits a solution* f *in* $C_s^{k,2+\alpha}(\overline{Q_{\tau_k}})$*, where* $Q_{\tau_k} = \cup_{0<t\le\tau_k}(\Omega(t)\times\{t\})$.

The proof of the Main Theorem stated in the Introduction will follow from the next regularity result:

4.3. Theorem. *Assume that for some* $T>0$ *and some number* α *in* $0<\alpha<1$, $f \in C_s^{2+\alpha}(\overline{Q_T})$ *is a solution of the free-boundary problem* (1.3) *and satisfies*

$$|Df(x,y,t)| \ge c > 0 \qquad \forall (x,y,t) \in \overline{Q}_T.$$

Then, for any positive integer k, $f \in C_s^{k,2+\alpha}(Q_T)$ *and for any* τ *in* $0<\tau<T$ *we have*

$$||f||_{C_s^{k,2+\alpha}(\overline{Q_{T,\tau}})} \le C_k(\tau, ||f^0||_{C_s^{2+\alpha}(\overline{\Omega})}),$$

where $Q_{\tau,T} = \cup_{\tau\le t\le T}(\Omega(t)\times\{t\})$.

Proof. We will only outline the proof, since the details can be easily adapted from the proof of Theorem III.4.1 in [DH]. In the interior on the domain Q_T, the function f is C^∞-smooth, by the classical theory of non-degenerate parabolic equations. To obtain the desired regularity near the free-boundary we will use the local coordinate change presented in Section 3 combined with the Schauder estimate in Theorem 2.4. We will just outline the proof in the case that $k=1$. The cases where $k>1$ follow by induction. Let $P_0(x_0,y_0,t_0)$ be a point at the free-boundary such that $\tau \le t_0 < \tau_1$, where τ_1 is the number given by Theorem 4.2 and τ a number n $0<\tau<\tau_1$. Lets assume that $f(P_0)=0$ (the proof is similar if $f(P_0)=1$). We can assume with no loss of generality (by rotating the coordinates) that

$$f_x(P_0)>0, \qquad f_y(P_0)=0, \qquad f_t(P_0)=(p-1)\,f_x^p>0.$$

Then, as we showed in Section 3, locally around the point P_0, we can solve the equation $z=f(x,y,t)$ with respect to x yielding to a function $x=h(z,y,t)$ defined on a small box $\mathcal{B}_\eta=\{\,0\le z\le\eta,\ |y-y_0|\le\eta,\ -\eta\le t-t_0\le 0\,\}$. We have computed in Section 3 that h satisfies the equation

$$\begin{aligned}h_t=\beta_p\,z\,(1-z)\,&\frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}}\\&\left(\frac{(1+h_y^2)^2}{h_z^2}\,h_{zz}-\frac{2(1+h_y^2)}{h_z}\,h_{zy}+((p-1)^{-1}+h_y^2)\,h_{yy}\right)\\&\qquad\qquad+\beta_p\,(2z-1)\frac{(1+h_y^2)^{\frac{p}{2}}}{h_z^{p-1}}.\end{aligned}$$

We will show that $h\in C_s^{1,2+\alpha}(\mathcal{B}_{\eta/4})$ and that

$$\|h\|_{C_s^{1,2+\alpha}(\mathcal{B}_{\eta/4})}\le C(\|f^0\|_{C_s^{2+\alpha}(\overline{\Omega})}).$$

We will first show that $h_y\in C_s^{2+\alpha}(\mathcal{B}_{\eta/2})$ and that

$$\|h_y\|_{C_s^{2+\alpha}(\mathcal{B}_{\eta/2})}\le C(\|f^0\|_{C_s^{2+\alpha}(\overline{\Omega})}).$$

To this end, we differentiate the above equation with respect to y in order to compute the evolution of $w=h_y$. After several calculations we conclude that $w=h_y$ satisfies an equation of the form

$$\begin{aligned}h_t=\beta_p\,z\,(1-z)\,&\frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}}\\&\left(\frac{(1+h_y^2)^2}{h_z^2}\,w_{zz}-\frac{2(1+h_y^2)}{h_z}\,w_{zy}+((p-1)^{-1}+h_y^2)\,w_{yy}\right)\\&+b^1(z,h,Dh,D^2h)\,w_z+b^2(z,h,Dh,D^2h)\,w_y+c(z,h,Dh,D^2h),\end{aligned}$$

where all the coefficients belong to the class $C_s^\alpha(\mathcal{B}_\eta)$, since $h\in C_s^{2+\alpha}(\mathcal{B}_\eta)$. Moreover, the coefficient b^1 is of the form

$$b^1=(p-1)\beta_p\,(1-2z)\,\frac{(1+h_y^2)^{\frac{p}{2}}}{h_z^p}+\mathcal{O}\big(z(1-z)\big),$$

where the function $\mathcal{O}$, which depends also on $||h||_{C_s^{2+\alpha}(\mathcal{B}_\eta)}$, satisfies

$$\mathcal{O}(z(1-z)) = 0, \text{ at } z = 0.$$

Hence, by the estimate in Theorem 2.4, we conclude that $h_y \in C_s^{2+\alpha}(\mathcal{B}_{\eta/2})$ and that

$$\|h_y\|_{C_s^{2+\alpha}(\mathcal{B}_{\eta/2})} \le C(||h||_{C_s^{2+\alpha}(\mathcal{B}_\eta)}) \le C(||f^0||_{C_s^{2+\alpha}(\overline{\Omega})}),$$

as desired. The same can be shown for $w = h_t$, via similar calculations. To show, finally, that $w = h_z \in C^{2+\alpha}(\mathcal{B}_{\eta/4})$, one differentiates with respect to z this time. The only difference is that now one obtains an equation of the form

$$\begin{aligned} h_t = {} & \beta_p \, z \, (1-z) \\ & \frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}} \left(\frac{(1+h_y^2)^2}{h_z^2} \, w_{zz} - \frac{2(1+h_y^2)}{h_z} \, w_{zy} + ((p-1)^{-1} + h_y^2) \, w_{yy} \right) \\ & \quad + b^1(z,h,Dh,D^2h) \, w_z + b^2(z,h,Dh,D^2h) \, w_y + c(z,h,Dh,D^2h) \\ & \qquad + \beta_p(1-2z) \, \frac{(1+h_y^2)^{\frac{p-4}{2}}}{h_z^{p-2}} \, ((p-1)^{-1} + h_y^2) \, h_{yy}, \end{aligned}$$

where all the coefficients belong to the class $C_s^\alpha(\mathcal{B}_\eta)$, if $h \in C_s^{2+\alpha}(\mathcal{B}_\eta)$, except of the last term for which we need that actually $h_{yy} \in C_s^\alpha(\mathcal{B}_\eta)$. But it was just shown that $h_y \in C_s^{2+\alpha}(\mathcal{B}_{\eta/2})$. Hence, $h_{yy} \in C_s^\alpha(\mathcal{B}_{\eta/2})$ and therefore, by Theorem 2.4, we finally conclude that $h_z \in C_s^\alpha(\mathcal{B}_{\eta/4})$ with

$$\|h_z\|_{(\mathcal{B}_{\eta/4})} \le C(||h||_{C_s^{2+\alpha}(\mathcal{B}_\eta)}) \le C(||f^0||_{C_s^{2+\alpha}(\overline{\Omega})})$$

as desired. This proves the Theorem in the case $k = 1$. The case $k > 1$, follows in a similar manner, by induction. □

We are finally ready for the proof of the C^∞-regularity of the free-boundary stated in the Introduction.

Proof of the Main Theorem. Observe first that $C^\alpha(\overline{\Omega}) \subset C_s^\alpha(\overline{\Omega})$. Hence, $f^0 \in C_s^{2+\alpha}(\overline{\Omega})$, by assumption. From Theorem 4.2, there exists a solution $f \in C_s^{2+\alpha}(\overline{Q_T})$ of the free-boundary problem (1.3), for some number $T > 0$. Moreover, since $Df^0 \neq 0$ along $\partial\Omega$, we can choose the number T so that

$$|Df(x,y,t)| \ge c > 0 \qquad \forall (x,y,t) \in Q_T$$

for some $c > 0$. But then, it follows from Theorem 4.3 that $f \in C_s^{k,2+\alpha}(\overline{Q_{\tau,T}})$, for any $\tau > 0$, and for all positive integers k, where we remind that we have denoted by $Q_{\tau,T}$ the domain $Q_{\tau,T} = \cup_{\tau \le t \le T}(\Omega(t) \times \{t\})$. We conclude that the function f is C^∞ smooth up to the interface, for $0 < t \le T$. In particular the free-boundary Γ_T is smooth. □

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