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VERTEX OPERATOR ALGEBRAS AND THE BLOWUP FORMULA FOR THE S-DUALITY CONJECTURE OF VAFA AND WITTEN

Wei-ping Li[1] and Zhenbo Qin[2]

1. Introduction

This is the announcement of our work [LQ3] which is the continuation of [LQ1, LQ2] on blowup formulae for the S-duality conjecture of Vafa and Witten. In [V-W], Vafa and Witten formulated some mathematical predictions about the Euler characteristics of instanton moduli spaces derived from the S-duality conjecture in physics. From these mathematical predictions, a blowup formula was proposed based upon the work of Yoshioka [Yos]. Roughly speaking, the blowup formula says that there exists a universal relation between the Euler characteristics of instanton moduli spaces for a smooth four manifold and the Euler characteristics of instanton moduli spaces for the blowup of the manifold. The universal relation is independent of the four manifold and related to some modular forms. Recently, relations between vector bundles over algebraic surfaces (or more generally, torsion-free sheaves) and vertex algebras have been discovered by Nakajima and Grojnowski [Gro, Na1, Na2]. For instance, the vertex algebra in section 9.4 of [Na2] constructed from the homology groups of Hilbert schemes of points on an algebraic surface X is the simple lattice vertex algebra associated to the Néron-Severi group $\mathrm{NS}(X)$ where X is an algebraic surface such that the intersection form on $\mathrm{NS}(X)$ is even. Implicitly the notion of vertex algebras was known to physists working on 2-dimensional conformal field theory [M-S] and to mathematicians working on Monster groups and Moonshine modules [FL1, FL2]. The first axioms of vertex algebras was introduced by Borcherds [Bor]. A very important property of vertex operator algebras is the modular invariance of their characters [Zhu] (up to some finiteness conditions on the vertex operator algebras). Hence the existence of a vertex operator algebra structure on the moduli spaces of semi-stable sheaves indicates the presence of some much richer mathematical structures of these moduli spaces and in particular will imply the modularity property in the S-duality conjecture of Vafa and Witten. In this paper, we try to reveal some relations between vertex operator algebras and the blowup formula for the S-duality conjecture of Vafa and Witten.

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2. The blowup formula

To state the blowup formula proved in [LQ1, LQ2], we recall some standard definitions and notations. Let $\phi : \widetilde{X} \to X$ be the blowing-up of an algebraic surface X at a point $x_0 \in X$, and E be the exceptional divisor. Fix a divisor c_1 on X, $\tilde{c}_1 = \phi^* c_1 - aE$ with $a = 0$ or 1, and an ample divisor H on X with odd $(H \cdot c_1)$. For an integer n, let $\mathfrak{M}_H(c_1, n)$ be the moduli space of Mumford-Takemoto H-stable rank-2 bundles with Chern classes c_1 and n, and $\mathfrak{M}_H^G(c_1, n)$ be the moduli space of Gieseker H-semistable rank-2 torsion-free sheaves with Chern classes c_1 and n. It is well-known that the Gieseker moduli spaces are projective and $\mathfrak{M}_H(c_1, n)$ is an open subset of $\mathfrak{M}_H^G(c_1, n)$. For $r \gg 0$, the divisors $H_r = r \cdot \phi^* H - E$ on $\widetilde{X}$ are ample; moreover, all the moduli spaces $\mathfrak{M}_{H_r}(\tilde{c}_1, n)$ (resp. $\mathfrak{M}_{H_r}^G(\tilde{c}_1, n)$) can be naturally identified [F-M, Bru, Qin]. So we shall use $\mathfrak{M}_{H_\infty}(\tilde{c}_1, n)$ (resp. $\mathfrak{M}_{H_\infty}^G(\tilde{c}_1, n)$) to denote the moduli space $\mathfrak{M}_{H_r}(\tilde{c}_1, n)$ (resp. $\mathfrak{M}_{H_r}^G(\tilde{c}_1, n)$) with $r \gg 0$.

For a complex algebraic scheme Y, let $e(Y; x, y)$ be the virtual Hodge polynomial of Y [Del, D-K, Che, Ful, LQ1]. Then $e(Y; 1, 1) = \chi_c(Y_{\text{red}})$ where $\chi_c(\cdot)$ denotes the Euler characteristic computed by using cohomology with compact supports (we refer to [LQ1, LQ2] for other properties of virtual Hodge polynomials). In particular, when Y is projective, $e(Y; 1, 1)$ is the topological Euler characteristic of Y_{red}. Our main theorems in [LQ1, LQ2] give the following blowup formula:

$$(2.1) \quad \sum_n e(\mathfrak{M}_{H_\infty}^G(\tilde{c}_1, n); x, y) q^{n - \frac{\tilde{c}_1^2}{4}} = (q^{\frac{1}{12}} \cdot \widetilde{\widetilde{Z}}_a) \cdot \sum_n e(\mathfrak{M}_H^G(c_1, n); x, y) q^{n - \frac{c_1^2}{4}},$$

where $\widetilde{\widetilde{Z}}_a = \widetilde{\widetilde{Z}}_a(x, y, q)$ is the universal function of x, y, q, a given by

$$(2.2) \qquad \frac{\sum_{n \in \mathbb{Z}} (xy)^{\frac{(2n+a)^2 - (2n+a)}{2}} q^{\frac{(2n+a)^2}{4}}}{[q^{\frac{1}{24}} \prod_{n \ge 1} (1 - (xy)^{2n} q^n)]^2}.$$

In particular, setting $x = y = 1$, we obtain the following blowup formula for the topological Euler characteristics of the Gieseker moduli spaces:

$$(2.3) \quad \sum_n \chi(\mathfrak{M}_{H_\infty}^G(\tilde{c}_1, n)_{\text{red}}) q^{n - \frac{\tilde{c}_1^2}{4}} = \left(q^{\frac{1}{12}} \cdot \frac{\theta_a(q)}{\eta(q)^2} \right) \cdot \sum_n \chi(\mathfrak{M}_H^G(c_1, n)_{\text{red}}) q^{n - \frac{c_1^2}{4}},$$

where $\eta(q) = q^{\frac{1}{24}} \prod\limits_{n \ge 1} (1 - q^n)$ is the Dedekind η-function, and $\theta_a(q) = \sum\limits_{n \in \mathbb{Z}} q^{(n + \frac{a}{2})^2}$ $(a = 0, 1)$ are theta functions. Formula (2.3) was conjectured by Vafa and Witten [V-W]. In this paper, we concentrate on the case when $a = 0$ and $\tilde{c}_1 = \phi^* c_1$:

(2.4)
$$\sum_n \chi(\mathfrak{M}_{H_\infty}^G(\phi^* c_1, n)_{\text{red}}) q^{n - \frac{c_1^2}{4}} = \left(q^{\frac{1}{12}} \cdot \frac{\theta_0(q)}{\eta(q)^2} \right) \cdot \sum_n \chi(\mathfrak{M}_H^G(c_1, n)_{\text{red}}) q^{n - \frac{c_1^2}{4}}.$$

Remark 2.5. Yoshioka [Yos] proved the blowup formula for the number of $\mathbb{F}_q$-rational points in the Gieseker moduli spaces. Using the Weil conjecture, one can prove the blowup formula (2.3) for Euler characteristics (see [Go2, Yos]). Göttsche in [Go2] also gave a proof of the blowup formula for Hodge numbers when the surface is rational, with the rank of bundles being an arbitrary positive integer.

3. Vertex operator algebras

The first axiom of a vertex algebra was introduced by Borcherds [Bor]. This is a rigorous mathematical definition of the chiral part of a 2-dimensional quantum field theory studied intensively by physicists. Here we adopt the definition of a vertex algebra from [Kac], in which the spaces of states are required to be superspaces. For a vertex algebra V, there exists a state-field correspondence Y which assigns $a \in V$ a field $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ with $a_{(n)} \in \mathrm{End}(V)$. Vertex operator algebras are a special type of vertex algebras. They admit decompositions into finite dimensional subspaces $V = \oplus_{n\in\mathbb{Z}} V^{(n)}$. For a vertex operator algebra $V = \oplus_{n\geq 0} V^{(n)}$ (i.e. $V^{(n)} = 0$ for all $n < 0$) with dim $V^{(0)} = 1$, we put

$$\mathfrak{g} = \{a_{(0)} | \quad a \in V^{(1)}\} \quad \text{and} \quad \hat{\mathfrak{g}} = \{a_{(n)} | \quad a \in V^{(1)}, n \in \mathbb{Z}\} + \mathbb{C} \cdot \mathrm{Id}_V \,. \tag{3.1}$$

Then it is known that $\mathfrak{g}$ is a Lie superalgebra of dimension $\dim(V^{(1)})$ with an invariant supersymmetric bilinear form and $\hat{\mathfrak{g}}$ is the affinization of $\mathfrak{g}$.

Now we briefly review simple lattice vertex algebras and refer the details to [Bor, FL1, Kac]. Let L be an integral lattice of rank-ℓ, i.e., $L = \mathbb{Z}^\ell$ is equipped with a $\mathbb{Z}$-valued symmetric bilinear form $(\cdot|\cdot)$. Choose a bimultiplicative 2-cocycle $\epsilon : L \times L \to \{\pm 1\}$ satisfying some properties. Let $\mathbb{C}[L]$ be the twisted group algebra,

$$e^\alpha \cdot e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}, \qquad e^0 = 1, \qquad\qquad (\alpha, \beta \in L),$$

and S_L be the symmetric algebra over the space $\mathfrak{h}^{<0} = \sum_{j<0} t^j \otimes \mathfrak{h}$ with $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ being the complexification of L. Then we can construct the simple lattice vertex algebra $V_L = S_L \otimes \mathbb{C}[L]$ associated to L. When the bilinear form $(\cdot|\cdot)$ is even and positive definite, V_L is actually a vertex operator algebra. More generally, for a sublattice Δ of L, we construct the vertex algebra $V_{\Delta\subset L} = S_L \otimes \mathbb{C}[\Delta]$. When the restricted bilinear form $(\cdot|\cdot)|_\Delta$ is even and positive definite, $V_{\Delta\subset L}$ is a vertex operator algebra with $\dim(V_{\Delta\subset L})^{(0)} = 1$ and $(V_{\Delta\subset L})^{(n)} = 0$ for all $n < 0$. The character of the vertex operator algebra $V_{\Delta\subset L}$ is given by

$$\mathrm{ch}(V_{\Delta\subset L}) = \frac{\sum_{\alpha\in\Delta} q^{\frac{1}{2}(\alpha|\alpha)}}{\eta(q)^\ell}.$$

Also, since $(\cdot|\cdot)|_\Delta$ is even, there is no odd part in $V_{\Delta\subset L}$ as a superspace. So the associated Lie superalgebra $\mathfrak{g}$ in (3.1) is a usual Lie algebra.

In particular, let $L = \mathbb{Z}^2$ with the standard bilinear form. Let the bimultiplicative 2-cocycle $\epsilon : L \times L \to \{\pm 1\}$ be defined as follows:

$$\epsilon(h^{(i)}, h^{(j)}) = \begin{cases} 1 & \text{if } i \le j, \\ -1 & \text{if } i > j, \end{cases}$$

where $\{h^{(1)}, h^{(2)}\}$ is the standard basis of L, and Δ be the diagonal in $\mathbb{Z}^2$. Then we obtain the vertex operator algebra $V_{\Delta\subset\mathbb{Z}^2} = S_{\mathbb{Z}^2} \otimes \mathbb{C}[\Delta]$. The relevant properties of $V_{\Delta\subset\mathbb{Z}^2}$ are summarized in the proposition below.

Proposition 3.2. (i) $\mathrm{ch}(V_{\Delta\subset\mathbb{Z}^2}) = \dfrac{\theta_0(q)}{\eta(q)^2}$;

(ii) *The associated Lie superalgebra* $\mathfrak{g}$ *in (3.1) is isomorphic to the Lie algebra* $\mathfrak{gl}_2$ *with the invariant symmetric bilinear form* $(\cdot|\cdot)$ *defined by* $(A|B) = \mathrm{tr}(AB)$.

So the character $\mathrm{ch}(V_{\Delta\subset\mathbb{Z}^2}) = \dfrac{\theta_0(q)}{\eta(q)^2}$ of the vertex operator algebra $V_{\Delta\subset\mathbb{Z}^2} = S_{\mathbb{Z}^2} \otimes \mathbb{C}[\Delta]$ is precisely the universal function $\widetilde{\widetilde{Z}}_0(1,1,q) = \dfrac{\theta_0(q)}{\eta(q)^2}$ appeared in our blowup formula (2.4) for the S-duality conjecture of Vafa-Witten.

4. Relations between vertex operator algebras and the blowup formula

Our goal is to study the relations between the vertex operator algebra $V_{\Delta\subset\mathbb{Z}^2}$ and the geometric constructions involved in our proofs of the blowup formula (2.4).

First of all, from the computation of the character $\mathrm{ch}(V_{\Delta\subset\mathbb{Z}^2})$, we know that the part $\eta(q)^2$ (resp. $\theta_0(q)$) in $\mathrm{ch}(V_{\Delta\subset\mathbb{Z}^2}) = \dfrac{\theta_0(q)}{\eta(q)^2}$ comes from the contribution of the part $S_{\mathbb{Z}^2}$ (resp. of $\mathbb{C}[\Delta]$) in $V_{\Delta\subset\mathbb{Z}^2} = S_{\mathbb{Z}^2} \otimes \mathbb{C}[\Delta]$. Next, recall from [LQ1, Yos] that there are two simple but fundamental geometric constructions in the proofs of the blowup formula (2.4). The first one is taking the double dual of a torsion free sheaf. Essentially this geometric construction gives rise to the part $\eta(q)^2$ in $\widetilde{\widetilde{Z}}_0(1,1,q) = \dfrac{\theta_0(q)}{\eta(q)^2}$. The second one is performing elementary modifications along the exceptional divisor E. This geometric construction gives rise to the theta function $\theta_0(q)$ in $\widetilde{\widetilde{Z}}_0(1,1,q)$. Therefore it is no surprise that the first (resp. second) geometric construction will correspond to the part $S_{\mathbb{Z}^2}$ (resp. $\mathbb{C}[\Delta]$) in $V_{\Delta\subset\mathbb{Z}^2} = S_{\mathbb{Z}^2} \otimes \mathbb{C}[\Delta]$. We remark that unlike [Gro, Na1, Na2] where homology groups of the Hilbert schemes and the Quot-schemes were used, we have to use the spaces of constructible functions, denoted by $\mathcal{C}(\cdot)$, over our relevant spaces. The reason is that while little is known about the homology groups of our relevant spaces for a general algebraic surface X, the spaces of constructible functions are much easier to handle (see [Grt, Mac] for constructible functions and their properties).

To state our main result, we introduce more notations related to the above-mentioned two geometric constructions in [LQ1, Yos]. Given a rank-2 torsion free sheaf V on the algebraic surface X, we have a canonical exact sequence

$$0 \to V \to V^{**} \to Q \to 0$$

where V^{**} is the double dual of V and Q is a torsion sheaf supported at finitely many points. It follows that abstractly, the first geometric construction (i.e. taking the double dual) is equivalent to working with the Quot-schemes $\mathrm{Quot}^n_{\mathcal{O}_X^{\oplus 2}}$. In [Gro, Na1], correspondences among these Quot-schemes $\mathrm{Quot}^n_{\mathcal{O}_X^{\oplus 2}}$ (i.e. subsets in $\mathrm{Quot}^n_{\mathcal{O}_X^{\oplus 2}} \times \mathrm{Quot}^m_{\mathcal{O}_X^{\oplus 2}}$) have been constructed. For our purpose, we work with certain constructible subsets $\mathrm{Quot}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}}$ of $\mathrm{Quot}^n_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}}$ where for $i = 1, 2$, $\mathbf{n}_{i,j}$ stands for the sequence of nonnegative integers $n_{i,1}, n_{i,2}, \ldots, n_{i,j}, \ldots$ satisfying $\sum_{i,j} j \cdot n_{i,j} = n$. Our correspondences $Z_k^{n,p}$ ($n \geq p > 0$ and $k = 1, 2$) among these subsets are modified over the correspondences in [Gro, Na1]. Using $Z_k^{n,p}$, we define operators $\mathbb{E}_k^{n,p}$ and $\mathbb{F}_k^{n,p}$ among the spaces $\mathcal{C}(\mathrm{Quot}^n_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}})$ of constructible functions over the Quot-schemes $\mathrm{Quot}^n_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}}$. We prove that there is a linear embedding

$$\lambda : S_{\mathbb{Z}^2} \to \bigoplus_{n=0}^{+\infty} \mathcal{C}(\mathrm{Quot}^n_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}}), \tag{4.1}$$

whose image is the subspace $\bigoplus_{n_{i,j} \geq 0; i=1,2; j \geq 1} \mathbb{C} \cdot \mathbf{1}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}$ generated by the characteristic functions $\mathbf{1}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}$ of the constructible subsets $\mathrm{Quot}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}}$ in $\mathrm{Quot}^n_{\mathcal{O}_{\widetilde{X}}^{\oplus 2}}$. Moreover, we interpret the standard operators (from the vertex operator algebra structure of $V_{\Delta \subset \mathbb{Z}^2}$) on $S_{\mathbb{Z}^2}$ in terms of our geometric operators $\mathbb{E}_k^{\mathbf{n}_{1,j},\mathbf{n}_{2,j},\, p}$ and $\mathbb{F}_k^{\mathbf{m}_{1,j},\mathbf{m}_{2,j},\, p}$ where $\mathbb{E}_k^{\mathbf{n}_{1,j},\mathbf{n}_{2,j},\, p}$ is the restriction of $\mathbb{E}_k^{\sum_{i,j} j \cdot n_{i,j},\, p}$ to $\mathbb{C} \cdot \mathbf{1}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}$, and $\mathbb{F}_k^{\mathbf{m}_{1,j},\mathbf{m}_{2,j},\, p}$ is the restriction of $\mathbb{F}_k^{p+\sum_{i,j} j \cdot m_{i,j},\, p}$ to $\mathbb{C} \cdot \mathbf{1}^{\mathbf{m}_{1,j},\mathbf{m}_{2,j}}$.

Similarly, using elementary modifications of vector bundles along the exceptional divisor E, we construct certain constructible subsets $\widetilde{\mathfrak{M}}^{n,p} (p \in \mathbb{Z})$ in the Mumford-Takemoto moduli space $\mathfrak{M}_{H_\infty}(\phi^* c_1, n)$. These subsets enable us to define certain correspondences $Z^{n,p} (n, p \in \mathbb{Z})$ among all the moduli spaces $\mathfrak{M}_{H_\infty}(\phi^* c_1, n)$. Again, geometric operators $\mathbb{E}^{n,p}$ and $\mathbb{F}^{n,p} (n, p \in \mathbb{Z})$ among the spaces $\mathcal{C}(\mathfrak{M}_{H_\infty}(\phi^* c_1, n))$ of constructible functions are constructed. Assume that $\mathfrak{M}_H(c_1, m) \neq \emptyset$ for some m. Then we show that there exists a linear embedding

$$\mu_m : \mathbb{C}[\Delta] \to \bigoplus_{n,p \in \mathbb{Z}} \mathcal{C}(\mathfrak{M}_{H_\infty}(\phi^* c_1, n + p^2)), \tag{4.2}$$

whose image is the subspace $\bigoplus_{p \in \mathbb{Z}} \mathbb{C} \cdot \mathbf{1}^{m+p^2,p}$ generated by the characteristic functions $\mathbf{1}^{m+p^2,p}$ of the constructible subsets $\widetilde{\mathfrak{M}}^{m+p^2,p}$ in $\mathfrak{M}_{H_\infty}(\phi^* c_1, m+p^2)$. Moreover, we interpret the standard operators (from the vertex operator algebra structure of $V_{\Delta\subset\mathbb{Z}^2}$) on $\mathbb{C}[\Delta]$ in terms of the geometric operators $\mathbb{E}^{m+p^2,p}$ and $\mathbb{F}^{m+p^2,p}$.

Our main result essentially follows from (4.1), (4.2) and Proposition 3.2 (ii).

Theorem 4.3. *Assume that $(H\cdot c_1)$ is odd and the Mumford-Takemoto moduli space $\mathfrak{M}_H(c_1,m)$ is nonempty. Then there exists a linear embedding:*

$$\lambda\otimes\mu_m : V_{\Delta\subset\mathbb{Z}^2} = S_{\mathbb{Z}^2}\otimes\mathbb{C}[\Delta] \to \bigoplus_{n,p\in\mathbb{Z}} \mathcal{C}(\mathfrak{M}^G_{H_\infty}(\phi^* c_1, n+p^2))$$

of the vertex operator algebra $V_{\Delta\subset\mathbb{Z}^2}$ in $\bigoplus_{n,p\in\mathbb{Z}} \mathcal{C}(\mathfrak{M}^G_{H_\infty}(\phi^* c_1, n+p^2))$ such that

(*i*) *the image of the embedding $\lambda\otimes\mu_m$ is the subspace*

$$\left(\bigoplus_{n_{i,j}\geq 0; i=1,2; j\geq 1} \mathbb{C}\cdot\mathbf{1}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}\right)\otimes\left(\bigoplus_{p\in\mathbb{Z}}\mathbb{C}\cdot\mathbf{1}^{m+p^2,p}\right);$$

(*ii*) *for every $a\in V_{\Delta\subset\mathbb{Z}^2}$ with the field $Y(a,z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$, the operators*

$$(\lambda\otimes\mu_m)\circ a_{(n)}\circ(\lambda\otimes\mu_m)^{-1} \quad (n\in\mathbb{Z})$$

on the subspace $\left(\bigoplus_{n_{i,j}\geq 0; i=1,2; j\geq 1} \mathbb{C}\cdot\mathbf{1}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}\right)\otimes\left(\bigoplus_{p\in\mathbb{Z}}\mathbb{C}\cdot\mathbf{1}^{m+p^2,p}\right)$ can be expressed in terms of the geometric operators $\mathbb{E}_k^{\mathbf{n}_{1,j},\mathbf{n}_{2,j},\ p'}, \mathbb{F}_k^{\mathbf{n}_{1,j},\mathbf{n}_{2,j},\ p'}, \mathbb{E}^{m+p^2,p}$ and $\mathbb{F}^{m+p^2,p}$ where the integers $k, n_{i,j}, p', p$ satisfy the conditions:

$$1\leq k\leq 2, n_{i,j}=0 \textit{ for } j\gg 0, \textit{and } 0<p'\leq\sum_{i,j} j\cdot n_{i,j};$$

(*iii*) *the embedding $\lambda\otimes\mu_m$ induces a representation ν_m of $\hat{\mathfrak{g}}$ in the subspace*

$$\left(\bigoplus_{n_{i,j}\geq 0; i=1,2; j\geq 1} \mathbb{C}\cdot\mathbf{1}^{\mathbf{n}_{1,j},\mathbf{n}_{2,j}}\right)\otimes\left(\bigoplus_{p\in\mathbb{Z}}\mathbb{C}\cdot\mathbf{1}^{m+p^2,p}\right)$$

where $\hat{\mathfrak{g}}$ is the affinization of the Lie algebra $\mathfrak{g}=\mathfrak{gl}_2$ with the invariant symmetric bilinear form $(\cdot|\cdot)$ defined by $(A|B)=\mathrm{tr}(AB)$.

It is well-known [Don, Fri] that for fixed c_1 and H, there exists a constant m_0 depending only on X, c_1, H such that the Mumford-Takemoto moduli space $\mathfrak{M}_H(c_1,m)$ is nonempty whenever $m>m_0$. It can be proved that there exists a family, parameterized by the integer m with $\mathfrak{M}_H(c_1,m)\neq\emptyset$, of embeddings of the vertex operator algebra $V_{\Delta\subset\mathbb{Z}^2}$ in the vector space $\bigoplus_{n,p\in\mathbb{Z}} \mathcal{C}(\mathfrak{M}^G_{H_\infty}(\phi^* c_1, n+p^2))$.

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Department of Mathematics, HKUST, Clear Water Bay, Kowloon, Hong Kong
E-mail address: mawpli@uxmail.ust.hk

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
E-mail address: zq@math.okstate.edu