

POINT PROCESSES AND THE INFINITE SYMMETRIC GROUP

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0. Introduction

Our work originated from a problem of harmonic analysis on the infinite symmetric group $S(\infty)$.

The central theme of noncommutative harmonic analysis is decomposition of natural unitary representations T into elementary ones (i.e., into irreducible or factor representations). When T is endowed with a distinguished cyclic vector, its decomposition is governed by a measure, called the spectral or Plancherel measure. For instance, one of the achievements of the classical representation theory is the explicit calculation by Gindikin and Karpelevich of the spectral measure for the natural representation in the L^2 space on an arbitrary Riemannian symmetric space.

The group $S(\infty)$ consisting of finite permutations of the natural numbers is the simplest example of a big group whose elementary representations depend on infinitely many continuous parameters. In such a situation a spectral measure may be supported by an infinite-dimensional space — a phenomenon which did not arise in the classical representation theory.

In [KOV], a two-parameter family of generalized regular representations $T = T_{zz'}$ of $S(\infty)$ was constructed. They are realized in certain L^2 spaces on a compactification of $S(\infty)$. Their spectral measures, denoted as $P_{zz'}$, live on an infinite-dimensional simplex Ω (a kind of dual object to $S(\infty)$). These are certain probability measures which are singular with respect to each other. The aim of the present work is to study the measures $P_{zz'}$.

Our strategy is as follows. We embed the simplex Ω into a space of countable point configurations on the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$; the points of a configuration accumulate near 0. This enables us to interpret the measures $P_{zz'}$ as certain point processes on \mathbb{R}^* . As the main tool for studying these processes we employ their correlation functions $\rho_n(x_1, \dots, x_n)$, $n = 1, 2, \dots$

Received October 23, 1998.

Supported by the Russian Foundation for Basic Research under grant 98-01-00303 (G. O) and by the Russian Program for Support of Scientific Schools under grant 96-15-96060 (A. B. and G. O.)

The correlation functions can be given by multidimensional integrals or expressed in terms of a multivariate hypergeometric series (the Lauricella function of type B). It turns out that their description is greatly simplified after a slight modification (‘lifting’) of the processes. Then the correlation functions take a determinantal form

$$(0.1) \quad \rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \leq i, j \leq n},$$

with a certain kernel $K(x, y)$ on $\mathbb{R}^* \times \mathbb{R}^*$.

The explicit knowledge of the correlation functions enables us to derive a number of results about the properties of the measures $P_{zz'}$. On the other hand, a rather special form (0.1) of the correlation functions makes it possible to understand the nature of the corresponding point processes and, consequently, of the initial measures. A variety of point processes with determinantal correlation functions arises in Random Matrix Theory (RMT, for short), and we were guided by this analogy.

We study in detail the kernel $K(x, y)$, which depends on the initial parameters z, z' . It is expressed through the classical Whittaker functions. On the other hand, it can be expressed through another kernel L which has a surprisingly simple form.¹ Thus, it turns out that all the information about the initial measure $P_{zz'}$ is encoded in this simple kernel L . It should be noted that the general structure of the kernel K is similar to that of kernels arising in two-matrix models [EM].

We also show that the kernel K admits various degenerations. They include the well-known in RMT sine and Bessel kernels as well as other Bessel-type kernels which, to our best knowledge, are new.

We believe that this work provides a new promising connection between RMT and Representation Theory.

A detailed exposition of the results is contained in five papers [Part I]–[Part V]. In the present paper we give a survey of the main results and indicate their location in [Part I]–[Part V].

1. The Thoma simplex [Part I, §1]

The starting point of our study was the work [KOV] on generalized regular representations of the infinite symmetric group. These representations depend on two parameters (further denoted by z and z'). The decomposition of the representations into irreducibles is governed by certain probability measures $P_{zz'}$ living on the infinite-dimensional simplex

$$\Omega = \left\{ \alpha_1 \geq \alpha_2 \geq \dots \geq 0; \beta_1 \geq \beta_2 \geq \dots \geq 0 \mid \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1 \right\},$$

¹In terms of the corresponding integral operators, $K = L(1 + L)^{-1}$.

called the *Thoma simplex* [VK], [KV]. Note that Ω is compact in the topology of pointwise convergence.

Our aim is to understand these measures. Our results show that the measures $P_{zz'}$ are close to stochastic point processes arising in scaling limits of random matrix ensembles.

2. Probability measures $P_{zz'}$ [Part I, §§1,2]

The measures $P_{zz'}$ are defined as follows. There exists a family $\{\tilde{s}_\lambda\}$ of continuous functions on Ω , indexed by the Young diagrams λ . Their linear span is dense in $C(\Omega)$. We know explicitly the integrals

$$(2.1) \quad \varphi_{zz'}(\lambda) = \int_{\Omega} \tilde{s}_\lambda(\omega) P_{zz'}(d\omega),$$

which determine $P_{zz'}$ uniquely. Essentially, this is the only information about $P_{zz'}$ that we possess.

Now we shall describe the functions \tilde{s}_λ and write down the formula for $\varphi_{zz'}$.

Let $\omega = (\alpha|\beta)$ range over Ω . Set

$$\tilde{p}_k(\omega) = \begin{cases} 1, & k = 1, \\ \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, & k \geq 2, \end{cases}$$

and for any partition $\rho = (\rho_1, \dots, \rho_l)$,

$$\tilde{p}_\rho(\omega) = \tilde{p}_{\rho_1}(\omega) \cdots \tilde{p}_{\rho_l}(\omega).$$

These are continuous functions on Ω . The functions \tilde{s}_λ are related to \tilde{p}_ρ 's in exactly the same way as the Schur functions s_λ are related to the power sums p_ρ , see [M, §1.7]. Specifically,

$$\tilde{s}_\lambda = \sum_{\rho} \chi_\rho^\lambda z_\rho^{-1} \tilde{p}_\rho,$$

where ρ ranges over the partitions of $|\lambda| = \lambda_1 + \lambda_2 + \dots$; χ^λ is the irreducible character of the symmetric group of degree $|\lambda|$, χ_ρ^λ is its value on the conjugacy class indexed by ρ , and $z_\rho^{-1} \cdot |\lambda|!$ is the cardinality of this conjugacy class.

Conversely,

$$(2.2) \quad \tilde{p}_\rho = \sum_{\lambda} \chi_\rho^\lambda \tilde{s}_\lambda.$$

The functions \tilde{s}_λ are called *extended Schur functions*, see [KV].

To define $\varphi_{zz'}(\lambda)$ we need Frobenius notation for a Young diagram λ :

$$\lambda = (p_1, \dots, p_d | q_1, \dots, q_d);$$

here d is the number of diagonal boxes of λ , and

$$p_i = \lambda_i - i, \quad q_i = \lambda'_i - i,$$

where λ' stands for the transposed diagram (see [M, §1.1]).

Set $n = |\lambda|$, $t = zz'$. Then

$$(2.3) \quad \varphi_{zz'}(\lambda) = \frac{t^d}{(t)_n} \prod_{i=1}^d \frac{(z+1)_{p_i} (z'+1)_{p_i} (-z+1)_{q_i} (-z'+1)_{q_i}}{p_i! q_i!} \times \frac{\prod_{i < j} [(p_i - p_j)(q_i - q_j)]}{\prod_{i,j} (p_i + q_j + 1)},$$

where $(a)_m = a(a+1) \cdots (a+m-1)$, $(a)_0 = 1$.

We shall always assume that z and z' satisfy one of the following conditions

$$(2.4) \quad \begin{aligned} (*) \quad & z' = \bar{z}, \quad z \in \mathbb{C} \setminus \mathbb{Z}; \\ (**) \quad & z \text{ and } z' \text{ are real and for a certain } m \in \mathbb{Z}, \quad m < z, z' < m + 1. \end{aligned}$$

Under this assumption there exists a probability measure $P_{zz'}$ such that (2.1) holds.

Note that the measure $P_{zz'}$ can be obtained as the limit, as $n \rightarrow \infty$, of certain statistics on partitions of n , see [KOO, proof of Theorem B], [Part II, Introduction].

3. Stochastic point processes [Part I, §4]

We shall interpret the measures $P_{zz'}$ as stochastic point processes on $I = [-1, 1] \setminus \{0\}$. With each point $\omega \in \Omega$ we associate a point configuration in I ,

$$\omega = (\alpha | \beta) \mapsto (\alpha_1, \alpha_2, \dots; -\beta_1, -\beta_2, \dots),$$

where we omit possible zeros in α and β . Note that any such configuration has no accumulation points in I , the points condensate near the origin which is not in I .

Thus, the measure $P_{zz'}$ becomes a probability measure on the space of point configurations in I , i.e., a stochastic point process on I . We shall denote this process by $\mathcal{P}_{zz'}$.

The n th correlation function $\rho_n(x_1, \dots, x_n)$ of a point process is the density of the probability to find a point in each of the infinitesimal intervals $[x_i, x_i + dx_i]$ (see [DVJ], [Part I, §4] for details).

Our aim is to compute the correlation functions $\rho_n^{(zz')}$ of our processes $\mathcal{P}_{zz'}$.

4. Moment problems [Part I, §3], [Part II, Section 1.1]

Our strategy to solve the problem is to find the probability measures $\sigma_n^{(zz')}$ on $[-1, 1]^n$, $n = 1, 2, \dots$, characterized by their moments

$$\int_{[-1,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma_n^{(zz')}(dx) = \int_{\Omega} \tilde{p}_{(l_1+1, \dots, l_n+1)}(\omega) P_{zz'}(d\omega).$$

Note that the RHS is explicitly known because of (2.1), (2.2):

$$(4.1) \quad \int_{[-1,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma_n^{(zz')}(dx) = \sum_{\lambda, |\lambda|=\sum_i (l_i+1)} \chi_{(l_1+1, \dots, l_n+1)}^\lambda \varphi_{zz'}(\lambda).$$

It turns out that outside the diagonals $x_i = x_j$ in I^n

$$(4.2) \quad \rho_n^{(zz')}(x_1, \dots, x_n) = \frac{1}{|x_1 \cdots x_n|} \{\text{density of } \sigma_n^{(zz')}(dx_1, \dots, dx_n)\}.$$

It is worth noting that the measure $\sigma_n^{(zz')}$ for $n \geq 2$ always has singular components living on the diagonals while (as is proved in [Part I, Proposition 4.2], [Part II, Theorem 2.5.1]) $\rho_n^{(zz')}$ has no such components².

Thus, we obtain the correlation functions from a more sophisticated object. However, we can not formulate a moment problem for $\rho_n^{(zz')}$ directly because of the absolute value sign in (4.2).

5. Integral representations for the correlation functions [Part I, §5, §6], [Part II, Chapter 2]

The moment problem (4.1) can be completely solved. In particular, for $n = 1$ we obtained a two dimensional integral representation of $\sigma_1^{(zz')}$ (see [Part I, Theorem 5.8]). A careful examination shows that $\sigma_1^{(zz')}$ has no atom at zero. This fact has an important corollary.

Theorem I ([Part I, Theorem 6.1]). *With probability 1,*

$$\sum_{i=1}^{\infty} (\alpha_i + \beta_i) = 1.$$

For $n > 1$ the solution of the moment problem (4.1) requires a lot of work ([Part II, Chapter 1]). The difficulties come from the fact that the RHS of (4.1)

²This just means that nonzero coordinates in α and β are pairwise distinct with probability 1.

involves the symmetric group characters for which there is no simple expression. We use Murnaghan–Nakayama rule ([M, §1.7, Ex. 5]) to handle the characters. In [Part II, Theorem 1.2.1] we obtain a more economic form of this rule which enables us to solve the moment problem.

The final expression for $\rho_n^{(zz')}$ is a linear combination of multidimensional integrals of various orders up to $3n$. The situation simplifies when all the variables x_1, \dots, x_n are of the same sign, say, positive.

Theorem II ([Part II, Theorem 2.2.1]). *Let $x_1, \dots, x_n > 0$. Then*

$$(5.1) \quad \rho_n^{(zz')}(x_1, \dots, x_n) = t^n \Gamma(t) \\ \times \int_{\substack{a_i, b_i > 0 \\ \sum_i x_i(a_i + b_i + 1) < 1}} \prod_{i=1}^n \frac{a_i^{-z}}{\Gamma(-z+1)} \frac{(a_i+1)^{z'}}{\Gamma(z'+1)} \frac{b_i^{-z'}}{\Gamma(-z'+1)} \frac{(b_i+1)^z}{\Gamma(z+1)} \\ \times \det \left(\frac{1}{a_i + b_j + 1} \right) \frac{(1 - \sum_i x_i(a_i + b_i + 1))^{t-n-1}}{\Gamma(t-n)} da db.$$

The RHS of (5.1) is well defined for z and z' such that

$$-1 < \Re z, \Re z' < 1; \quad t = zz' > n.$$

For other values of z, z' we use analytic continuation.

We tacitly assume that $\sum_i x_i < 1$. Actually, the correlation measure $\rho_n^{(zz')}(x) dx$ always lives on the set $\sum_i |x_i| \leq 1$; additional considerations show that there are no singular components on the faces $\sum_i |x_i| = 1$, see the beginning of the proof of Theorem 3.3.1 in [Part II].

The RHS of (5.1) can be expressed via the multivariate Lauricella hypergeometric function of type B, see [Part II, Section 2.4]. In particular, $\rho_1^{(zz')}$ can be expressed in terms of the Appell's two-dimensional hypergeometric function F_3 , see [Part II, Corollary 2.4.2]. Another expression of $\rho_1^{(zz')}$ through the Lauricella function in three variables is given in [Part I, Theorem 5.12].

6. Lifting [Part II, Chapter 3]

A surprising fact is that the correlation functions are greatly simplified after a 'lifting' of our processes $\mathcal{P}_{zz'}$ to a slightly bigger space.

According to Theorem I, the measure $P_{zz'}$ is concentrated on the face

$$\Omega_0 = \{\omega = (\alpha | \beta) \mid \sum_i (\alpha_i + \beta_i) = 1\} \subset \Omega.$$

Let $\tilde{\Omega}_0 = \Omega_0 \times \mathbb{R}_+$. We pass from the measure $P_{zz'}$ on Ω_0 to the measure

$$\widetilde{P}_{zz'} = P_{zz'} \otimes \left\{ \frac{s^{t-1}}{\Gamma(t)} e^{-s} ds \right\}$$

on $\tilde{\Omega}_0$. In other words, we tensor $P_{zz'}$ by the gamma-distribution with parameter t .

We associate to a point $\tilde{\omega} = (\omega, s) \in \tilde{\Omega}_0$ a point configuration in $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ as follows

$$((\alpha|\beta), s) \mapsto (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots; -\tilde{\beta}_1, -\tilde{\beta}_2, \dots) = (s\alpha_1, s\alpha_2, \dots; -s\beta_1, -s\beta_2, \dots).$$

Thus, we get a probability measure on the space of point configurations in \mathbb{R}^* , i.e., a point process. We denote it by $\widetilde{\mathcal{P}}_{zz'}$.

The process $\widetilde{\mathcal{P}}_{zz'}$ is obtained from $\mathcal{P}_{zz'}$ by multiplying the random configuration in I by the scalar random factor s with gamma-distribution.

The lifting is invertible via the map

$$(\tilde{\alpha}|\tilde{\beta}) \mapsto (\alpha|\beta) = \left(\frac{\tilde{\alpha}_1}{s}, \frac{\tilde{\alpha}_2}{s}, \dots; -\frac{\tilde{\beta}_1}{s}, -\frac{\tilde{\beta}_2}{s}, \dots \right),$$

where $s = \sum_i (\tilde{\alpha}_i + \tilde{\beta}_i)$.

Note that s is finite almost surely with respect to $\widetilde{P}_{zz'}$.

Let $\tilde{\rho}_n^{(zz')}(x_1, \dots, x_n)$ be the n th correlation function of $\widetilde{\mathcal{P}}_{zz'}$.

The 'lifted' correlation functions are related to the initial ones by the following simple transformation

$$(6.1) \quad \tilde{\rho}_n^{(zz')}(x_1, \dots, x_n) = \int_0^\infty \frac{s^{t-1} e^{-s}}{\Gamma(t)} \rho_n^{(zz')}(x_1 s^{-1}, \dots, x_n s^{-1}) \frac{ds}{s^n},$$

see [Part II, Proposition 3.1.1]. This implies that the moments of the correlation measures $\rho_n^{(zz')}(x)dx$ and $\tilde{\rho}_n^{(zz')}(x)dx$ are related in a very simple way

$$\begin{aligned} & \int x_1^{l_1} \dots x_n^{l_n} \tilde{\rho}_n^{(zz')}(x_1, \dots, x_n) dx_1 \dots dx_n \\ &= (t)_{l_1 + \dots + l_n} \int x_1^{l_1} \dots x_n^{l_n} \rho_n^{(zz')}(x_1, \dots, x_n) dx_1 \dots dx_n \end{aligned}$$

(the moments are finite if $l_1, \dots, l_n \geq 1$).

Note also that the transformation (6.1) is invertible, and there exists a complex inversion formula similar to that for the Laplace transform.

**7. Determinantal formula. Matrix
Whittaker kernel [Part IV, Sections 1,2]**

Now we shall show that the correlation functions $\tilde{\rho}_n^{(zz')}$ have determinantal form with a kernel expressed through the Whittaker function $W_{\kappa,\mu}(x)$, $x > 0$.

This function can be characterized as the only solution of the Whittaker equation

$$y'' - \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{\mu^2 - \frac{1}{4}}{x^2} \right) y = 0,$$

such that $y \sim x^\kappa e^{-\frac{x}{2}}$ as $x \rightarrow +\infty$ (see [E1, Chapter 6]). Here κ and μ are complex parameters. Note that

$$W_{\kappa,\mu} = W_{\kappa,-\mu}.$$

We shall employ the Whittaker function for real κ and real or pure imaginary μ ; then $W_{\kappa,\mu}$ is real.

Set

$$(7.1) \quad a = \frac{z + z'}{2}, \quad \mu = \frac{z - z'}{2}, \quad \sigma = \sqrt{\sin(\pi z) \sin(\pi z')} > 0.$$

It is often convenient to consider (a, μ) as new parameters of our processes, instead of (z, z') . In terms of (a, μ) the basic assumptions (2.4) take the following form

- a is always real
- either μ is pure imaginary
- or μ is real, $|\mu| < \frac{1}{2}$, and there exists $m \in \mathbb{Z}$ such that $m + |\mu| < a < m + 1 - |\mu|$
- a is not an integer when $\mu = 0$.

Introduce the following functions on \mathbb{R}_+

$$\begin{aligned} A_+(x) &= \frac{1}{\sqrt{x}} W_{a+\frac{1}{2},\mu}(x), & A_-(x) &= \frac{1}{\sqrt{x}} W_{-a+\frac{1}{2},\mu}(x), \\ B_+(x) &= \frac{1}{\sqrt{x}} W_{a-\frac{1}{2},\mu}(x), & B_-(x) &= \frac{1}{\sqrt{x}} W_{-a-\frac{1}{2},\mu}(x). \end{aligned}$$

Theorem III ([Part IV, Theorem 2.7]). *The correlation functions of the lifted process $\widetilde{\mathcal{P}}_{zz'}$ have the form*

$$(7.2) \quad \begin{aligned} \tilde{\rho}_n^{(zz')}(u_1, \dots, u_n) &= \det [K(u_i, u_j)]_{i,j=1}^n, \\ n &= 1, 2, \dots; \quad u_1, \dots, u_n \in \mathbb{R}^*, \end{aligned}$$

where the kernel $K(u, v)$ is conveniently written in the block form

$$K(u, v) = \begin{cases} K_{++}(u, v), & u, v > 0; \\ K_{+-}(u, -v), & u > 0, v < 0; \\ K_{-+}(-u, v), & u < 0, v > 0; \\ K_{--}(-u, -v), & u, v < 0; \end{cases}$$

with

$$\begin{aligned}
 (7.3) \quad K_{++}(x, y) &= \frac{1}{\Gamma(z)\Gamma(z')} \frac{A_+(x)B_+(y) - B_+(x)A_+(y)}{x - y}, \\
 K_{+-}(x, y) &= \frac{\sigma}{\pi} \frac{A_+(x)A_-(y) + tB_+(x)B_-(y)}{x + y}, \\
 K_{-+}(x, y) &= -\frac{\sigma}{\pi} \frac{A_+(y)A_-(x) + tB_+(y)B_-(x)}{x + y}, \\
 K_{--}(x, y) &= \frac{1}{\Gamma(-z)\Gamma(-z')} \frac{A_-(x)B_-(y) - B_-(x)A_-(y)}{x - y},
 \end{aligned}$$

(recall that $t = zz'$).

This is one of our main results.

The matrix representation of the kernel

$$(7.4) \quad K = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix}$$

corresponds to the splitting $\mathbb{R}^* = \mathbb{R}_+ \sqcup \mathbb{R}_-$ of the phase space and subsequent identification of \mathbb{R}_- with the second copy of \mathbb{R}_+ . We call (7.4) the *matrix Whittaker kernel*.

All the blocks of (7.4) are real valued kernels on \mathbb{R}_+ . They possess the following symmetry

$$K_{++}(x, y) = K_{++}(y, x), \quad K_{--}(x, y) = K_{--}(y, x), \quad K_{+-}(x, y) = -K_{-+}(y, x).$$

Note the minus sign in the last relation. It means that the kernel (7.4) is formally J -symmetric where J is the operator $\text{id} \oplus (-\text{id})$ in $L^2(\mathbb{R}_+, dx) \oplus L^2(\mathbb{R}_+, dx)$.

Another symmetry property: the change of parameters $(z, z') \rightarrow (-z, -z')$ is equivalent to the transform of the kernel

$$\begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix} \longrightarrow \begin{bmatrix} K_{--} & -K_{-+} \\ -K_{+-} & K_{++} \end{bmatrix}.$$

Determinantal form for the correlation functions (like (7.2)) appears in different problems of random matrix theory and mathematical physics, see, e.g., [Dy, Me1, Ma1, Ma2, TW1-3, L, KBI]. In most situations the kernel K is symmetric or Hermitian (see, however, [B]). Appearance of J -symmetric kernels seems to be new.

8. The L -kernel [Part V, §2]

Consider the operator K in the Hilbert space

$$(8.1) \quad L^2(\mathbb{R}^*, du) \simeq L^2(\mathbb{R}_+, dx) \oplus L^2(\mathbb{R}_+, dx),$$

given by the kernel (7.4).

Theorem IV ([Part V, Theorem 2.4]). *Assume $|a| < \frac{1}{2}$. Then*

$$(8.2) \quad K = \frac{L}{1+L},$$

where L is bounded and given by the kernel

$$(8.3) \quad L(x, y) = \begin{bmatrix} 0 & \frac{\sigma}{\pi} \left(\frac{x}{y}\right)^a \frac{\exp\left(-\frac{x+y}{2}\right)}{x+y} \\ -\frac{\sigma}{\pi} \left(\frac{y}{x}\right)^a \frac{\exp\left(-\frac{x+y}{2}\right)}{x+y} & 0 \end{bmatrix}.$$

Recall that a and σ were defined in (7.1).

It is worth noting that (8.3) involves no special functions. Note also that $\mu = \frac{z-z'}{2}$ enters only in the scalar factor σ .

The formulas (7.2), (7.3), (8.2), (8.3) give a precise description of the process $\widetilde{\mathcal{P}}_{zz'}$ and, thus, of the initial process $\mathcal{P}_{zz'}$. We consider these formulas as our main result.

9. Spectral analysis [Part V, §3]

In this section we shall diagonalize the operators K and L . Denote by A the operator in $L^2(\mathbb{R}_+, dx)$ with the kernel

$$(9.1) \quad A(x, y) = \frac{\sigma}{\pi} \left(\frac{x}{y}\right)^a \frac{\exp\left(-\frac{x+y}{2}\right)}{x+y}.$$

It is bounded provided that $|a| < \frac{1}{2}$. Let A' denote the operator in $L^2(\mathbb{R}_+, dx)$ with the transposed kernel

$$A'(x, y) = A(y, x).$$

Then (8.2) implies that the blocks of the matrix Whittaker kernel are expressed through A and A' as follows

$$(9.2) \quad \begin{aligned} K_{++} &= AA'(1+AA')^{-1}, & K_{--} &= A'A(1+A'A)^{-1}, \\ K_{+-} &= (1+AA')^{-1}A, & K_{-+} &= -(1+A'A)^{-1}A'. \end{aligned}$$

Consider the following Sturm–Liouville differential operator on (\mathbb{R}_+, dx) depending on a as a parameter:

$$(9.3) \quad \mathcal{D}_a = -\frac{d}{dx} x^2 \frac{d}{dx} + \left(a - \frac{x}{2}\right)^2.$$

We have

$$\mathcal{D}_a A = A \mathcal{D}_{-a},$$

which implies that \mathcal{D}_a (formally) commutes with AA' and K_{++} while \mathcal{D}_{-a} commutes with $A'A$ and K_{--} .

Consider the following functions on \mathbb{R}_+ :

$$f_{a,m}(x) = \frac{1}{x} W_{a,im}(x), \quad m > 0.$$

We have

$$\mathcal{D}_a f_{a,m} = \left(a^2 + \frac{1}{4} + m^2 \right) f_{a,m}.$$

According to [W], the functions $f_{a,m}$ with a fixed form a continual basis in $L^2(\mathbb{R}_+, dx)$ diagonalizing \mathcal{D}_a . Moreover, the Plancherel formula looks as follows. For good enough functions $f(x)$ and $g(x)$

$$(f, g) = \frac{1}{\pi^2} \int_0^\infty (f, f_{a,m})(f_{a,m}, g) \cdot \Gamma\left(\frac{1}{2} - a + im\right) \Gamma\left(\frac{1}{2} - a - im\right) dm,$$

where (\cdot, \cdot) stands for the inner product in $L^2(\mathbb{R}_+, dx)$.

We have

$$Af_{-a,m} = \frac{\sigma}{\pi} \Gamma\left(\frac{1}{2} - a + im\right) \Gamma\left(\frac{1}{2} - a - im\right) f_{a,m},$$

$$A'f_{a,m} = \frac{\sigma}{\pi} \Gamma\left(\frac{1}{2} + a + im\right) \Gamma\left(\frac{1}{2} + a - im\right) f_{-a,m}.$$

Returning to the decomposition (8.1), we take $\{f_{a,m}\}_{m>0}$ as a basis in the first summand $L^2(\mathbb{R}_+, dx)$ and $\{f_{-a,m}\}$ as a basis in the second one.

Then we get a basis in the whole space $L^2(\mathbb{R}^*, du)$ diagonalizing both

$$L = \begin{bmatrix} 0 & A \\ -A' & 0 \end{bmatrix} \text{ and } K = \begin{bmatrix} AA'(1 + AA')^{-1} & (1 + AA')^{-1}A \\ -(1 + A'A)^{-1}A' & A'A(1 + A'A)^{-1} \end{bmatrix}.$$

In particular, we get the diagonalization of K_{++} :

$$K_{++}f_{a,m} = \frac{\cos(2\pi\mu) - \cos(2\pi a)}{\cos(2\pi\mu) + \cos(2\pi im)} f_{a,m}.$$

Note that for $a = 0$ the integral transform inverse to $f \mapsto \{(f, f_{a,m})\}_{m>0}$ is the Kontorovich–Lebedev transform, see [E2].

The above results show that the operators K and L with a fixed and μ varying form a commutative family.

10. Applications [Part III, Sections 2,5]

Now we shall give applications of the main results. These applications concern our initial object — the probability measures $P_{zz'}$ on Ω .

Theorem V ([Part III, Theorem 5.1]³). *Consider $\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots$ as random variables on the probability space $(\Omega, P_{zz'})$. Then with probability 1 there exist the limits*

$$(10.1) \quad \lim_{k \rightarrow \infty} \alpha_k^{1/k} = \lim_{k \rightarrow \infty} \beta_k^{1/k} = \exp\left(-\frac{\pi \sin(2\pi\mu)}{2\mu\sigma^2}\right) = \prod_{n \in \mathbb{Z}} e^{-1/(z-n)(z'-n)}.$$

This result is a kind of the strong law of large numbers. Roughly speaking, it means that both α_i 's and β_i 's decay as geometric progressions with the same exponent. Similar situation occurs for the Poisson–Dirichlet process, see [VS].

The same limit relation as (10.1) holds for $\tilde{\alpha}_k$'s and $\tilde{\beta}_k$'s.

The proof of (10.1) is based on the examination of $\tilde{\rho}_1^{(zz')}$ and $\tilde{\rho}_2^{(zz')}$, and it incorporates Kingman's argument from [Ki, Section 4.2].

Observe that the exponent of the decay does not change under the shifts $(z, z') \rightarrow (z + N, z' + N)$, $N \in \mathbb{Z}$. Moreover, the whole process describing the tails of the sequences $\{\alpha_i\}$ and $\{\beta_i\}$ turns out to be invariant with respect to these shifts. This periodicity is quite unexpected: the initial formula (2.3) has no apparent periodicity.

Another application relies on the examination of $\tilde{\rho}_1^{(zz')}$ alone.

Theorem VI ([Part III, Proposition 2.2]). *Let*

$$|\alpha| = \sum_{i=1}^{\infty} \alpha_i, \quad |\beta| = \sum_{i=1}^{\infty} \beta_i, \quad \psi(\cdot) = \frac{\Gamma'(\cdot)}{\Gamma(\cdot)}.$$

Then

$$\mathbb{E} |\alpha| = \sin(\pi z) \sin(\pi z') \left[\frac{z - z'}{2\pi \sin(\pi(z - z'))} \frac{z + z' - 1}{z z'} + \frac{\psi(-z') - \psi(-z)}{\pi \sin(\pi(z - z'))} \right],$$

and $\mathbb{E} |\beta|$ is given by the same formula with (z, z') replaced by $(-z, -z')$.

Here \mathbb{E} is the symbol of expectation.

³The last equality in (10.1) was kindly communicated to us by I. G. Macdonald (letter of December 2, 1998).

11. Tail process [Part III, Sections 3,4], [Part V, §4]

Here we study the asymptotic behavior of the process $\widetilde{\mathcal{P}}_{zz'}$ near the origin. The starting observation is that

$$(11.1) \quad \widetilde{\rho}_1^{(zz')}(u) \sim \frac{c}{|u|}, \quad u \rightarrow 0,$$

where

$$(11.2) \quad c = \frac{2\mu\sigma^2}{\pi \sin(2\pi\mu)}.$$

This asymptotic relation agrees with the rate of decay of α_i 's and β_i 's, see (10.1).

As before, we identify the phase space \mathbb{R}^* with the disjoint union of two copies of \mathbb{R}_+ and then in each copy of \mathbb{R}_+ we make the following change of variable

$$(11.3) \quad x \mapsto \xi = -c \ln x.$$

Theorem VII ([Part V, Theorem 4.1]). *Let $K'(\xi, \eta)$ denote the matrix Whittaker kernel on \mathbb{R} obtained from the matrix Whittaker kernel (7.4) by the change of variable (11.3). Then there exists the limit*

$$\lim_{M \rightarrow +\infty} K'(\xi + M, \eta + M) = \mathcal{K}(\xi, \eta).$$

Here $\mathcal{K}(\xi, \eta)$ is a translation invariant matrix kernel with $\mathcal{K}(\xi, \xi) \equiv 1$,

$$\mathcal{K}(\xi, \eta) = \begin{bmatrix} \mathcal{K}_{++}(\xi, \eta) & \mathcal{K}_{+-}(\xi, \eta) \\ \mathcal{K}_{-+}(\xi, \eta) & \mathcal{K}_{--}(\xi, \eta) \end{bmatrix} = \begin{bmatrix} F(\xi - \eta) & G(\xi - \eta) \\ -G(\eta - \xi) & F(\xi - \eta) \end{bmatrix},$$

where

$$(11.4) \quad F(\zeta) = \frac{B \sinh(A\zeta)}{A \sinh(B\zeta)},$$

$$G(\zeta) = \frac{1}{2\mu\sigma} \frac{(\sin(\pi\mu) \cos(\pi a)) \cosh(A\zeta) + (\sin(\pi a) \cos(\pi\mu)) \sinh(A\zeta)}{\cosh(B\zeta)},$$

and the constants A, B are as follows

$$B = \frac{\pi \sin(2\pi\mu)}{4\mu\sigma^2} > 0, \quad A = 2\mu B = \frac{\pi \sin(2\pi\mu)}{2\sigma^2}.$$

We call the point process on $\mathbb{R} \sqcup \mathbb{R}$ with the correlation functions given by the determinantal formula with the matrix kernel $\mathcal{K}(\xi, \eta)$ the *tail process* for $\widetilde{\mathcal{P}}_{zz'}$.

By the construction, it describes the behavior of $\tilde{\alpha}_i$'s and $\tilde{\beta}_i$'s with large i after the appropriate rescaling.

In particular, the kernel $\mathcal{K}_{++}(\xi, \eta)$ on \mathbb{R} describes the tail of $\{\tilde{\alpha}_i\}$ alone, and $\mathcal{K}_{--}(\xi, \eta)$ does the same for $\{\tilde{\beta}_i\}$. Since $\mathcal{K}_{++} = \mathcal{K}_{--}$, the tail properties of $\{\tilde{\alpha}_i\}$ and $\{\tilde{\beta}_i\}$ are identical.

The same kernel $\mathcal{K}_{++}(\xi, \eta)$ appears in the scaling limit of the correlation functions for the unlifted process $\mathcal{P}_{zz'}$ restricted to $(0, 1]$. However, this requires more sophisticated considerations, see [Part II, Sections 4.2, 4.3].

Recall that μ is either real or pure imaginary. According to this, the constant A is also either real or pure imaginary. In the latter case the hyperbolic sine in the numerator of (11.4) turns into the ordinary sine.

The next result is parallel to Theorem IV.

Theorem VIII ([Part V, Proposition 4.2]). *Let $|a| < \frac{1}{2}$. Then*

$$\mathcal{K} = \frac{\mathcal{L}}{1 + \mathcal{L}}$$

where \mathcal{L} is a bounded operator in $L^2(\mathbb{R}, d\xi) \oplus L^2(\mathbb{R}, d\xi)$ with the kernel

$$(11.5) \quad \mathcal{L}(\xi, \eta) = \begin{bmatrix} 0 & \frac{\sigma}{2\pi} \frac{\exp(-2aB(\xi-\eta))}{\cosh(B(\xi-\eta))} \\ -\frac{\sigma}{2\pi} \frac{\exp(-2aB(\eta-\xi))}{\cosh(B(\eta-\xi))} & 0 \end{bmatrix}$$

where B is as above.

Finally, note that all formulas of this section are invariant under the shifts $(z, z') \rightarrow (z + N, z' + N)$, $N \in \mathbb{N}$ (or, equivalently, $(a, \mu) \rightarrow (a + N, \mu)$), cf. Section 10.

12. Formalism of fermion point processes

[Part III, Section 1], [Part V, §1]

Here we shall discuss elementary general properties of the point processes with determinantal correlation functions [DVJ, Ma1, Ma2].

Let \mathfrak{X} be a phase space with reference measure dx , $K(x, y)$ be a kernel on \mathfrak{X} , and \mathcal{P} be a point process on \mathfrak{X} with the correlation functions

$$\rho_n(x_1, \dots, x_n) = \det [K(x_i, x_j)]_{i,j=1}^n.$$

Let \mathfrak{Y} be a subset of \mathfrak{X} such that

$$\int_{\mathfrak{Y}} \rho_1(x) dx < \infty.$$

Then the number of points in \mathfrak{Y} is finite with probability 1. Moreover, the probability to find exactly n points located in the infinitesimal volumes dx_1, \dots, dx_n around points x_1, \dots, x_n equals

$$(12.1) \quad \pi_n(x_1, \dots, x_n) dx_1 \dots dx_n = \frac{\det [L_{\mathfrak{Y}}(x_i, x_j)]_{i,j=1}^n}{\det(1 + L_{\mathfrak{Y}})} dx_1 \dots dx_n.$$

Here $L_{\mathfrak{Y}}(x, y)$ is the kernel of the operator $L_{\mathfrak{Y}}$ in $L^2(\mathfrak{Y}, dx)$ such that

$$K_{\mathfrak{Y}} = \frac{L_{\mathfrak{Y}}}{1 + L_{\mathfrak{Y}}}$$

where $K_{\mathfrak{Y}}$ is the operator in $L^2(\mathfrak{Y}, dx)$ with the kernel $K(x, y)$ restricted to \mathfrak{Y} .

Thus, in case of finite point configurations the operator $L = \frac{K}{1-K}$ has a clear probabilistic meaning — it gives the distribution functions π_n .

It is tempting to apply (12.1) in our case, when the operator L has especially simple form. Unfortunately, we cannot do this for the whole space because the point configurations are infinite. Of course, we can restrict ourselves to an appropriate region \mathfrak{Y} , but then the simple form of our $L(x, y)$ will be lost.

Again, in the case of finite configurations, if the space is a disjoint union of two pieces and L is written in block form according to this splitting, then vanishing of the diagonal blocks of L (as in (8.3) and (11.5)) exactly means that the random configuration has equally many points in each of the pieces (see [Part V, Proposition 1.7]). We do not know how to interpret such vanishing in our situation.

13. Distribution of $\tilde{\alpha}_1$ [Part III, Section 2]

Here we shall consider the lifted process $\widetilde{\mathcal{P}}_{zz'}$ restricted to $\mathbb{R}_+ \subset \mathbb{R}^*$; it is governed by the ‘++’ block of the matrix Whittaker kernel, see (7.3). We call $K_{++}(x, y)$ the *Whittaker kernel*.

Note that for any $\tau > 0$ the intersection of the random configuration with $\mathfrak{Y}(\tau) = [\tau, +\infty)$ is finite. By (12.1)

$$(13.1) \quad \text{Prob}\{\tilde{\alpha}_1 < \tau\} = \frac{1}{\det(1 + L_{\mathfrak{Y}(\tau)})} = \det(1 - K_{\mathfrak{Y}(\tau)}).$$

Here, following the notation of Section 12, the kernel of $K_{\mathfrak{Y}(\tau)}$ is obtained by restricting $K_{++}(x, y)$ on $[\tau, +\infty)$. As was pointed out by Tracy [T2], a modification of the argument from [TW3, Section V.B] allows to express the Fredholm determinant (13.1) through the Painlevé transcendent V.

14. Comparison with random matrices. Degenerations of the Whittaker kernel [Part III, Sections 1,6], [Part V, §5]

There are a lot of similarities between the processes $\widetilde{\mathcal{P}}_{zz'}$ and point processes arising from random matrices. Random matrix theory leads to various kernels: the sine kernel, the Bessel kernel, the Laguerre kernel, etc. (see [Me1, TW1–3]). All of them have the form

$$(14.1) \quad \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y},$$

and so does the Whittaker kernel $K_{++}(x, y)$, see (7.3). Here $\varphi(\cdot)$ and $\psi(\cdot)$ are solutions of certain linear second-order differential equations. The Fredholm determinants of many kernels of this form are expressed through Painlevé transcendents, see [TW3, Sections III and V]. Various kernels (14.1) restricted to suitable intervals commute with Sturm–Liouville operators (see [G], [Me1, §5.3], [TW1], [TW2]). The same is true for the Whittaker kernel restricted to $[\tau, +\infty)$, see [Part III, Section 6].

The Whittaker kernel degenerates to the Laguerre kernel of order N and parameter $\alpha > -1$ if we formally substitute $z' = N$, $z = N + \alpha$, see [Part III, Remark 2.4]. The Bessel kernel is the scaling limit of the Laguerre kernel as $N \rightarrow +\infty$. Similar scaling procedure applied to the Whittaker kernel leads to a two-parametric family of Bessel-type kernels, see [Part V, Theorem 5.1].

Likewise, the stationary kernels from Theorem VII generalize the sine kernel: if $\mu \rightarrow i\infty$, (11.4) tends to $\frac{\sin(\pi\zeta)}{\pi\zeta}$ irrespective to the behavior of a . The kernels $\mathcal{K}_{++}(\xi, \eta)$ have already appeared in [BCM, MCIN] in connection with so-called q -Hermite ensemble.

As for the matrix Whittaker kernel, it has a similarity with matrix kernels arising from two-matrix ensembles ([EM, Me2, MS]), see [Part IV, Section 3].

Acknowledgements

We would like to thank Craig A. Tracy for his letter [T2] and for drawing our attention to the works [MTW] and [T1]. We are also grateful to Ian G. MacDonald for valuable remarks (see, in particular, the footnote to Theorem V).

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