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MAXIMAL FUNCTIONS AND HILBERT TRANSFORMS ALONG VARIABLE FLAT CURVES

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Abstract. In this work we establish L^p boundedness for maximal functions and Hilbert transforms along variable curves in the plane, via L^2 estimates for certain singular integral operators with oscillatory terms.

§1. Introduction

In this paper, we study the $L^p(\mathbb{R}^2)$ boundedness for the maximal function $\mathcal{M}$ and the Hilbert transform $\mathcal{H}$ along variable curves. In our discussions, these are defined a priori on functions in $C_0^\infty(\mathbb{R}^2)$ by

$$\mathcal{M}f(x) = \sup_{0<h<\infty} \frac{1}{h} \left| \int_0^h f(x_1 - t, x_2 - S(x_1, x_1 - t))dt \right|$$

and

$$\mathcal{H}f(x) = \; p.v. \int_{-\infty}^{\infty} f(x_1 - t, x_2 - S(x_1, x_1 - t)) \frac{dt}{t},$$

where $S(x, y)$ is a suitable real-valued function vanishing on the diagonal.

We shall also consider the singular integral operators T_λ (acting on functions on the real line) which are of the form

$$T_\lambda f(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} e^{i\lambda S(x,y)} (x-y)^{-1} f(y)dy.$$

Local versions of the operators T_λ have been studied by Phong and Stein [PS], and by Pan [P] who proved the L^p boundedness of T_λ with bounds independent of λ when the mixed derivative S''_{xy} does not vanish to infinite order at any diagonal point (x_0, x_0). In [S] Seeger showed that for a certain class of phases S without this finite type property, the associated operators T_λ are also uniformly bounded on L^2. Here we extend the Seeger-type result, for a different, closely related (but not always directly comparable) class of phases, to all other p, $1 < p < \infty$.

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Theorem 1. *Let S be an antisymmetric function in $C^3(\mathbb{R}^2)$. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying that there exists a constant $B \geq 1$ such that*

$$g(Bt) \geq 2g(t),$$

and suppose that for some $E \geq 1$ and $A \geq 1$,

$$\begin{aligned} A^{-1}g(|x-y|) &\leq |S_1'(x,y)| \leq Ag(E|x-y|), \\ A^{-1}g(|x-y|) &\leq |S_2'(x,y)| \leq Ag(E|x-y|). \end{aligned} \tag{1}$$

Suppose furthermore that S_{112}''' is single-signed on $\mathbb{R}^2$. Then $\mathcal{M}$ is bounded on L^p for all $1 < p \leq \infty$, and $\mathcal{H}$ is bounded on L^p for all $1 < p < \infty$.

A local version of the theorem where the hypotheses are assumed on S in a neighbourhood of the diagonal and $\mathcal{M}$ and $\mathcal{H}$ are suitably modified also holds. Included in this setting are examples such as $S(x,y) = e^{-(x-y)^{-2}}g(x,y)$, $g(x,x) \neq 0$, or $S(x,y) = e^{-(x-y)^{-2}h(x,y)}$, $h(x,x) > 0$, (defined thus for $y > x$ and extended to be antisymmetric) where g and h are smooth. Thus the theorem covers certain "flat" curves which is a point of principal interest.

It is well-known that estimates for $\mathcal{H}$ yield uniform estimates for T_λ. Indeed, if $\mathcal{F}_2$ denotes the Fourier transform in the second variable then $\mathcal{F}_2\mathcal{H}f(x_1,\lambda) = T_\lambda(\mathcal{F}_2 f(\cdot,\lambda))(x_1)$. By applying Plancherel's theorem we see that

$$\sup_{\lambda\in\mathbb{R}} \|T_\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|\mathcal{H}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}.$$

Moreover, a variant of de Leeuw's theorem implies that if $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^2)$ then

$$\sup_{\lambda\in\mathbb{R}} \|T_\lambda\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leq \|\mathcal{H}\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}.$$

Thus, an immediate consequence of Theorem 1 is the following corollary:

Corollary 2. *If S satisfies the same hypotheses as in Theorem 1, then for any p, $1 < p < \infty$, we have*

$$\sup_{\lambda\in\mathbb{R}} \|T_\lambda\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leq C.$$

In the work of Seeger [S], S, in addition to (1), is assumed to satisfy a condition on its second order derivatives (related to the so-called k-quasimonotonicity condition.) We, in contrast, demand a condition on the third-order derivative S_{112}'''.

In the translation-invariant setting, when $S(x,y) = \gamma(x-y)$, our condition reduces to $\gamma''' \geq 0$ (see [DR]) which implies the infinitessimal doubling (see [CCVWW]) and hence doubling of γ', (see [NVWW1], [NVWW2] and [CCCDRVWW]). On the other hand, the single-signedness of S_{112}''' is the minimal hypothesis for our proof — based upon smoothing estimates for relatives of T_λ where the Hilbert singularity is replaced by a smooth cut-off function — to work according to what is currently known about smoothing for oscillatory integral operators. (See for example [PS1] and [CCW]). A general result for more general versions of $\mathcal{H}$ in which the variable curve has rotational curvature not vanishing to infinite order has recently been established in [CNSW]. In our setting, that condition reduces to S''_{xy} not vanishing to infinite order on the diagonal. The first results for the non-translation invariant flat case (i.e. in which the rotational curvature may vanish to infinite order on the diagonal) were obtained in [CWW2].

The idea of the proof is to consider pieces of the operators like the following

$$S_j f(x) = \frac{1}{2^{j+1}} \int_{2^j<|t|\le 2^{j+1}} f(x_1-t, x_2 - S(x_1, x_1-t))dt,$$

and

$$T_j f(x) = \int_{2^j<|t|\le 2^{j+1}} f(x_1-t, x_2 - S(x_1, x_1-t))\frac{dt}{t}.$$

Thus $\mathcal{H}f(x) = \sum_j T_j f(x)$ and $\mathcal{M}$ is bounded for a non-negative function f by $\mathcal{M}f(x) \le C \sup_j |S_j f(x)|$. And then we can apply the following generalization of the Cotlar-Stein lemma, see [C].

Proposition 3. *(Almost-Orthogonality Principle) Assume that $\{Q_j\}$ satisfies $\sum_{j\in\mathbb{Z}} Q_j = I$. Assume that*

$$\|Q_j^* Q_k\|_{2-2} + \|Q_j Q_k^*\|_{2-2} \le C 2^{-\epsilon|j-k|},$$

and that

$$\|\sum_j \pm Q_j\|_{p_0-p_0} + \|\sum_j \pm Q_j^*\|_{p_0-p_0} \le C,$$

for some $p_0 \in (1,2)$. Suppose that $Q_j = P_j - P_{j+1}$ where $P_j \ge 0$ and $\|\sup_j |P_j f|\|_r \le C_r \|f\|_r$ for $p_0 \le r \le 2$. Assume also that

$$\|(\sum_k |Q_k g_k|^2)^{1/2}\|_{p_0'} \le C\|(\sum_k |g_k|^2)^{1/2}\|_{p_0'}.$$

Suppose that $\{T_j\}$, $\{S_j\}$ satisfy

$$|T_j f| \le S_j |f|$$

where $S_j \ge 0$. Assume that $\|S_j\|_{r-r} \le C$ for $p_0 \le r \le 2$. Moreover, assume that

$$\|(S_j - P_j)Q_{j+k}^*\|_{2-2} + \|(S_j-P_j)^* Q_{j+k}\|_{2-2} \le C 2^{-\epsilon|k|}, \tag{2}$$

and

$$\|T_j Q_{j+k}^*\|_{2-2} + \|T_j^* Q_{j+k}\|_{2-2} \leq C 2^{-\epsilon|k|}. \tag{3}$$

Then $f \to \sup_j |S_j f(x)|$ *and* $\sum_j T_j$ *are bounded on* L^p, $p_0 < p \leq 2$.

In order to define the appropriate Littlewood-Paley decomposition $I = \sum_j Q_j$ we define the dilations,

$$A(t) = \begin{pmatrix} t & 0 \\ 0 & G(t) \end{pmatrix} \quad \text{with} \quad G(t) = \int_0^t g(s)ds.$$

Similar dilations were first explicitly used in the flat translation invariant case in [CCVWW]. The collection $\{A(t)\}$ satisfies the Rivière condition

$$\|A(s)^{-1}A(t)\| \leq C\left(\frac{t}{s}\right)^{\epsilon} \quad \text{for} \quad s \geq t, \quad \text{for some} \quad \epsilon > 0.$$

In fact in this case it is true with $\epsilon = 1$ and it is enough to show that $\frac{G(t)}{G(s)} \leq C\frac{t}{s}$ for $s \geq t$. Observe that if $s \geq t$ there exists a natural number k such that $B^k t \leq s \leq B^{k+1}t$ and we have

$$\begin{aligned} \frac{1}{t}G(t) &= \frac{1}{t}\int_0^t g(u)du \leq \frac{1}{2t}\int_0^t g(Bu)du \leq \cdots \leq \frac{1}{2^k t}\int_0^t g(B^k u)du \\ &= \frac{1}{B^k 2^k t}\int_0^{B^k t} g(u)du \leq \frac{1}{B^k 2^k t}G(s) = \frac{s}{t}\frac{1}{B^k 2^k}\frac{G(s)}{s} \leq B\frac{G(s)}{s}. \end{aligned}$$

Let ϕ be a nonnegative $C_0^\infty(\mathbb{R}^2)$ function such that $\int \phi = 1$. We set the initial averaging operator

$$P_0 f(x) = \int \phi(x-y)f(y)dy;$$

then an approximation of the identity (with $P_j \to I$ as $j \to -\infty$ and $P_j \to 0$ as $j \to \infty$) is given by

$$P_j f(x) = \int (det A_j)^{-1}\phi(A_j^{-1}(x-y))f(y)dy \quad \text{with} \quad A_j = A(2^j).$$

The natural Littlewood-Paley difference operators Q_j are then $Q_j = P_j - P_{j+1}$.

According to [CVWW] and [CWW1], the conditions on the operators P_j and Q_j in the almost orthogonality lemma are satisfied for any p_0, $1 < p_0 < \infty$; just the Rivière condition is required. Therefore, subject to having verified (2) and (3), Proposition 3 shows that $\mathcal{M}$ and $\mathcal{H}$ are bounded on $1 < p \leq 2$. But the maximal function is trivially bounded on L^∞, thus it maps continuously L^p into L^p for $1 < p \leq \infty$. And for $\mathcal{H}$ we notice that the original problem itself is selfadjoint, so the boundedness of $\mathcal{H}$ for $1 < p \leq 2$ implies its boundedness for $2 \leq p < \infty$.

It hence remains to prove (2) and (3).

2. The curves and their normalization

Let $S(x,y)$ be an antisymmetric function. Then it is easy to see that $S_1'(z,x) = -S_2'(x,z)$, $S_{11}''(z,x) = -S_{22}''(x,z)$, $S_{12}''(z,x) = -S_{12}''(x,z)$ (consequently $S_{12}''(x,x)=0$), and $S_{112}'''(z,x) = -S_{122}'''(x,z)$. Since we also assume that S_{112}''' does not change sign then S_{122}''' does not change its sign either and its sign is opposite to that of S_{112}'''. On the other hand, by applying the mean value theorem we get

$$\operatorname{sgn} S_{12}''(z,x) = \operatorname{sgn} S_{112}''' \operatorname{sgn}(z-x). \tag{4}$$

Finally, by using $S_1'(x,x) = S_2'(x,x) = 0$ one may see that

$$\operatorname{sgn} S_1' = -\operatorname{sgn} S_{112}''' \quad \text{and} \quad \operatorname{sgn} S_2' = \operatorname{sgn} S_{112}''', \tag{5}$$

since $S_1'(z,x) = S_{12}''(z,\nu)(x-z)$ for some $\nu \in \overline{zx}$ (the line segment joining z to x) and so $\operatorname{sgn} S_1'(z,x) = \operatorname{sgn} S_{112}''' \operatorname{sgn}(z-\nu)\operatorname{sgn}(x-z)$).

Lemma 4. *If S_{112}''' is single-signed and S is antisymmetric then for any x and y*

$$|S_1'(x,y)| \le |x-y||S_{12}''(x,y)| \quad \textit{and} \quad |S_2'(x,y)| \le |x-y||S_{12}''(x,y)|.$$

Proof. We use the mean value theorem:

$$|S_1'(x,y)| = |S_1'(x,y) - S_1'(x,x)| = |S_{12}''(x,u)||x-y| \quad \text{for some} \quad u \in \overline{xy}$$

If $x<y$ then $x<u<y$ and $|S_{12}''(x,u)| = -\operatorname{sgn} S_{112}''' S_{12}''(x,u)$. Since $\operatorname{sgn} S_{112}''' = -\operatorname{sgn} S_{122}'''$ this function is increasing in u and so $|S_{12}''(x,u)| \le |S_{12}''(x,y)|$. When $x>y$, $|S_{12}''(x,u)| = \operatorname{sgn} S_{112}''' S_{12}''(x,u)$ is decreasing in u and thus also $|S_{12}''(x,u)| \le |S_{12}''(x,y)|$.

To prove the estimate for $|S_2'(x,y)|$ we can repeat the proof, or realize that $|S_2'(x,y)| = |S_1'(y,x)| \le |y-x||S_{12}''(y,x)| = |y-x||S_{12}''(x,y)|$, since S_{12}'' is also antisymmetric. □

In our development we shall need to work with normalized versions of $S(.,.)$, that is, for fixed j

$$\tilde{S}(x,y) = \frac{S(2^j x, 2^j y)}{G(2^j)}.$$

It is easy to check several facts concerning them that we shall need later on. First,

$$\text{for } i=1,2 \quad |\tilde{S}_i'(x,y)| \ge A^{-1} \quad \text{whenever} \quad |x-y| \ge 1. \tag{6}$$

To see this observe that

$$|\tilde{S}_i'(x,y)| = \frac{2^j|S_i'(2^j x, 2^j y)|}{G(2^j)} \ge \frac{A^{-1}2^j g(|2^j x - 2^j y|)}{G(2^j)} \ge \frac{A^{-1}2^j g(2^j)}{G(2^j)} \ge A^{-1},$$

where the last inequality is true because g is a non-decreasing function.

By Lemma 4 whenever $|x-y| \le C_0$ then

$$|\tilde{S}'_1(x,y)| \le C_0 |\tilde{S}''_{12}(x,y)| \quad \text{and} \quad |\tilde{S}'_2(x,y)| \le C_0 |\tilde{S}''_{12}(x,y)| \tag{7}$$

since

$$|\tilde{S}'_1(x,y)| = 2^j \frac{|S'_1(2^j x, 2^j y)|}{G(2^j)} \le 2^j |2^j x - 2^j y| \frac{|S''_{12}(2^j x, 2^j y)|}{G(2^j)} = |x-y||\tilde{S}''_{12}(x,y)|.$$

With this observation we can prove the following lemma:

Lemma 5. *If S is antisymmetric and S'''_{112} is single-signed then, for any x, y and z such that either $-C_1 \le z-x, z-y \le 0$ with $x \le y$, or $0 \le z-x, z-y \le C_1$ for $x \ge y$, there exists a constant C such that*

$$\frac{|\tilde{S}'_1(z,y)| + |\tilde{S}'_1(z,x)| + |\tilde{S}'_2(z,x)|}{|\tilde{S}'_1(z,x) - \tilde{S}'_1(z,y)|} \le \frac{C}{|x-y|}.$$

Proof. We consider the case $-C_1 \le z-x, z-y \le 0$ which implies $|x-y| \le 2C_1$ with $x \le y$ (the proof for the case $0 \le z-x, z-y \le C_1$ for $x \ge y$ is a repetition of the following arguments). We have that

$$\begin{aligned} |\tilde{S}'_1(z,x) - \tilde{S}'_1(z,y)| &= \int_x^y -\operatorname{sgn} \tilde{S}'''_{112} \cdot \tilde{S}''_{12}(z,u) du \\ &\ge -\operatorname{sgn} \tilde{S}'''_{112} \cdot \tilde{S}''_{12}(z,x)(y-x) = |\tilde{S}''_{12}(z,x)||y-x|, \end{aligned}$$

where we have used that the function inside the integral is increasing in u. By (7), as $|z-x| \le C_1$ then

$$|\tilde{S}'_1(z,x) - \tilde{S}'_1(z,y)| \ge c|\tilde{S}'_1(z,x)||y-x|$$

and

$$|\tilde{S}'_1(z,x) - \tilde{S}'_1(z,y)| \ge c|\tilde{S}'_2(z,x)||y-x|.$$

Then we just need to prove the lemma for $|\tilde{S}'_1(z,y)|$. If $|\tilde{S}'_1(z,x)| \ge \frac{|\tilde{S}'_1(z,y)|}{2}$ then with the previous estimate we get also $|\tilde{S}'_1(z,x) - \tilde{S}'_1(z,y)| \ge c|\tilde{S}'_1(z,y)||y-x|$. But otherwise $|\tilde{S}'_1(z,x) - \tilde{S}'_1(z,y)| \ge \frac{1}{2}|\tilde{S}'_1(z,y)| \ge c|\tilde{S}'_1(z,y)||y-x|$, since $|y-x| \le 2C_1$. □

3. The heart of the proof

If T is an integral operator on $\mathbb{R}^n$ with distribution kernel $K(x,y)$, and $A \in GL(n,\mathbb{R})$, we let A_*T be the operator whose kernel is $(det A)^{-1} K(A^{-1}x, A^{-1}y)$. Thus $\|A_*T\|_{p-p} = \|T\|_{p-p}$ for all $1 \le p \le \infty$. In the case that T is the Hilbert transform along a curve $\Gamma(x,t)$, then A_*T becomes the Hilbert transform along the curve $A_*\Gamma$, where $(A_*\Gamma)(x,t) = A[\Gamma(A^{-1}x,t)]$.

We just need to prove estimates (2) and (3). By the essential self-adjointness of the problem, it suffices to prove either the first or the second inequalities in (2) and (3). For $k > 0$, they are a direct consequence of the smoothness of $\{P_j\}$, the support properties of $\{T_j, S_j, P_j\}$ and the fact that $T_j 1 = T_j^* 1 = (S_j - P_j)1 = (S_j - P_j)^* 1 = 0$. For instance, we indicate how to prove that $\|T_j^* Q_{j+k}\|_{2-2} \leq C2^{-\epsilon k}$; for this it suffices to show that $\|T_j^* P_{j+k}\|_{2-2} \leq C2^{-\epsilon k}$. Moreover, by setting $T_{jk}^* f(x) = A_{j+k_*}^{-1} T_j^* f(x)$ it is equivalent to the estimate $\|T_{jk}^* P_0\|_{2-2} \leq C2^{-\epsilon k}$. To prove that we just need the cancellation property $T_{jk}^* 1 = 0$ and that T_{jk}^* has its distribution kernel supported in $\{(x,y) : |x-y| \leq C2^{-\epsilon k}\}$. This reduces to seeing that if $|t| \leq 2^{j+1}$, $|x - A_{j+k_*}^{-1}\Gamma(x,t)| \leq C2^{-\epsilon k}$. Now

$$|x - A_{j+k_*}^{-1}\Gamma(x,t)| \leq \left|\frac{t}{2^{j+k}}\right| + \left|\frac{S(2^{j+k}x_1 + t, 2^{j+k}x_1)}{G(2^{j+k})}\right|.$$

To handle the second term, notice that since $S_1'(x,x) = 0$ then

$$\begin{aligned} |S(2^{j+k}x_1 + t, 2^{j+k}x_1)| &= \left|\int_0^t S_1'(2^{j+k}x_1 + s, 2^{j+k}x_1)ds\right| \\ &\leq \int_0^{|t|} Ag(Es)ds \leq CG(E|t|), \end{aligned}$$

which is smaller than or equal to $CG(E2^{j+1})$. The support condition now follows from the Rivière property. (The estimate $G(E2^{j+1})/G(2^{j+k}) \leq C2^{-\epsilon k}$ holds for any $k \geq k_0$ with k_0 such that $2^{k_0-1} \geq E$, but otherwise $\|T_j^* Q_{j+k}\|_{2-2} \leq C \leq C2^{-k}$.)

When $k \leq 0$, since $\|Q_{j+k}P_j^*\|_{2-2} \leq C2^{\epsilon k}$ then $\|Q_{j+k}(S_j - P_j)^*\|_{2-2} \leq C2^{\epsilon k}$ is equivalent to $\|Q_{j+k}S_j^*\|_{2-2} \leq C2^{\epsilon k}$, and the bound for $\|Q_{j+k}T_j^*\|_{2-2}$ will follow exactly the same argument.

Now we have to break up the operator S_j^* into two pieces determined by whether or not t is positive, and we work with $(\tilde{S_j^*}) = A_{j\ *}^{-1} S_j^*$. Then we set the normalized "positive" part of the operator S_j^* as follows

$$(\tilde{S_j^*})^+ f(x) = \int f(x_1+t, x_2+\tilde{S}(x_1+t,x_1))\alpha^+(t)dt, \text{ with } \tilde{S}(x,y) = \frac{S(2^j x, 2^j y)}{G(2^j)},$$

where α^+ is a real-valued smoothed-out version of $\chi_{[1,2]}$. (The corresponding kernel for the case $(S_j^*)^-$ is with α^- being a smoothed-out version of $\chi_{[-2,-1]}$). We write $\tilde{Q}_{j+k} = A_{j\ *}^{-1} Q_{j+k}$. Therefore, we need to show that

$$\|\tilde{Q}_{j+k}(\tilde{S_j^*})^+\|_{2-2} \leq C2^{\epsilon k} \quad \text{and} \quad \|\tilde{Q}_{j+k}(\tilde{S_j^*})^-\|_{2-2} \leq C2^{\epsilon k}.$$

Since the two estimates are similar we concentrate only on the first.

Let $K : \mathbb{R}^m \to C$ be a kernel, and $A : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^q$ be a function. Let $Tf(x) = \int_{\mathbb{R}^m} f(x_1 - y_1, x_2 - A(x_1, y))K(y)dy$ where $(x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and $(y_1, y_2) \in \mathbb{R}^p \times \mathbb{R}^{m-p}$. Define $T_\lambda h(x) = \int_{\mathbb{R}^m} h(x - y_1)e^{i\lambda A(x,y)}K(y)dy$ where now $x \in \mathbb{R}^p$. Then $(TS^*)_\lambda = T_\lambda S^*_\lambda$, and Plancherel's theorem in the $x_2 \in \mathbb{R}^q$ variable shows that $\|T\|_{2-2} = \sup_\lambda \|T_\lambda\|_{2-2}$. Thus, in our case at hand, it suffices to prove

$$\|(\tilde{Q}_{j+k})_\lambda((\tilde{S^*_j})^+)_\lambda\|_{2-2} \leq C2^{\epsilon k},$$

uniformly in λ, or indeed

$$\|(\tilde{Q}_{j+k})_\lambda((\tilde{S^*_j})^+)_\lambda((\tilde{S_j})^+)_\lambda(\tilde{Q}^*_{j+k})_\lambda\|_{2-2} \leq C2^{\epsilon k}. \tag{8}$$

Now the convolution kernel of $\tilde{P}_{j+k}$ can be written as

$$\frac{1}{2^k}\Phi_1\left(\frac{x_1}{2^k}\right)\frac{G(2^j)}{G(2^{j+k})}\Phi_2\left(\frac{G(2^j)x_2}{G(2^{j+k})}\right),$$

for some even functions Φ_1 and Φ_2 such that Φ_1 is supported in $[-2, 2]$, and Φ_2 is such that $\widehat{\Phi_2}$ is identically one in $[-1, 1]$ and also supported in $[-2, 2]$. By taking the Fourier transform in the second variable we have

$$\begin{aligned} Ker(\tilde{Q}_{j+k})_\lambda(x_1) &= \frac{1}{2^k}\Phi_1\left(\frac{x_1}{2^k}\right)\widehat{\Phi_2}\left(\tfrac{G(2^{j+k})\lambda}{G(2^j)}\right) \\ &\qquad - \frac{1}{2^{k+1}}\Phi_1\left(\frac{x_1}{2^{k+1}}\right)\widehat{\Phi_2}\left(\tfrac{G(2^{j+k+1})\lambda}{G(2^j)}\right) \\ &= \frac{1}{2^{k+1}}\Phi_1\left(\frac{x_1}{2^{k+1}}\right)\left[\widehat{\Phi_2}\left(\tfrac{G(2^{j+k})\lambda}{G(2^j)}\right) - \widehat{\Phi_2}\left(\tfrac{G(2^{j+k+1})\lambda}{G(2^j)}\right)\right] \\ &\qquad + \frac{1}{2^k}\Psi\left(\frac{x_1}{2^k}\right)\widehat{\Phi_2}\left(\tfrac{G(2^{j+k})\lambda}{G(2^j)}\right) \\ &\equiv I_\lambda(x_1) + II_\lambda(x_1) \end{aligned}$$

where $\Psi(x) = \Phi_1(x) - \frac{1}{2}\Phi_1(\frac{x}{2})$ and so $\int \Psi = 0$.

Since $\widehat{\Phi_2}$ is identically one in $[-1, 1]$, $I_\lambda(x_1) = 0$ unless $\frac{G(2^{j+k+1})|\lambda|}{G(2^j)} \geq 1$, and as $\left|\widehat{\Phi_2}\left(\frac{G(2^{j+k})\lambda}{G(2^j)}\right) - \widehat{\Phi_2}\left(\frac{G(2^{j+k+1})\lambda}{G(2^j)}\right)\right| \leq 2$, the estimate (8) when we consider the part I_λ of the $Ker(\tilde{Q}_{j+k})_\lambda$ follows from $\|(\tilde{Q}^*_{j+k})_\lambda\|_{2-2} \leq C$ and

$$\|((\tilde{S^*_j})^+)_\lambda((\tilde{S_j})^+)_\lambda\|_{2-2} \leq C2^{\epsilon k}, \quad \text{when} \quad \frac{G(2^{j+k+1})|\lambda|}{G(2^j)} \geq 1. \tag{9}$$

It is not difficult to see that the kernel of $((\tilde{S}^*_j)^+)_\lambda((\tilde{S}_j)^+)_\lambda$ is, as a function of x and y,

$$K^+_\lambda(x, y) = \int e^{i\lambda[\tilde{S}(z,y) - \tilde{S}(z,x)]}\alpha^+(z - y)\alpha^+(z - x)dz.$$

To show (9), since K_λ^+ is supported in $|x-y| \leq 5$, it suffices to prove that $\int |K_\lambda^+(x,y)|^2 dx \leq C/|\lambda|$ uniformly in y, since both the Rivière condition and $G(2^{j+k+1})|\lambda|/G(2^j) \geq 1$ then imply $\int |K_\lambda^+(x,y)|^2 dx \leq C2^{\epsilon k}$. In order to do that we first observe that by Van der Corput's lemma $|K_\lambda(x,y)| \leq C/(|\lambda||x-y|)$. Indeed, set $u(z) = \tilde{S}(z,y) - \tilde{S}(z,x)$ then $u'(z) = \tilde{S}_1'(z,y) - \tilde{S}_1'(z,x)$ and $u''(z) = \tilde{S}_{11}''(z,y) - \tilde{S}_{11}''(z,x) = \tilde{S}_{112}'''(z,\nu)(y-x)$ for fixed x and y and since $\tilde{S}_{112}'''$ is single-signed, u'' is single-signed and $|u'(z)| = |\tilde{S}_1'(z,y) - \tilde{S}_1'(z,x)| \geq C|x-y||\tilde{S}_1'(z,x)| \geq c|x-y|$ (see Lemma 5 and (6)). Then,

$$\int |K_\lambda^+(x,y)|^2 dx \leq \int_{\{x:|x-y|<\delta\}} C dx + \int_{\{x:|x-y|>\delta\}} \frac{1}{\lambda^2|x-y|^2} dx \leq C\delta + C\frac{1}{\lambda^2\delta} \leq C\frac{1}{|\lambda|},$$

by taking $\delta = 1/|\lambda|$.

Thus the contribution to (8) arising from I_λ is under control.

Now we need to consider, for technical reasons, separately the cases $\chi_{x\geq y}$ and $\chi_{x\leq y}$. Let A be the operator with kernel $K_\lambda^+(x,y)\chi_{\{x\geq y\}}$ and let B be the operator with kernel $K_\lambda^+(x,y)\chi_{\{x\leq y\}}$; since $\overline{K_\lambda^+}(y,x) = K_\lambda^+(x,y)$ we have that $A^* = B$, and in order to prove (8) it is enough to prove it either for A or for B. Since we have a trivial estimate for the L^∞ operator norm it suffices to show that the L^1 norm of the operator has the decay we want, and in fact it is enough to show that

$$\int \left| \int II_\lambda(x-x')K_\lambda'(x',y)dx' \right| dx \leq C2^{\epsilon k}$$

uniformly in $y \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and K_λ' denotes K_λ^+ restricted to $x \geq y$. But $\int II_\lambda(x-x')K_\lambda'(x',y)dx' = C\Psi_k *_1 K_\lambda'(\cdot\,,y)$ ($*_1$ means convolution in the first variable and $\Psi_k(x) = \frac{1}{2^k}\Psi(\frac{x}{2^k})$), and therefore since $\int \Psi_k = 0$ the following lemma finishes the proof.

Lemma 6. *For $K_\lambda'(x,y) = K_\lambda^+(x,y)\chi_{\{x\geq y\}}(y)$, we have*

$$\int |K_\lambda'(x+h,y) - K_\lambda'(x,y)|\, dx \leq C|h|^{\frac{1}{2}}.$$

Proof. Let us assume $|h| \leq \frac{1}{4}$ otherwise the conclusion is clear, and let us assume for simplicity that $h > 0$. Then

$$\begin{aligned}
&\int |K'_\lambda(x+h,y) - K'_\lambda(x,y)|\, dx \\
&\leq \int \left| \int \left[e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x+h)]} - e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]} \right] \alpha^+(z-y)\alpha^+(z-x-h) dz \right| dx \\
&\quad + \int \left| \int e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]} \Big(\alpha^+(z-y)\alpha^+(z-x) \right. \\
&\qquad\qquad\qquad\qquad \left. - \alpha^+(z-y)\alpha^+(z-x-h) \Big) dz \right| dx \\
&= I + II.
\end{aligned}$$

The second term is fine because, since we are working with normalized pieces of curves the regions of integration are finite, and the function α^+ is smooth enough, so it is clearly $O(|h|)$. The first term satisfies

$$\begin{aligned}
I &= \int \left| \int \int_0^h \frac{\partial}{\partial t} e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x+t)]} dt\ \alpha^+(z-y)\alpha^+(z-x-h) dz \right| dx \\
&= \int \left| \int \int_0^h i\lambda \tilde{S}'_2(z,x+t) e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x+t)]} dt\ \alpha^+(z-y)\alpha^+(z-x-h) dz \right| dx \\
&\leq |h| \sup_{0\leq t\leq h} \int \left| \int \lambda \tilde{S}'_2(z,x+t) e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x+t)]} \alpha^+(z-y)\alpha^+(z-x-h) dz \right| dx \\
&= |h| \sup_{0\leq t\leq h} \int \left| \int \lambda \tilde{S}'_2(z,x) e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]}\ \alpha^+(z-y)\alpha^+(z-x-h+t) dz \right| dx.
\end{aligned}$$

Then, it suffices to show that

$$\int_{x:|x-y|>\delta} \left| \int \lambda \tilde{S}'_2(z,x) e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]} \alpha^+(z-y)\alpha^+(z-x-h+t) dz \right| dx \leq \frac{C}{\delta},$$

independently of $0 \leq t \leq h$, because then

$$\int |K'_\lambda(x+h,y) - K'_\lambda(x,y)|\, dx \leq C\delta + C\frac{|h|}{\delta} \leq C|h|^{\frac{1}{2}},$$

by taking $\delta = |h|^{\frac{1}{2}}$. Now we integrate by parts with respect to z and obtain

$$\begin{aligned}
&\int \lambda \tilde{S}'_2(z,x) e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]} \alpha^+(z-y)\alpha^+(z-x-h+t) dz \\
&\quad = -\frac{1}{i} \int \frac{\partial}{\partial z} \left(\tfrac{\tilde{S}'_2(z,x)}{\tilde{S}'_1(z,y)-\tilde{S}'_1(z,x)} \right) e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]}\ \alpha^+(z-y)\alpha^+(z-x-h+t) dz \\
&\qquad - \frac{1}{i} \int \frac{\partial}{\partial z} \left(\alpha^+(z-y)\alpha^+(z-x-h+t) \right) \tfrac{\tilde{S}'_2(z,x)}{\tilde{S}'_1(z,y)-\tilde{S}'_1(z,x)} e^{i\lambda[\tilde{S}(z,y)-\tilde{S}(z,x)]} dz.
\end{aligned}$$

Since for $K'_\lambda(x,y) = K^+_\lambda(x,y)\chi_{\{x\geq y\}}(y)$ we are under the hypothesis of Lemma 5 and then we get $|\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)| \geq C|x-y||\tilde{S}'_2(z,x)|$. We shall be finished if we can show

$$\int_{x:|x-y|>\delta} \int \left| \frac{\partial}{\partial z} \left(\frac{\tilde{S}'_2(z,x)}{\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)} \right) \right| dz dx \leq \frac{C}{\delta}$$

for $1 < z-y < 2$ and $0 < z-x < 3$ (recall that $0 < t < h$ and $h \leq \frac{1}{4}$). But

$$\begin{aligned} \frac{\partial}{\partial z} \frac{\tilde{S}'_2(z,x)}{\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)} &= \frac{-\tilde{S}''_{12}(z,x)\tilde{S}'_1(z,x)}{[\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)]^2} + \frac{\tilde{S}''_{12}(z,x)\tilde{S}'_1(z,y)}{[\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)]^2} \\ &\quad - \frac{\tilde{S}'_2(z,x)[\tilde{S}''_{11}(z,y) - \tilde{S}''_{11}(z,x)]}{[\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)]^2} \\ &= M + N + L. \end{aligned}$$

Now, it is very important to check that each of the terms has single sign and that $\operatorname{sgn} M = \operatorname{sgn} L$; fortunately we know the signs precisely in terms of the sign of $\tilde{S}'''_{112}$ (we use that $\tilde{S}''_{11}(z,y) - \tilde{S}''_{11}(z,x) = \tilde{S}'''_{112}(z,\nu)(y-x)$.) Indeed then, by (4) and (5)

$$\begin{aligned} \operatorname{sgn} M &= \operatorname{sgn}(-\tilde{S}''_{12}(z,x)\tilde{S}'_1(z,x)) = \operatorname{sgn} \tilde{S}'''_{112} \operatorname{sgn}(x-z)(-\operatorname{sgn} \tilde{S}'''_{112}) \\ &= \operatorname{sgn}(z-x), \\ \operatorname{sgn} N &= \operatorname{sgn}(\tilde{S}''_{12}(z,x)\tilde{S}'_1(z,y)) = -\operatorname{sgn} \tilde{S}'''_{112} \operatorname{sgn}(x-z)(-\operatorname{sgn} \tilde{S}'''_{112}) \\ &= \operatorname{sgn}(x-z), \quad \text{and} \\ \operatorname{sgn} L &= -\operatorname{sgn} \tilde{S}'''_{112} \operatorname{sgn} \tilde{S}'''_{112} \operatorname{sgn}(y-x) \\ &= \operatorname{sgn}(x-y). \end{aligned}$$

So since $z - x > 0$ then M and N have single sign. Also since we need to prove the lemma only for $K'_\lambda(x,y) = K^+_\lambda(x,y)\chi_{\{x\geq y\}}(y)$ then $\operatorname{sgn} M = \operatorname{sgn} L$. Therefore, we can use that

$$\iint |M+N+L| dz dx \leq \left| \iint M dz dx + \iint L dz dx \right| + \left| \iint N dz dx \right|.$$

The double integral of N is, for some boundary points x^* and x^{**}, by using Lemma 5 and the fact that we are always integrating over bounded intervals, controlled by

$$\left| \iint N dx dz \right| \leq \int \left| \left. \frac{\tilde{S}'_1(z,y)}{\tilde{S}'_1(z,y) - \tilde{S}'_1(z,x)} \right]_{x^*}^{x^{**}} \right| dz \leq \frac{C}{\delta} \int dz \leq \frac{C}{\delta}.$$

Now we integrate M, first in the variable x and then with respect to z,

$$\iint M dx dz = \int \frac{-\tilde{S}_1'(z,x)}{\tilde{S}_1'(z,y) - \tilde{S}_1'(z,x)}\Bigg]_{x^*}^{x^{**}} dz + \iint \frac{\tilde{S}_{12}''(z,x)}{\tilde{S}_1'(z,y) - \tilde{S}_1'(z,x)} dx dz.$$

In the same way, but first in the variable z and then with respect to x,

$$\iint L dz dx = \int \frac{\tilde{S}_2'(z,x)}{\tilde{S}_1'(z,y) - \tilde{S}_1'(z,x)}\Bigg]_{z^*}^{z^{**}} dx - \iint \frac{\tilde{S}_{12}''(z,x)}{\tilde{S}_1'(z,y) - \tilde{S}_1'(z,x)} dz dx,$$

for suitable boundary points z^* and z^{**}. Therefore, again Lemma 5 gives

$$\left|\iint (M+L) dz dx\right| = \left|\int \frac{-\tilde{S}_1'(z,x)}{\tilde{S}_1'(z,y)-\tilde{S}_1'(z,x)}\Bigg]_{x^*}^{x^{**}} dz + \int \frac{\tilde{S}_2'(z,x)}{\tilde{S}_1'(z,y)-\tilde{S}_1'(z,x)}\Bigg]_{z^*}^{z^{**}} dx\right| \leq \frac{C}{\delta},$$

as required. □

References

[C] A. Carbery, *A version of Cotlar's lemma for L^p spaces and some applications*, Contemp. Math. **189** (1995), 117–134.

[CCVWW] A. Carbery, M. Christ, J. Vance, S. Wainger and D. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods*, Duke Math. J. **59** (1989), 675–700.

[CCW] A. Carbery, M. Christ and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc., to appear.

[CVWW] A. Carbery, J. Vance, S. Wainger and J. Wright, *A variant of the notion of a space of homogeneous type*, J. Funct. Anal. **132** (1995), 119–140.

[CWW1] A. Carbery, S. Wainger and J. Wright, *Hilbert transforms and maximal functions associated to flat curves*, J. Fourier Anal. Appl. **Special Issue** (1995), 119–139.

[CWW2] A. Carbery, S. Wainger and J. Wright, *Hilbert transforms and maximal functions associated to flat curves on the Heisenberg group*, J. Amer. Math. Soc. **8,1** (1995), 141–179.

[CCCDRVWW] H. Carlsson, M. Christ, A. Córdoba, J. Duoandikoetxea, J.L. Rubio de Francia, J. Vance, S. Wainger and D. Weinberg, *L^p estimates for maximal functions and Hilbert transforms along variable flat convex curves in $\mathbb{R}^2$*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 263–267.

[CNSW] M. Christ, A. Nagel, E. M. Stein and S. Wainger, *Singular and maximal Radon transforms: analysis and geometry.*, preprint, to appear.

[DR] J. Duoandikoetxea and J.L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.

[NVWW1] A. Nagel, J. Vance, S. Wainger and D. Weinberg, *Hilbert transforms for convex curves*, Duke Math. J. **50** (1983), 735–744.

[NVWW2] A. Nagel, J. Vance, S. Wainger and D. Weinberg, *Maximal functions for convex curves*, Duke Math. J. **52** (1985), 715–722.

[P] Y. Pan, *Uniform estimates for oscillatory integral operators*, J. Funct. Anal. **100** (1991), 207–220.

[PS] D.H. Phong and E.M. Stein, *Hilbert integrals, singular integrals and Radon transforms,I*, Acta Math. **157** (1986), 99–157.

[PS1] D.H. Phong and E.M. Stein, *On a stopping process for oscillatory integrals*, J. Geom. Anal. **4** (1994), 104–120.

[S] A. Seeger, L^2 *estimates for a class of singular oscillatory integrals*, Math. Res. Lett. **1** (1994), 65–73.

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