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FOURIER BASES AND A DISTANCE PROBLEM OF ERDŐS

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Abstract. We prove that no ball admits a non-harmonic orthogonal basis of exponentials. We use a combinatorial result, originally studied by Erdős, which says that the number of distances determined by n points in $\mathbb{R}^d$ is at least $C_d n^{\frac{1}{d}+\epsilon_d}$, $\epsilon_d > 0$.

Introduction and statement of results

Fourier bases. Let D be a domain in $\mathbb{R}^d$, i.e., D is a Lebesgue measurable subset of $\mathbb{R}^d$ with finite non-zero Lebesgue measure. We say that D is a *spectral set* if $L^2(D)$ has orthogonal basis of the form $E_\Lambda = \{e^{2\pi i x\cdot\lambda}\}_{\lambda\in\Lambda}$, where Λ is an infinite subset of $\mathbb{R}^d$. We shall refer to Λ as a *spectrum* for D.

We say that a family $D+t$, $t \in T$, of translates of a domain D *tiles* $\mathbb{R}^d$ if $\cup_{t\in T}(D+t)$ is a partition of $\mathbb{R}^d$ up to sets of Lebesgue measure zero.

Conjecture. It has been conjectured (see [Fug]) that a domain D is a spectral set if and only if it is possible to tile $\mathbb{R}^d$ by a family of translates of D.

This conjecture is nowhere near resolution, even in dimension one. It has been the subject of recent research, see for example [JoPe2], [LaWa], and [Ped].

In this paper we address the following special case of the conjecture. Let $B_d = \{x \in \mathbb{R}^d : |x| \leq 1\}$ denote the unit ball. We prove that

Theorem 1. *An affine image of $D = B_d$, $d \geq 2$, is not a spectral set.*

If A is a (possibly unbounded) self-adjoint operator acting on some Hilbert space, then we may define $\exp\left(-\sqrt{-1}A\right)$ using the Spectral Theorem. We say that two (unbounded) self-adjoint operators A and B acting on the same Hilbert space *commute* if the bounded unitary operators $\exp\left(-\sqrt{-1}sA\right)$ and $\exp\left(-\sqrt{-1}tB\right)$ commute for all real numbers s and t. See, for example, [ReSi] for more details on the needed operator theory. As an immediate consequence of [Fug] and Theorem 1 we have:

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Corollary. *There do not exist commuting self-adjoint operators H_j acting on $L^2(B_d)$ such that $H_j f = -\sqrt{-1}\,\partial f/\partial x_j$ for f in the domain of the unbounded operator H_j and $1 \leq j \leq d$. The derivatives $\partial/\partial x_j$ act on $L^2(B_d)$ in the distribution sense.*

In other words, there do not exist commuting self-adjoint restrictions of the partial derivative operators $-\sqrt{-1}\,\partial/\partial x_j$, $j = 1, \ldots, d$, acting on $L^2(B_d)$ in the distribution sense.

The two-dimensional case of Theorem 1 was proved by Fuglede in [Fug]. Our proof uses the following combinatorial result. See for example [AgPa], Theorem 12.13.

Theorem 2. *Let $g_d(n)$, $d \geq 2$, denote the minimum number of distances determined by n points in $\mathbb{R}^d$. Then*

$$g_d(n) \geq C_d n^{\frac{3}{3d-2}}. \tag{*}$$

Remark. The study of the problem addressed in Theorem 2 was initiated by Erdős. He proved that $g_2(n) \geq Cn^{\frac{1}{2}}$. See [Erd]. Moser proved in [Mos] that $g_2(n) \geq Cn^{\frac{2}{3}}$. More recently, Chung, Szeremedi, and Trotter proved that $g_2(n) \geq C\frac{n^{\frac{4}{5}}}{\log^c(n)}$ for some $c > 0$. See [CST]. Theorem 2 above is proved by induction using the $g_2(n) \geq Cn^{\frac{3}{4}}$ result proved by Clarkson et al. in [C].

As the reader shall see, Theorem 1 does not require the full strength of Theorem 2. We just need the fact $g_d(n) \geq C_d n^{\frac{1}{d}+\epsilon}$, for some $\epsilon > 0$.

It is interesting to contrast the case of the ball with the case of the cube $[0,1]^d$. It was proved in [IoPe1], (and, independently, in [LRW]; for $d \leq 3$ this was established in [JoPe2]), that Λ is a spectrum for $[0,1]^d$, in the sense defined above, if and only if Λ is a tiling set for $[0,1]^d$, in the sense that $[0,1]^d + \Lambda = \mathbb{R}^d$ without overlaps. It follows that $[0,1]^d$ has lots of spectra. The standard integer lattice $\Lambda = \mathbb{Z}^d$ is an example, though there are many non-trivial examples as well. See [IoPe1] and [LaSh].

Our method of proof is as follows. We shall argue that if B_d were a spectral set, then any corresponding spectrum Λ would have the property $\#\{\Lambda \cap B_d(R)\} \approx R^d$, where $B_d(R)$ denotes a ball of radius R and $f(R) \approx g(R)$ means that there exist constants $c \leq C$ so that $c\,f(R) \leq g(R) \leq C\,f(R)$ for R sufficiently large. On the other hand, we will show that the number of distinct distances between the elements of $\{\Lambda \cap B_d(R)\}$ is $\approx R$. Theorem 2 implies that if R is sufficiently large, this is not possible.

Kolountzakis ([Kol]) recently proved that if D is any convex non-symmetric domain in $\mathbb{R}^d$, then D is not a spectral set. Theorem 1 is a step in the direction of proving that if D is a convex domain such that ∂D has at least one point where the Gaussian curvature does not vanish, then D is not a spectral set. This, in its turn, would be a step towards proving the conjecture of Fuglede mentioned above.

Orthogonality

For a domain D let

$$Z_D = \{\xi \in \mathbb{R} : \widehat{\chi}_D(\xi) = 0\}.$$

Consider a set of exponentials E_Λ. Observe that

$$\widehat{\chi}_D(\lambda - \lambda') = \int_D e_\lambda(x)\overline{e_{\lambda'}(x)}\,dx.$$

It follows that the exponentials E_Λ are orthogonal in $L^2(D)$ if and only if

$$\Lambda - \Lambda \subseteqq Z_D \cup \{0\}.$$

Proposition 1. *If E_Λ is an orthogonal subset of $L^2(D)$ then there exists a constant C depending only on D such that*

$$\#\,(\Lambda \cap B_d(R)) \le C\,R^d,$$

for any ball $B_d(R)$ of radius R in $\mathbb{R}^d$.

Proof. Since $\widehat{\chi}_D$ is continuous and $\widehat{\chi}_D(0) = |D|$ it follows that

$$\inf\{|\xi| : \widehat{\chi}_D(\xi) = 0\} = r > 0.$$

If $\xi_1, \ldots, \xi_n$ are in $\Lambda \cap B_d(R)$ then the balls $B(\xi_j, r/2)$ are disjoint and contained in $B_d(R + r/2)$. Since r only depends on D the desired inequality follows. □

To study the exact possibilities for sets Λ so that E_Λ is orthogonal it is of interest to us to compute the set Z_D. We will without loss of generality assume that $0 \in \Lambda$. We again compare the sets Z_D for the cases where D is the cube and the ball.

Let $Q_d = [0,1]^d$ be the cube in $\mathbb{R}^d$. The zero set Z_Q for $\widehat{\chi}_Q$ is the union of the hyperplanes $\{x \in \mathbb{R}^d : x_i = z\}$, where the union is taken over $1 \le i \le d$, and over all non-zero integers z.

Let $B_d = \{x \in \mathbb{R}^d : \|x\| \le 1\}$ be the unit ball in $\mathbb{R}^d$. The zero set Z_{B_d} for $\widehat{\chi}_B$ is the union of the spheres $\{x \in \mathbb{R}^d : \|x\| = r\}$, where the union is over all the positive roots r of an appropriate Bessel function.

For the cube Q_d it is easy to find a large set $\Lambda \subseteqq Z_{Q_d} \cup \{0\}$ so that $\Lambda - \Lambda \subseteqq Z_{Q_d} \cup \{0\}$. For example, we may take $\Lambda = \mathbb{Z}^d$. In the case of the ball B_d, we will show that only relatively small sets $\Lambda \subseteqq Z_{B_d} \cup \{0\}$ satisfy $\Lambda - \Lambda \subseteqq Z_{B_d} \cup \{0\}$.

Proof of Theorem 1

We shall need the following result.

Theorem 3. *Suppose that D is a spectral set and that Λ is a spectrum for D in the sense defined above, where D is a bounded domain. There exists an $r > 0$ so that any ball of radius r contains at least one point from Λ.*

Proof. This is a special case of [IoPe2]. See also [Beu], [Lan], and [GrRa]. □

It is a consequence of Theorem 3 that if D is a spectral set then there exists a constant $C > 0$ such that if Λ is a spectrum for D then $\#\{\Lambda \cap B_d(R)\} \geq C\, R^d$ for any ball $B_d(R)$ of radius R provided that R is sufficiently large. Combining this with Proposition 1 we see that $\#\{\Lambda \cap B_d(R)\} \approx R^d$.

Suppose Λ is a spectrum for the unit ball B_d centered at the origin in $\mathbb{R}^d$. Let $B_d(R)$ be a ball of radius R. Since $\#\{\Lambda \cap B_d(R)\} \approx R^d$ it follows from Theorem 2 that

$$(**) \qquad \#\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \cap B_d(R)\} \geq C\, R^{\frac{3d}{3d-2}}.$$

Now, since $\widehat{\chi}_{B_d}$ is an analytic radial function, it follows that if f is given by $f(|\xi|) = \widehat{\chi}_{B_d}(\xi)$, then the number of zeros of f in the interval $[-R, R]$ is bounded above by a multiple of R. In fact an explicit calculation shows that $\widehat{\chi}_{B_d}(\xi) = |\xi|^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi|\xi|)$, where J_ν denotes the usual Bessel function of order ν. See, for example, [BCT, p. 265].

If λ, $\lambda' \in \Lambda$ then

$$f(|\lambda - \lambda'|) = \widehat{\chi}_{B_d}(\lambda - \lambda') = 0.$$

Combining the upper bound on the number of zeros of f in $[-R, R]$ with the lower bound (**) we derived from Theorem 2 above we have

$$C'\, R \geq \#\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \cap B_d(R)\} \geq C\, R^{\frac{3d}{3d-2}}.$$

Since $1 < \frac{3d}{3d-2}$ this leads to a contradiction by choosing R sufficiently large. This completes the proof of Theorem 1. □

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