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BOUNDARY LAYERS OF 2D INVISCID FLUIDS FROM A HAMILTONIAN VIEWPOINT

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Abstract. In this paper we study how the modulation of Lyapounov functionnals of Hamiltonian systems leads to nontrivial norms which can be used to investigate the linear and nonlinear stability of boundary layers. We apply this method to Euler equations of incompressible and compressible ideal fluids in 2D.

1. Introduction

One classical problem in Fluid Mechanics is the study of the inviscid limit of Navier Stokes in an half plane. From a mathematical point of view, it is widely open, except for analytic initial data in small time [4]. Let us consider a sequence of solutions u^ν of incompressible Navier Stokes equations in $R \times R_+$

$$\partial_t u^\nu + (u^\nu.\nabla)u^\nu - \nu\Delta u^\nu + \nabla p^\nu = 0, \tag{1.1}$$

$$\text{div } u^\nu = 0, \tag{1.2}$$

with $u^\nu(t,x,y) = 0$ when $y = 0$, and let us assume that u^ν has a Prandtl type boundary layer behavior, namely

$$|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma u^\nu| \le C_{\alpha,\beta,\gamma} + \frac{C_{\alpha,\beta,\gamma}}{\sqrt{\nu}^\beta}\Phi_{\alpha,\beta,\gamma}(t,x,\frac{y}{\sqrt{\nu}}) \tag{1.3}$$

where $\Phi_{\alpha,\beta,\gamma}$ are rapidly decreasing functions. Let us make the isotropic change of variables

$$\tilde{u}^\nu(t,x,y) = u^\nu(\sqrt{\nu}t, \sqrt{\nu}x, \sqrt{\nu}y) \tag{1.4}$$

to get a sequence of functions $\tilde{u}^\nu$ satisfying

$$\partial_t \tilde{u}^\nu + (\tilde{u}^\nu.\nabla)\tilde{u}^\nu - \sqrt{\nu}\Delta\tilde{u}^\nu + \nabla\tilde{p}^\nu = 0, \tag{1.5}$$

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with div $\tilde{u}^\nu = 0$ and $\tilde{u}^\nu = 0$ when $y = 0$, and

$$(1.6) \qquad |\partial_x^\alpha \partial_y^\beta \partial_t^\gamma \tilde{u}^\nu| \le C_{\alpha,\beta,\gamma} \sqrt{\nu}^{\alpha+\gamma} + C_{\alpha,\beta,\gamma} \sqrt{\nu}^{\alpha+\gamma} \Phi_{\alpha,\beta,\gamma}(t,x,y).$$

Now assuming (1.2) and (1.6), formally, $\tilde{u}^\nu$ converge to a time independent function u, solution of Euler equations. The linear and nonlinear stability of this limit solution is then a crucial point to know whether the formal analysis is true or not. In fact if the limit u is an unstable stationnary solution of Euler equations, in some cases, we can construct a sequence of solutions of Navier Stokes equations which does *not* converge in H^1 to the sum of a solution of Euler equations and of a solution of Prandtl equations [7], due to the apparition of small scale instabilities, growing like $\exp(t/\sqrt{\nu})$. The aim of this paper is conversely to give criteria which insure linear and nonlinear stability of solutions of Euler equations satisfying a priori estimates of the form (1.3), where ν is simply the index of the sequence.

An other way to look at things is to say that in some cases solutions of Navier Stokes equations do not converge in strong sense to solutions of Euler and Prandtl equations, because of short time instabilities in which viscosity does not play any significant role (see the construction of [7]): the instabilities are completely driven by the inviscid part. To be short, if $u^\nu(0)$ is an unstable sequence of initial data for Euler equations, viscosity is not strong enough to stop the instability, and instabilities develop. Therefore one first step in looking for cases when Navier Stokes do converge to Euler even for Sobolev type initial data is to look for initial data $u^\nu(0)$ which are linearly and nonlinearly stable for Euler equations, which is the aim of this paper, both in the incompressible and compressible cases.

A third way to consider the problem is to note with Rayleigh that the viscosity can have a destabilizing role, and even when viscosity is stabilizing, its strength may not be sufficient to completely stabilize the flow if there is an inviscid instability. It is then tempting to throw it away in a first approach [5].

Moreover, to study stability of solutions of Euler equations with a priori estimates of the form (1.3) is a toy model for semigeostrophic asymptotics in frontogenesis which is algebraically more complex. We believe this method can be applied to other physical relevent issues such as semigeostrophic or hydrostatic dynamics in Meteorology [2],[3] or plasma erosion.

Let us turn to the principle of the method. This problem will be considered in the more general setting of stability problems of solutions of Hamiltonian partial differential equations. Namely let H be a Hamiltonian, with Poisson bracket $\{.,.\}$ and let us consider a family of solutions $u^\varepsilon(t,x_1,x_2)$ of this Hamiltonian system in $\{x_1 \in T, x_2 \ge 0\}$, having a boundary layer behavior near $x_2 = 0$. By "boundary layer behavior" we mean that u^ε varies on time scales $O(1)$, and on space scales $O(1)$ except near $x_2 = 0$ where it varies on x_2 scales $O(\varepsilon)$ when $\varepsilon \to 0$:

$$|\partial_t^\alpha \partial_1^\beta \partial_2^\gamma u^\varepsilon| \le \frac{C_{\alpha\beta\gamma}}{\varepsilon^\gamma} \Phi_{\alpha,\beta,\gamma}\Big(\frac{x_2}{\varepsilon}\Big) + C_{\alpha,\beta,\gamma}$$

for $\alpha, \beta, \gamma \geq 0$, some positive constants $C_{\alpha,\beta,\gamma}$ and some rapidly decreasing functions $\Phi_{\alpha,\beta,\gamma}$. Our first aim is to prove linear stability of such a sequence of solutions, that is to find a family of norms $|||.|||$ depending on ε, such that

$$\|.\|_{L^2} \leq |||.||| \leq \varepsilon^{-\eta}\|.\|_{H^s} \tag{1.7}$$

for some $\eta \geq 0$ and $s \geq 0$, and such that

$$\partial_t |||v||| \leq C|||v||| \tag{1.8}$$

for every solution v of the linearized system around u^ε (with $0 < \varepsilon \leq 1$), with C independent on v and ε. Notice, and this is an important issue, that we do not require $|||.|||$ to be equivalent to some Sobolev norm. If there were no boundary we could take $|||.||| = \|.\|_{L^2}$, but we have to look for more refined norms in order to handle large gradients near $x_2 = 0$. The main observation is that if we make a "blow up" near a boundary point $(x_1^0, 0)$, that is if we set

$$\tilde{u}^\varepsilon(t, x_1, x_2) = u^\varepsilon\Big(\frac{t}{\varepsilon^\nu}, x_1^0 + \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big),$$

then $\tilde{u}^\varepsilon$ (for a suitable choice of ν) is again a sequence of solutions of the *same* Hamiltonian system, sequence of solutions which may presumably converge to some *time independent solution* u_e as $\varepsilon \to 0$. Therefore the stability of u^ε "near" $(x_1^0, 0)$ in times of order $O(1)$ is linked to the stability in large times of u_e.

But the stability of u_e can be studied using Lyapounov method, following V.I. Arnold [1] and D. Holm, J. Marsden, T. Ratiu [9] [10]. The classical approach is to find a Casimir C such that u_e is a critical point of $H + C$. If moreover u_e is a local extremum of $H + C$ we get stability properties. In particular the Hessian $D^2(H + C)$ provides a norm which is preserved by the linearized dynamics. The principle of our approach is to make the Casimir depend on the point. We thereby loose any critical point or extremum property, but the Hessian gives a non trivial norm which controls the linearized dynamics in the sense that (1.8) is true.

It is then straightforward to prove nonlinear stability results in the following sense : we say that a sequence of solutions u^ε in $\cap_{s'} L^\infty([0, T], H^{s'})$ is nonlinearly stable if there exists s_1 and η_1 (large enough), two functions $\tilde{s}(s, \eta)$ and $\tilde{\eta}(s, \eta)$ with

$$\lim_{(s,\eta)\to(+\infty,+\infty)} \tilde{s} = \lim_{(s,\eta)\to(+\infty,+\infty)} \tilde{\eta} = +\infty,$$

and constants $C_{s,\eta}$ such that any other family of solutions $v^\varepsilon \in \cap_{s'} L^\infty([0, T], H^{s'})$ with

$$\|u^\varepsilon - v^\varepsilon\|_{H^s} \leq \varepsilon^\eta$$

at $t = 0$ ($s \geq s_1$ and $\eta \geq \eta_1$) satisfies

$$\|u^\varepsilon - v^\varepsilon\|_{H^{\tilde{s}(s,\eta)}} \leq C_{s,\eta}\varepsilon^{\tilde{\eta}(s,\eta)}$$

for $0 \leq t \leq T$. Notice that this notion of nonlinear stability is very weak, but turns out to be sufficient to study and justify asymptotic expansions (see for instance [7],[8]), which is our main goal.

We will now illustrate the method by studying boundary layers of incompressible and compressible Euler equations.

2. Application to incompressible 2D fluids

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a sequence of solutions of incompressible Euler equations

$$\partial_t u^\varepsilon + (u^\varepsilon.\nabla)u^\varepsilon + \nabla p^\varepsilon = 0, \tag{2.1}$$

$$\nabla . u^\varepsilon = 0, \tag{2.2}$$

in $x_1 \in T$, $x_2 \geq 0$, with $u_2^\varepsilon = 0$ on $x_2 = 0$. The stability of *time independent* solutions u_e has been investigated by V.I. Arnold [1] who introduced the functionnal

$$H_C = \int \frac{|u|^2}{2} + F(\omega), \tag{2.3}$$

where $\omega = \operatorname{curl} u$ and $F'' = u_e / \nabla^\perp \operatorname{curl} u_e$ (which is well defined since u_e and $\nabla^\perp \operatorname{curl} u_e$ are colinear). Notice that

$$D^2 H_C(v) = \int |v|^2 + \frac{u_e}{\nabla^\perp \operatorname{curl} u_e} |\operatorname{curl} v|^2 \tag{2.4}$$

is preserved by the linearized Euler equations around u_e.

Here we modulate (2.3) and introduce

$$H_C^\varepsilon = \int \frac{|u + (\tilde{u}, 0)|^2}{2} + \int F^\varepsilon(t, x, \operatorname{curl} u), \tag{2.5}$$

where $F^\varepsilon(t, x, \operatorname{curl} u)$ is a sequence of functions to be choosen carefully, and where $\tilde{u}$ is a given time and space independent scalar, introduced to take into account the Galilean invariance of the equations with respect to x_1 translations of the referential. Notice that $\int F^\varepsilon(t, x, \operatorname{curl} u)$ plays the role of a time and space "modulated" Casimir.

Theorem 2.1. *Let $u^\varepsilon(t)$ be a sequence of solutions of* (2.1) *and* (2.2) *on* $[0, T]$, *and let us assume that there exists $\eta > 0$, a smooth function $\tilde{u}(t)$, exponentially rapidly decreasing positive functions $\Psi_{\alpha,\beta}$ and constants $C_{\alpha,\beta} > 0$ for $\alpha \geq 0$ and $\beta \geq 0$ such that for every $0 < \varepsilon \leq 1$, on* $[0, T]$,

$$|\partial_1^\alpha \partial_2^\beta u^\varepsilon| \leq C_{\alpha,\beta} \varepsilon^{-\beta} \Psi_{\alpha,\beta}(\varepsilon^{-1} x_2) + C_{\alpha,\beta} \quad \textit{for} \quad \alpha \geq 0,\ \beta \geq 0, \tag{H1}$$

(H2) $$\Big|\frac{\partial_t u_1^\varepsilon + (u^\varepsilon.\nabla)u_1^\varepsilon}{u_1^\varepsilon + \tilde u}\Big| + \Big|\frac{\partial_t \partial_{22}^2 u_1^\varepsilon + (u^\varepsilon.\nabla)\partial_{22}^2 u_1^\varepsilon}{\partial_{22}^2 u_1^\varepsilon}\Big| \le C_0 \quad \textit{for} \quad 0 \le x_2 \le \eta,$$

(H3)
$$0 < \frac{u_1^\varepsilon + \tilde u}{\partial_{22}^2 u_1^\varepsilon} \le C_0, \quad |u_1^\varepsilon + \tilde u| \ge C_0^{-1} \quad \textit{and} \quad |\partial_2 \omega^\varepsilon| \ge C_0^{-1} \quad \textit{for} \quad 0 \le x_2 \le \eta,$$

(H4) $$|\partial_{11}^2 u_1^\varepsilon| + |\partial_{11}^2 u_2^\varepsilon| + \Big|\frac{(\partial_{22}^2 u_2^\varepsilon)^2}{\partial_{22}^2 u_1^\varepsilon}\Big| + \Big|(u_2^\varepsilon)^2 \partial_{22}^2 u_1^\varepsilon\Big| + \frac{|\partial_2 u_1|^2}{|\partial_{22}^2 u_1|} + |\partial_1 u_2|^2 |\partial_{22}^2 u_1| \le C_0 \quad \textit{for} \quad 0 \le x_2 \le \eta.$$

Then u^ε *is a linearly and nonlinearly stable sequence of solutions of* (2.1) *and* (2.2).

Assumptions (H1) to (H4) can be fulfilled by flows of the form $\big(u_1^\varepsilon(t,x_1,x_2,x_2/\varepsilon), u_2^\varepsilon(t,x_1,x_2,x_2/\varepsilon)\big)$ (rapidly decreasing in $Y = x_2/\varepsilon$), where u_1^ε and u_2^ε are smooth functions with uniformly in ε bounded derivatives in x_1, x_2 and Y (see [6] for the existence of such solutions). (H1) is straightforward. For (H2) and (H4) we use incompressibility condition: as $|\partial_1 u_1^\varepsilon| \le C$, $|u_2^\varepsilon| \le C x_2$. (H3) is a classical assumption (there is no inflexion point in the boundary layer) and is linked to Fjortoft's Theorem [5]. If (H3) is not satisfied, u^ε can be linearly and nonlinearly unstable [6], [7].

Notice also that in the boundary layer, u_1^ε is a *concave* function (if $u_1^\varepsilon \ge 0$), hence $u_1^\varepsilon / \partial_{22} u_1^\varepsilon$ is negative and has the *wrong* sign in the estimates. The introduction of $\tilde u$ is therefore a crucial point in order to reverse the sign of $(u_1^\varepsilon + \tilde u)/\partial_{22} u_1^\varepsilon$ and to make the Hessian of H_C^ε convex.

2.1 Linear stability

Let us consider $v = (v_1, v_2)$, solution of

$$\partial_t v + (u^\varepsilon.\nabla)v + (v.\nabla)u^\varepsilon + \nabla q = 0, \tag{2.6}$$

$$v_2 = 0 \qquad \text{on} \qquad x_2 = 0. \tag{2.7}$$

Let $F^\varepsilon(t,x,\omega)$ be a given sequence of functions with $\partial_{\omega\omega}^2 F^\varepsilon \ge 0$, let $\tilde u(t)$ be a given smooth function and let us define a sequence of norms N^ε by

$$\big(N^\varepsilon(v)\big)^2 = \int |v|^2 + \int \partial_{\omega\omega}^2 F^\varepsilon(t,x,\operatorname{curl} u^\varepsilon) |\operatorname{curl} v\,|^2.$$

Lemma 2.2. *The sequence u^ε of solutions of* (2.6) *and* (2.7) *is linearly stable for the norms N^ε provided*

$$\text{(H1}'\text{)} \qquad |\partial_t \partial^2_{\omega\omega} F^\varepsilon + (u^\varepsilon.\nabla)\partial^2_{\omega\omega} F^\varepsilon| \leq C \partial^2_{\omega\omega} F^\varepsilon,$$

and

$$\text{(H2}'\text{)} \qquad |\partial^2_{\omega\omega} F^\varepsilon \nabla^\perp \operatorname{curl} u^\varepsilon - u^\varepsilon - (\tilde{u}, 0)| \leq C \sqrt{\partial^2_{\omega\omega} F^\varepsilon},$$

for some constant C independent on ε, and provided (1.7) *holds true.*

Proof. This Lemma can be proved directly by deriving in time N^ε and using the equation, but this approach can not be extended to algebraically more complex systems like compressible Euler equations since it leads to untractable calculations. Therefore we will present an alternative proof, based on formal calculations.

Let us fix ε and omit it in the notation. Let u^δ be the solution of (nonlinear) Euler equations with initial data $u + \delta \bar{u}$. We have $u^\delta = u + \delta \bar{u} + \delta^2 \hat{u}$ where $\bar{u}$ solves (2.6), (2.7) and where $\hat{u}$ solves $\hat{u}(0) = 0$ and

$$\partial_t \hat{u} + [(u + \delta \bar{u} + \delta^2 \hat{u}).\nabla]\hat{u} + (\hat{u}.\nabla)(u + \delta \bar{u}) + (\bar{u}.\nabla)\bar{u} + \nabla \hat{p} = 0.$$

Let $\bar{\omega} = \operatorname{curl} \bar{u}$, $\omega^\delta = \operatorname{curl} u^\delta$ and let

$$H_C(t, \delta) = \int \frac{|u^\delta + (\tilde{u}, 0)|^2}{2} + F(t, x, \operatorname{curl} u^\delta).$$

We will compute $\partial_t \partial^2_\delta H_C$ at $t = 0$ and $\delta = 0$ using two different methods. First,

$$\begin{aligned} \partial_t H_C = \int \partial_t F + \int \partial_\omega F \partial_t \operatorname{curl} u^\delta &= \int \partial_t F - \int \partial_\omega F u_i^\delta \partial_i \omega^\delta \\ &= \int \partial_t F + \int u_i^\delta \partial_i F(t, x, \operatorname{curl} u^\delta), \end{aligned}$$

hence, using $\hat{u} = 0$ at $t = 0$,

$$\partial^2_\delta \partial_t H_C(0,0) = \int \partial_t \partial^2_{\omega\omega} F \bar{\omega}^2 + \int u_i \partial_i \partial^2_{\omega\omega} F \bar{\omega}^2 + 2 \int \bar{u}_i \partial_i \partial_\omega F \bar{\omega}.$$

On the other side, for $\delta = 0$,

$$\partial^2_\delta H_C = \int \bar{u}^2 + 2 \int u \hat{u} + 2 \int \tilde{u} \hat{u}_1 + \int \partial^2_{\omega\omega} F \bar{\omega}^2 + 2 \int \partial_\omega F \hat{\omega},$$

hence at $t = 0$, using $\hat{u} = 0$,

$$\partial_t \partial^2_\delta H_C(t = 0) = \partial_t (N^\varepsilon(\bar{u}))^2 + 2 \int u \partial_t \hat{u} + 2 \int \tilde{u} \partial_t \hat{u}_1 + 2 \int \partial_\omega F \partial_t \hat{\omega}.$$

Equivalating the two expressions of H_C we get

$$\begin{aligned}\partial_t(N^\varepsilon(\bar u))^2 &= 2\int \bar u_i\partial_i\partial_\omega F\bar\omega + 2\int u(\bar u.\nabla)\bar u + 2\int \tilde u(\bar u.\nabla)\bar u_1 \\ &\quad + 2\int \partial_\omega F\partial_i(\bar u_i\bar\omega) + \int(\partial_t\partial^2_{\omega\omega}F + u_i\partial_i\partial^2_{\omega\omega}F)\bar\omega^2 \\ &= \int\Big(\partial_t\partial^2_{\omega\omega}F + u_i\partial_i\partial^2_{\omega\omega}F\Big)\bar\omega^2 + \int \bar\omega\bar u_1(u_2 - \partial^2_{\omega\omega}F\partial_1\omega) \\ &\qquad - \int \bar\omega\bar u_2(u_1 + \tilde u + \partial^2_{\omega\omega}F\partial_2\omega).\end{aligned}$$

Therefore under (H′1) and (H′2), $\partial_t(N^\varepsilon(\bar u))^2 \le C(N^\varepsilon(\bar u))^2$, at $t = 0$ and in fact for every $0 \le t \le T$, which ends the proof of the Lemma. □

(H1′), (H′2) are of course "ad hoc" assumptions to play the game of modulation. Let us show that (H1), (H2), (H3) and (H4) imply the existence of F^ε such that (H1′) and (H′2) hold true. Let $\chi(x_2)$ be a smooth positive function which equals 1 for $x_2 \le \eta/2$ and 0 for $x_2 \ge \eta$. We will take

$$F^\varepsilon(t,x,\omega) = \chi(x_2)F_1^\varepsilon(t,x_1,\omega) + (1-\chi(x_2))\omega^2,$$

and choose carefully F_1^ε. Namely

$$\partial^2_{\omega\omega}F_1^\varepsilon\big(t,x_1,\operatorname{curl} u^\varepsilon(t,x_1,x_2)\big) = g^\varepsilon(t,x_1,x_2), \tag{2.8}$$

where

$$g^\varepsilon = \frac{u_1^\varepsilon + \tilde u}{\partial^2_{22}u_1^\varepsilon}.$$

Notice that as $\partial_2\omega^\varepsilon$ does not vanish for $x_2 \le \eta$, F_1^ε is well defined by (2.8) for $x_2 \le \eta$. By (H1) and (H3), g^ε is strictly positive and there exists C such that

$$C^{-1}\|v\|^2_{L^2} + C^{-1}\varepsilon^2\|\operatorname{curl} v\|^2_{L^2} \le (N^\varepsilon)^2(v) \le C\|v\|^2_{L^2} + C\|\operatorname{curl} v\|^2_{L^2}. \tag{2.9}$$

We can make other choices of g^ε : for instance we can take $g^\varepsilon = -(u_1^\varepsilon + \tilde u)/\partial_2\omega^\varepsilon$, which leads to slightly modified assumptions (H1) to (H4).

Lemma 2.3. *The sequence u^ε of solutions of* (2.6) *and* (2.7) *is linearly stable for the norms N^ε under assumptions* (H1), (H2), (H3), *and* (H4).

Let $x_2 \le \eta/2$. (H2) directly implies (H′1). Moreover, dropping the ε indices,

$$g\nabla^\perp\omega - u - \begin{pmatrix}\tilde u\\ 0\end{pmatrix} = \begin{pmatrix} \dfrac{u_1+\tilde u}{\partial^2_{22}u_1}(\partial^2_{11}u_1 + \partial^2_{22}u_1) - u_1 - \tilde u \\ \dfrac{u_1+\tilde u}{\partial^2_{22}u_1}(\partial^2_{11}u_2 + \partial^2_{22}u_2) - u_2\end{pmatrix},$$

the first component is bounded by Cg using (H4) and thus by $C\sqrt{g}$ using (H3), and the second component is bounded by $C\sqrt{g}$ using (H4).

Therefore (H′1) and (H′2) hold true for $x_2 \le \eta/2$. For $x_2 \ge \eta/2$ we simply use $|g| \ge C > 0$ and $|\partial_1^\alpha\partial_2^\beta u^\varepsilon| \le C$ with C independent on ε for $\alpha+\beta \le 3$, to get (H1′) and (H2′).

2.2. Estimates on first order derivatives

In order to control the nonlinearity of Euler equations we now need estimates on higher order derivatives. This step is straightforward and makes little use of the precise structure of Euler equations. Let, as in [8],

$$(N_n^\varepsilon)^2 = \sum_{\alpha+\beta=n} \int (g^\varepsilon)^\beta |\partial_1^\alpha \partial_2^\beta \operatorname{curl} v|^2.$$

Lemma 2.4. *Let us assume that for $\alpha + \beta \le n$ we have*

$$|\partial_1^\alpha \partial_2^\beta u^\varepsilon| \le C\varepsilon^{-\beta}, \tag{2.10}$$

and that

$$|\partial_1 u_1^\varepsilon| + \sqrt{g^\varepsilon}|\partial_2 u_1^\varepsilon| + \Big|\frac{\partial_1 u_2^\varepsilon}{\sqrt{g^\varepsilon}}\Big| + |\partial_2 u_2^\varepsilon| + \Big|\frac{\partial_t g^\varepsilon + u_1^\varepsilon \partial_1 g^\varepsilon + u_2^\varepsilon \partial_2 g^\varepsilon}{g^\varepsilon}\Big| \le C, \tag{2.11}$$

then for $n \ge 1$, we have

$$\partial_t N_n^2 \le C N_n^2 + \varepsilon^{-4n-4} \sum_{i=0}^{n-1} N_i^2. \tag{2.12}$$

Just differentiate $\partial_t \bar\omega + (u.\nabla)\bar\omega + (\bar u.\nabla)\omega = 0$ and estimate one by one the terms appearing in $\partial_t (N_n^\varepsilon)^2$, using in particular for $\alpha' + \beta' \ge 1$,

$$\|\partial_1^{\alpha'} \partial_2^{\beta'} v_1\|_{L^2}^2 + \|\partial_1^{\alpha'} \partial_2^{\beta'} v_2\|_{L^2}^2 \le C\| \operatorname{curl} v\|^2_{H^{\alpha'+\beta'-1}} \le \varepsilon^{-2\alpha'-2\beta'+2} N^2_{\alpha'+\beta'-1}.$$

2.3. Nonlinear stability

To obtain nonlinear stability from Lemmas 3.2 and 3.3 is now routine work (see [7],[8] for instance). Let

$$\tilde N_s^2 = \sum_{n\le s} \varepsilon^{8n^2} N_n^2.$$

Making intensive use of Sobolev injections in order to handle the nonlinear term $(v.\nabla v)v$ we prove for s large enough

$$\partial_t \tilde N_s^2(u^\varepsilon - v^\varepsilon) \le C\tilde N_s^2(u^\varepsilon - v^\varepsilon) + \frac{\tilde N_s(u^\varepsilon - v^\varepsilon)^3}{\varepsilon^{16s^2}}, \tag{2.13}$$

where u^ε and v^ε are two solutions of (3.1),(3.2). The end of the proof of Theorem 2.1 is then straightforward.

3. Application to 2D barotropic fluids

Let us turn to the stability of boundary layers of compressible 2D fluids

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{3.1}$$

$$\partial_t u + (u.\nabla)u + \nabla h(\rho) = 0, \tag{3.2}$$

where ρ is the density and h a given increasing function. Notice that $\Omega = \omega/\rho$ where $\omega = \operatorname{curl} u$ satisfies

$$\partial_t \Omega + (u.\nabla)\Omega = 0. \tag{3.3}$$

The stability of stationnary solutions has been investigated in [9]. Let $e(\rho)$ such that $d(\rho e(\rho))/d\rho = h(\rho)$. The energy is

$$H = \int \frac{1}{2}\rho|u + (\tilde{u}, 0)|^2 + \rho e(\rho),$$

and the Casimir functions are

$$C = \int \rho F(\Omega),$$

where F is an arbitrary function, and $\tilde{u}(t)$ an arbitrary scalar (smooth in t) to take into account invariance with respect to x_1 translations. As previously we introduce the family of modulated functionnals

$$H_C^\varepsilon = \int \rho \frac{|u + (\tilde{u}, 0)|^2}{2} + \rho e(\rho) + G^\varepsilon(t, x, \rho, \Omega),$$

where $G^\varepsilon = \rho F^\varepsilon(t, x, \Omega)$, and the related norms (with $\bar{\omega} = \operatorname{curl} \bar{u}$,)

$$(N^\varepsilon(\bar{\rho}, \bar{u}))^2 = D^2 H_C \begin{pmatrix} \bar{\rho} \\ \bar{u} \end{pmatrix} \begin{pmatrix} \bar{\rho} \\ \bar{u} \end{pmatrix} = \int \rho^\varepsilon |\bar{u} + \frac{(u^\varepsilon + (\tilde{u}, 0))}{\rho^\varepsilon} \bar{\rho}|^2$$
$$+ (\rho^\varepsilon)^{-1} \begin{pmatrix} (c^\varepsilon)^2 - |u^\varepsilon + (\tilde{u}, 0)|^2 + (\Omega^\varepsilon)^2 \partial^2_{\Omega\Omega} F^\varepsilon & -\Omega^\varepsilon \partial^2_{\Omega\Omega} F^\varepsilon \\ -\Omega^\varepsilon \partial^2_{\Omega\Omega} F^\varepsilon & \partial^2_{\Omega\Omega} F^\varepsilon \end{pmatrix} \begin{pmatrix} \bar{\rho} \\ \bar{\omega} \end{pmatrix} \begin{pmatrix} \bar{\rho} \\ \bar{\omega} \end{pmatrix},$$

where $(c^\varepsilon)^2 = \rho^\varepsilon dh(\rho^\varepsilon)/d\rho$. Notice that

$$\int \bar{\rho}^2 + \int |\bar{u}|^2 + \int |\partial^2_{\Omega\Omega} F^\varepsilon|^2 \bar{\omega}^2 \le C_0 (N^\varepsilon(\bar{\rho}, \bar{u}))^2$$

for C_0 large enough, provided $|u^\varepsilon + (\tilde{u}, 0)|^2 \le (c^\varepsilon)^2 - \sigma$ everywhere (the flow is subsonic), $\rho^\varepsilon \ge \sigma$ for some $\sigma > 0$, and $0 < \partial^2_{\Omega\Omega} F^\varepsilon < C$. Notice that the natural norm N^ε which will control the linearized equation is very intricated and almost impossible to infer directly. Let us first derive modulations inequalities, equivalent of conditions (H′1) and (H′2) of the previous section.

Lemma 3.1. *The sequence $(\rho^\varepsilon, u^\varepsilon)$ of solutions of (3.1) and (3.2) is linearly stable for the norms N^ε, provided*

$$C^{-1}\|(\rho,u)\|_{L^2} \le N^\varepsilon \le C\varepsilon^{-\eta}\|(\rho,u)\|_{H^s}$$

for some $\eta \ge 0$, $C > 0$ and $s \ge 0$, and provided

$$\text{(H}'1\text{)} \qquad |\partial_t D^2_{\rho,\omega} G^\varepsilon + u_i^\varepsilon \partial_i D^2_{\rho,\omega} G^\varepsilon| \le C D^2_{\rho,\omega} G^\varepsilon,$$

$$\text{(H}'2\text{)} \qquad \Big|\partial_i\Big(\frac{|u^\varepsilon + (\tilde u, 0)|^2}{2} + e(\rho^\varepsilon) + \rho^\varepsilon e'(\rho^\varepsilon)\Big) - \partial^2_{\Omega\Omega} F^\varepsilon \Omega \partial_i \Omega\Big| \le C, \quad i = 1, 2,$$

$$\text{(H}'3\text{)} \qquad \Big|\rho u_2^\varepsilon - \partial^2_{\Omega\Omega} F^\varepsilon \partial_1 \Omega^\varepsilon\Big| + \Big|\rho u_1^\varepsilon + \rho \tilde u(t) + \partial^2_{\Omega\Omega} F^\varepsilon \partial_2 \Omega^\varepsilon\Big| \le C\sqrt{\partial^2_{\Omega\Omega} F^\varepsilon},$$

$$\text{(H}'4\text{)} \qquad |\operatorname{div}\Big(\rho^\varepsilon(u^\varepsilon + (\tilde u, 0))\Big)| \le C,$$

the constant C being independent on ε.

Proof. Let $(\bar\rho, \bar u)$ be a solution of linearized compressible Euler equations. As in the previous section, to compute $\partial_t N^\varepsilon(\bar\rho, \bar u)$ at $t = 0$ we introduce (ρ^δ, u^δ) solution of (nonlinear) compressible Euler equations with initial data $\rho + \delta\bar\rho$ and $u + \delta\bar u$, with $\rho^\delta = \rho + \delta\bar\rho + \delta^2\hat\rho$ and similarly for u^δ. Let $H_C^\varepsilon(t,\delta) = H_C^\varepsilon(\rho^\delta, u^\delta)$. We have, considering G^ε as a function of ρ and Ω, and dropping the ε indices,

$$\begin{aligned}
\partial_t H_C &= \int \partial_t G - \int \partial_\rho G \partial_i(u_i\rho) - \int \partial_\Omega G u_i \partial_i \Omega \\
&= \int G\partial_i u_i + \int u_i \partial_i G + \int \partial_t G - \int \partial_\rho G \rho \partial_i u_i \\
&= \int \partial_t G + u_i \partial_i G,
\end{aligned}$$

therefore at $t = 0$ and $\delta = 0$, considering G as a function of ρ and ω, as $\hat\rho(0) = \hat u(0) = 0$,

$$\begin{aligned}
\partial_t \partial_\delta^2 H_C = \int (\partial_t D^2_{\rho,\omega} G + u_i \partial_i D^2_{\rho,\omega} G) \begin{pmatrix} \bar\rho \\ \bar\omega \end{pmatrix} \begin{pmatrix} \bar\rho \\ \bar\omega \end{pmatrix} \\
+ 2\int \partial_i \partial_\rho G \bar\rho \bar u_i + 2\int \partial_i \partial_\omega G \bar\omega \bar u_i.
\end{aligned}$$

On the other side, for $\delta = 0$,

$$\begin{aligned}
\partial_t \partial_\delta^2 H_C = \partial_t (N^\varepsilon)^2 + \int |u + (\tilde u, 0)|^2 \partial_t \hat\rho + 2\int \rho u \partial_t \hat u + 2\int \rho \tilde u \partial_t \hat u_1 \\
+ 2\int e(\rho)\partial_t\hat\rho + 2\int \rho e'(\rho)\partial_t\hat\rho + 2\int \partial_\rho G \partial_t \hat\rho + 2\int \partial_\omega G \partial_t \hat\omega.
\end{aligned}$$

Therefore at $t=0$ and $\delta=0$,

$$\partial_t(N^\varepsilon)^2 = -2\int \rho u\partial_t\hat{u} - 2\int \rho\tilde{u}\partial_t\hat{u}_1 - 2\int \partial_\rho G\partial_t\hat{\rho} - 2\int \partial_\omega G\partial_t\hat{\omega} + 2\int \partial_i\partial_\rho G\bar{\rho}\bar{u}_i - 2\int \partial_t\hat{\rho}[\frac{|u+(\tilde{u},0)|^2}{2} + e(\rho) + \rho e'(\rho)] + 2\int \partial_i\partial_\omega G\bar{\omega}\bar{u}_i + \int (\partial_t D^2_{\rho,\omega}G + u_i\partial_i D^2_{\rho,\omega}G)\begin{pmatrix}\bar{\rho}\\ \bar{\omega}\end{pmatrix}\begin{pmatrix}\bar{\rho}\\ \bar{\omega}\end{pmatrix}.$$

This leads to

$$\begin{aligned}\partial_t(N^\varepsilon)^2 &= -2\int \partial_t\hat{\rho}\Big[\frac{|u+(\tilde{u},0)|^2}{2} + e(\rho) + \rho e'(\rho) + F - \Omega\partial_\Omega F\Big] \\ &\quad + 2\int\Big(\partial_i F - \Omega\partial_i\partial_\Omega F\Big)\bar{\rho}\bar{u}_i - 2\int \rho u\partial_t\hat{u} - 2\int \rho\tilde{u}\partial_t\hat{u}_1 - 2\int \partial_\Omega F\partial_t\hat{\omega} \\ &\quad + 2\int \partial_i\partial_\Omega F\bar{u}_i\bar{\omega} + \int(\partial_t D^2_{\rho,\omega}G + u_i\partial_i D^2_{\rho,\omega}G)\begin{pmatrix}\bar{\rho}\\ \bar{\omega}\end{pmatrix}\begin{pmatrix}\bar{\rho}\\ \bar{\omega}\end{pmatrix} \\ &= -2\int \bar{\rho}\bar{u}_i\Big[\partial_i\Big(\frac{|u+(\tilde{u},0)|^2}{2} + e(\rho) + \rho e'(\rho) + F - \partial_\Omega F\Omega\Big) - \partial_i F + \partial_i\partial_\Omega F\Omega\Big] \\ &\quad + 2\int \bar{\omega}\bar{u}_1\Big(\rho u_2 - \partial^2_{\Omega\Omega}F\partial_1\Omega\Big) - 2\int \bar{\omega}\bar{u}_2\Big(\rho u_1 + \rho\tilde{u} + \partial^2_{\Omega\Omega}F\partial_2\Omega\Big) \\ &\quad - \int \operatorname{div}(\rho u + \rho\begin{pmatrix}\tilde{u}\\ 0\end{pmatrix})(\bar{u}_1^2 + \bar{u}_2^2 + h''\bar{\rho}^2) \\ &\quad + \int(\partial_t D^2_{\rho,\omega}G + u_i\partial_i D^2_{\rho,\omega}G)\begin{pmatrix}\bar{\rho}\\ \bar{\omega}\end{pmatrix}\begin{pmatrix}\bar{\rho}\\ \bar{\omega}\end{pmatrix},\end{aligned}$$

since

$$\int \rho u(\bar{u}.\nabla)\bar{u} = \int \rho(u_2\bar{u}_1 - u_1\bar{u}_2)\operatorname{curl}\bar{u} - \frac{1}{2}\int \operatorname{div}(\rho u)(\bar{u}_1^2 + \bar{u}_2^2),$$

hence (H′1), (H′2), (H′3), and (H′4) imply $\partial_t N^\varepsilon \le CN^\varepsilon$ at $t=0$ and similarly for $0\le t\le T$, which ends the proof of the Lemma. □

We will now translate conditions (H′1), (H′2), (H′3), and (H′4) into conditions on ρ^ε and u^ε. There is not a unique way to do this since we have some freedom on F^ε. Near the boundary we will take here $\partial^2_{\Omega\Omega}F^\varepsilon = -\rho^\varepsilon(u_1^\varepsilon + \tilde{u})/\partial_2\Omega^\varepsilon$.

Theorem 3.2. *Let $(\rho^\varepsilon, u^\varepsilon)$ be a sequence of solutions of (3.1) and (3.2) on $[0,T]$, and let us assume that there exists $C>0$, $\eta>0$, a smooth function $\tilde{u}(t)$, rapidly decreasing positive functions $\Psi_{\alpha,\beta}$, and constants $C_{\alpha,\beta}>0$ for $\alpha\ge 0$ and $\beta\ge 0$, such that for $0<\varepsilon\le 1$, for all $0\le t\le T$, $|\rho^\varepsilon|\ge C^{-1}$, and $|u^\varepsilon + (\tilde{u},0)|^2 \le (c^\varepsilon)^2 - C^{-1}$,*
(H1)

$$|\partial_2\Omega^\varepsilon| \le C^{-1}, \quad C \ge -\frac{u_1^\varepsilon + \tilde{u}}{\partial_2\Omega^\varepsilon} > 0, \quad \textit{and} \quad |u_1^\varepsilon + \tilde{u}| \ge C^{-1} \quad \textit{for} \quad 0\le x_2\le\eta,$$

$$\text{(H2)}\quad |\rho^\varepsilon| + |u_1^\varepsilon| + |u_2^\varepsilon| + |\partial_1 u_1^\varepsilon| + |\partial_1 u_2^\varepsilon| + |\partial_2 u_2^\varepsilon| + |\operatorname{div} u^\varepsilon| + |\operatorname{div}(\rho^\varepsilon u^\varepsilon)| + \Big|\frac{\partial_t u_1^\varepsilon + (u^\varepsilon.\nabla)u_1^\varepsilon}{u_1^\varepsilon + \tilde u}\Big| + \Big|\frac{\partial_t \Omega^\varepsilon + (u^\varepsilon.\nabla)\Omega^\varepsilon}{\Omega^\varepsilon}\Big| + \Big|\frac{\partial_t \partial_2\Omega^\varepsilon + (u^\varepsilon.\nabla)\partial_2\Omega^\varepsilon}{\partial_2\Omega^\varepsilon}\Big| + \Big|(u_2^\varepsilon)^2\partial_2\Omega^\varepsilon\Big| + \Big|\frac{(\partial_1\Omega^\varepsilon)^2}{\partial_2\Omega^\varepsilon}\Big| + |\partial_t u^\varepsilon| + |u_2^\varepsilon \operatorname{curl} u^\varepsilon| + \Big|\frac{\Omega^\varepsilon\partial_1\Omega^\varepsilon}{\partial_2\Omega^\varepsilon}\Big| \le C \quad \text{for} \quad 0 \le x_2 \le \eta,$$

$$\text{(H3)}\quad |\partial_1^\alpha\partial_2^\beta u^\varepsilon| + |\partial_1^\alpha\partial_2^\beta \rho^\varepsilon| \le C_{\alpha,\beta}(1 + \varepsilon^{-\beta}\Psi_{\alpha,\beta}(\varepsilon^{-1}x_2)) \quad \text{for} \quad \alpha \ge 0, \beta \ge 0,$$

then $(\rho^\varepsilon, u^\varepsilon)$ *is a linearly and nonlinearly stable sequence of solutions of* (3.1) *and* (3.2).

Let $F^\varepsilon = \chi(x_2)F_1^\varepsilon + (1-\chi(x_2))\Omega^2$ where $\partial^2_{\Omega\Omega}F_1^\varepsilon = -\rho^\varepsilon(u_1^\varepsilon + \tilde u)/\partial_2\Omega^\varepsilon$ (which is meaningful by (H1)). (H2) directly implies (H′1), (H′3), and (H′4). Moreover, using the equation (H′2) gives

$$|u_2\omega + \tilde u\partial_1 u_1 - \partial_t u_1 + \frac{\rho(u_1+\tilde u)}{\partial_2\Omega}\Omega\partial_1\Omega| + |-u_1\omega + \tilde u\partial_2 u_1 - \partial_t u_2 + \rho(u_1+\tilde u)\Omega|,$$

which is bounded using (H2). Nonlinear stability can be obtained as in sections 2.2 and 2.3 and will not be detailed here. Notice again that the introduction of $\tilde u$ is crucial in order to make $\partial^2_{\Omega\Omega}F^\varepsilon$ positive.

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