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STARK ZEROS IN CERTAIN TOWERS OF FIELDS

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1. Introduction

It has long been known that zeros of the Dedekind zeta function near $s = 1$ have a mitigating effect on the growth of the residue at $s = 1$. In some cases, this can be translated into an effect on the growth of class numbers. All attempts thus far to prove that such zeros do not exist have been unsuccesful. However, in some applications, it suffices to know that the appearance of these zeros is a rare phenomenon. The main results of this paper consider a tower of fields, each member of which has solvable normal closure over its predecessor. In this case, we show that (in a sense to be made precise below), zeros near $s = 1$ cannot persist as we move up the tower.

To state this precisely, we introduce some notation. Let K be a number field and denote by d_K the absolute value of the discriminant of $K/\mathbb{Q}$. Denote by ρ_K the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of K. It is a well-known result of Stark [10] that for $0 < c < 1/4$, $\zeta_K(s)$ has at most one zero in the region

$$1 - \frac{c}{\log d_K} \ \le\ \sigma\ \le\ 1, \quad |t| \le \frac{c}{\log d_K}. \tag{1}$$

This zero, if it exists, is necessarily real and simple. We shall call such a zero a *Stark zero* (or more precisely, a *c-Stark zero*) and denote it by $\beta = \beta_K$.

The concept of a Stark zero is similar to, but different from, that of a *Siegel zero*. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_q)$. It is known that $\zeta_K(s)$ has at most one zero in the region

$$1 - \frac{c}{\log q} \ \le\ \sigma\ \le\ 1, \quad |t| \le \frac{c}{\log q}, \tag{2}$$

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provided $c > 0$ is sufficiently small. The best known value of c (due to Heath-Brown [5]) is $c = .348$. This exceptional zero, if it exists, is called a Siegel zero. Note that this region is much wider than the region in which a Stark zero can occur. Indeed, in this case,

$$\log d_K \ \sim\ q \log q.$$

For a general field, the concept of a Siegel zero should be defined in terms of Artin conductors. We give such a definition in §3.

Stark showed that if L/K is a (finite) Galois extension and $\zeta_L(s)$ has a *simple* zero at $s = s_0$, then there is a field F with

$$K \subseteq F \subseteq L, \tag{3}$$

such that

(i) F/K is cyclic,
(ii) $\zeta_F(s_0) = 0$,
(iii) if $K \subseteq E \subseteq L$ and $\zeta_E(s_0) = 0$, then $F \subseteq E$.

Moreover, if $s_0 \in \mathbb{R}$, then $[F : K] \leq 2$. This paper is based on a generalization (Theorem 4.1) of Stark's theorem to certain non-normal extensions.

The problem of weakening the requirement that L/K be normal in the theorem quoted above was also considered by Stark [10] himself. Denote by $\tilde{L}$ the normal closure of L/K. By the Aramata-Brauer theorem, $\zeta_L(s)$ divides $\zeta_{\tilde{L}}(s)$, in the sense that the quotient $\zeta_{\tilde{L}}(s)/\zeta_L(s)$ is entire. Thus, $\zeta_{\tilde{L}}(s_0) = 0$. However, in general, it may no longer be a simple zero and so the above theorem of Stark no longer applies. To deal with this, Stark considered zeros sufficiently near $s = 1$ where simplicity could be forced. In particular, this would be the case if s_0 were a Stark zero of $\zeta_{\tilde{L}}(s)$. Let $n = [K : \mathbb{Q}]$ and $g(n) = n!$. If $s_0 = \beta_K$ satisfies the stronger condition

$$\beta \ \geq\ 1 - \frac{1}{4n! \log d_K}, \tag{4}$$

then Stark showed that there is an extension F with $\mathbb{Q} \subseteq F \subseteq K$, $[F : \mathbb{Q}] \leq 2$ and such that $\zeta_F(\beta) = 0$.

By considering special classes of extensions, several authors have attempted to weaken the condition (4) in which this conclusion can be made. In particular, Odlyzko and Skinner [9] considered towers of radical extensions

$$\mathbb{Q} = K_0 \ \subseteq\ K_1 \ \subseteq\ \cdots\ \subseteq K_m \ =\ K. \tag{5}$$

Thus for each i, $K_{i+1} = K_i(\alpha_i^{1/n_i})$ for some $\alpha_i \in K_i$ and $1 \leq n_i \in \mathbb{Z}$. Assuming that

$$[K_{i+1} : K_i] \text{ is odd for all } i, \tag{6}$$

they proved that $\zeta_K(s)$ has no Stark zeros. They asked whether a similar result could be proved if radical extensions were replaced with extensions having solvable normal closure. In §4, we shall prove such a result. Our method is actually simpler than [9].

The Brauer-Siegel theorem asserts that the residue ρ_K of $\zeta_K(s)$ at $s = 1$ satisfies

$$\frac{\log \rho_K}{\log d_K} \longrightarrow 0$$

as we range through a sequence of fields K with

$$\frac{1}{n_K} \log d_K \longrightarrow \infty.$$

Here, $n_K = [K : \mathbb{Q}]$. Thus, for example, if we range through all imaginary quadratic fields, then this says that

$$\frac{\log h_K/\sqrt{d_K}}{\log d_K} \longrightarrow 0,$$

or what is the same,

$$\frac{\log h_K}{\log d_K} \longrightarrow \frac{1}{2} \quad \text{as } d_K \longrightarrow \infty.$$

When one attempts to make this effective, difficulties are encountered because of possible zeros of $\zeta_K(s)$ near $s = 1$. Though such zeros cannot be eliminated, Stark [10] used his result on simple zeros to show that they do not propagate in towers of fields, each element of which is *normal* over the predecessor. We prove (Proposition 5.1) an analogue of this for certain non-normal towers.

The relation between zeros near $s = 1$ and ρ_K has been made very explicit in Stark [10]. In particular, he shows that

$$\rho_K > c_1(1 - \beta_1), \tag{7}$$

where $c_1 > 0$ is an absolute and effective constant and

$$\beta_1 = \begin{cases} \beta_K & \text{if it exists,} \\ 1 - \frac{1}{4 \log d_K} & \text{otherwise,} \end{cases}$$

and β_K is the possible zero in (1). Moreover, if $K/\mathbb{Q}$ is Galois, the existence of F in (3) with $\zeta_F(\beta_K) = 0$ implies that

$$\beta_K < 1 - \frac{c_2}{d_F^{1/2}}, \tag{8}$$

for $c_2 > 0$ absolute and effective. Since

$$d_K \geq d_F^{n_K/2}, \tag{9}$$

this gives

$$\rho_K \geq \frac{c_2}{d_F^{1/2}} > \frac{c_2}{d_K^{1/n_K}} = c_2 \exp(-\frac{1}{n_K} \log d_K). \tag{10}$$

In §5, we note how the result of §4 can be similarly applied.

2. Group theory

Let F be a field and denote by F^s a separable algebraic closure. Let V be a vector space of dimension r over F. A subgroup G of $\mathrm{GL}(V) \simeq \mathrm{GL}_r(F)$ is *irreducible* if the only F-subspaces of V invariant under G are $\{0\}$ and V. We say that G is *completely reducible* if $V = W_1 \oplus \cdots \oplus W_s$, with each W_i irreducible.

A subgroup H of G is called *subnormal* if there is a finite chain of subgroups

$$G = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_k = H, \tag{11}$$

with H_{i+1} normal in H_i, $i = 0, 1, \ldots, k-1$. We need the following result of Dixon ([2], Theorem 2).

Theorem 2.1. *Let G be a solvable, completely reducible subgroup of $\mathrm{GL}_r(F^s)$. Then G has a subnormal Abelian subgroup A with*

$$[G : A] \leq \gamma(r), \tag{12}$$

where

$$\gamma(r) = 12^{r-1} 3^{1/3}. \tag{13}$$

Actually, we need the following small variant.

Proposition 2.1. *Suppose H is an irreducible, solvable subroup of $\mathrm{GL}_r(\mathbb{F}_p)$. Then H has a subnormal Abelian subgroup of index $\leq \gamma(r)$.*

Proof. (Dixon) As H is irreducible, it does not contain any normal p-subgroup larger than $\{1\}$. Hence, by [3], Corollary 2.4, H is completely reducible, viewed as a subgroup of $\mathrm{GL}_r(\overline{\mathbb{F}}_p)$. Now the conclusion follows from Theorem 2.1.

3. Zero-free regions for L-functions

We begin by defining the concept of a Siegel zero of $\zeta_K(s)$ for any (finite) extension $K/\mathbb{Q}$. Let $\tilde{K}/\mathbb{Q}$ be the normal closure of $K/\mathbb{Q}$ and let H denote $\mathrm{Gal}(\tilde{K}/K)$. Let χ be an irreducible character of $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$ and denote by $\mathfrak{f}_\chi$ the corresponding Artin conductor. Set

$$A = \max |\mathfrak{f}_\chi|,$$

where the maximum ranges over those characters χ for which $(\chi|_H, 1) > 0$. We make the following plausible conjecture.

Conjecture 3.1. *There exists $c > 0$, absolute and effective, such that $\zeta_K(s)$ has at most one zero in the region*

$$1 - \frac{c}{\log A} \leq \sigma \leq 1, \quad |t| \leq \frac{c}{\log A}. \tag{14}$$

This zero, if it exists, is real and simple.

Definition 3.1. *A zero in the region* (14) *is called a* Siegel zero *(or more precisely, a* c-Siegel zero*).*

This usage is then consistent with what is known to hold in the cyclotomic case as explained in the Introduction. Conjecture 3.1 is known to hold for $\tilde{K} = K$ Abelian over $\mathbb{Q}$.

To state what is known in general, we need some notation. Let L/K be a Galois extension of number fields. For a character χ of $\mathrm{Gal}(L/K)$, let us set

$$A_\chi \;=\; d_K^{\chi(1)} \mathrm{N}_{K/\mathbb{Q}} \mathfrak{f}_\chi.$$

Definition 3.2. *We shall say that Artin's conjecture holds for L/K if the Artin L-function $L(s,\chi)$ is entire for each irreducible non-identity character χ of* $\mathrm{Gal}(L/K)$.

Under the assumption of Artin's conjecture for L/K, one has the following result ([6], Proposition 3.8).

Proposition 3.1. *Let L/K be a Galois extension and assume that Artin's conjecture holds for L/K. Set*

$$A \;=\; \max A_\chi, \quad d \;=\; \max\; \chi(1).$$

where the maximum is taken over all irreducible characters χ of $\mathrm{Gal}(L/K)$. *Then there is a constant $0 < c_1 \leq 1$, absolute and effective, such that $\zeta_L(s)$ has at most one zero in the region*

$$1 - \frac{c_1}{d^3 \log A} \;\leq\; \sigma \;\leq\; 1, \quad |t| \leq \frac{c_1}{d^3 \log A}.$$

If it exists, this zero, β_1 (say), is necessarily simple, real and belongs to a character χ_1 (say) which is Abelian and real.

Remark. It would be of interest to eliminate the factor d^3 in the above.

A conjecture related to Conjecture 3.1 is the following.

Conjecture 3.2. *There exists a constant $c > 0$, absolute and effective, such that $\zeta_K(s)$ has at most one zero in the region*

$$1 - \frac{cn_K}{\log d_K} \;\leq\; \sigma \;\leq\; 1, \quad |t| \leq \frac{cn_K}{\log d_K}. \tag{15}$$

Next, we need some estimates for A_χ in the Abelian case.

Proposition 3.2. *Let L/K be an Abelian extension. Then*

$$\max_\chi \; \log A_\chi \;\leq\; \frac{2}{[L:K]} \log d_L,$$

where the maximum ranges over all irreducible characters of $\mathrm{Gal}(L/K)$.

Proof. First, consider the case of L/K cyclic of prime power order. The maximum on the left is attained by a character ψ which generates the group of characters. From this, we see that

$$A_{\psi^i} \;=\; A_\psi,$$

if ψ^i is also a generator. Let $t = [L : K]$. By the conductor-discriminant formula,

$$A_\psi^{\phi(t)} \leq d_L.$$

If $t = p^\alpha$, this gives

$$\log A_\psi \leq \frac{1}{p^{\alpha-1}(p-1)} \log d_L = \frac{1}{t}\left(1 + \frac{1}{p-1}\right) \log d_L \leq \frac{2}{t} \log d_L.$$

Second, consider the case of a general cyclic extension. Then, L is the compositum of disjoint extensions $L_1, \ldots, L_r$ with L_i/K cyclic of order $t_i = p_i^{\alpha_i}$, and $t = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Moreover, we can write $\psi = \psi_1 \cdots \psi_r$, with each ψ_i a character of $\mathrm{Gal}(L_i/K)$. If ψ is a generator, so is each ψ_i a generator of the character group of $\mathrm{Gal}(L_i/K)$. Moreover, A_ψ divides $A_{\psi_1} \cdots A_{\psi_r}$. Hence

$$\log A_\psi \leq \sum \log A_{\psi_i} \leq \sum \frac{2}{t_i} \log d_{L_i} = \frac{2}{t} \log d_L.$$

Third, in the general case, let us say that

$$\log A_\psi = \max_\chi \log A_\chi.$$

The fixed field of the kernel of ψ gives a cyclic extension L_ψ of K. Then by the previous case,

$$\begin{aligned} [L : K] \log A_\psi &\leq 2[L : L_\psi] \log d_{L_\psi} \\ &\leq 2 \log d_L. \end{aligned}$$

4. Solvable Galois extensions

The main result of this section generalizes the result of Odlyzko-Skinner [9] to extensions with solvable normal closure.

We need some notation. Let

$$n = [L : K],$$

and

$$e(n) = \max_{p^\alpha || n} \alpha.$$

Then set

$$\delta(n) = (e(n) + 1)^2 \gamma(e(n)) = (e(n) + 1)^2 3^{1/3} 12^{e(n)-1}.$$

Since

$$e(n) \leq \frac{\log n}{\log 2},$$

we certainly have

$$\delta(n) \ll n^4.$$

Let $c_1 > 0$ be as in Proposition 3.1.

Theorem 4.1. *Let $0 < c < c_1/2$ and set $n = [L : K]$. Suppose that n is odd and that the Galois closure $\tilde{L}$ of L/K is solvable over K. Suppose that $\zeta_L(s)$ has a zero β in the range*

$$1 - \frac{c}{\delta(n)\log d_L} \le \sigma \le 1, \quad |t| \le \frac{c}{\delta(n)\log d_L}.$$

Then, $\zeta_K(\beta) = 0$.

Remark. Note that this zero β is necessarily real and simple because it lies in the Stark region (1).

Proof. Let $G = \mathrm{Gal}(\tilde{L}/K)$ and $H = \mathrm{Gal}(\tilde{L}/L)$. We shall proceed by induction on the degree $n = [L : K]$. As $\tilde{L}$ is the normal closure, H does not contain any nontrivial normal subgroup of G. Assume first that H is maximal. Let A be a minimal normal subgroup. Then $G = HA$. Moreover, $H \cap A = \{1\}$. Indeed, $H \cap A$ is normalized by H (as H normalizes itself and A is normal in G) and by A (as A is Abelian and so commutes with $H \cap A$). Thus, it is normal in G and as it is contained in H, it must be trivial.

The action of H on A is given by a map

$$H \longrightarrow \mathrm{Aut}(A).$$

This action is faithful, as the kernel is a normal subgroup of G contained in H. As G is solvable, A is elementary Abelian, say $A \simeq (\mathbb{Z}/p)^r$. Notice that

$$p^r = |A| = [L : K].$$

Thus we have an inclusion

$$H \hookrightarrow \mathrm{GL}_r(\mathbb{Z}/p).$$

This identifies H with an irreducible subgroup of $\mathrm{GL}_r(\mathbb{Z}/p)$, for any invariant subspace would be normal in G and contained in A. Now, we can apply Proposition 2.1 to deduce that there is a subnormal Abelian subgroup H_1 of H (say) with

$$[H : H_1] \le \gamma(r).$$

Let L_1 denote the subfield of $\tilde{L}$ fixed by H_1. We have

$$|H_1| \log d_{L_1} \le \log d_{\tilde{L}}.$$

On the other hand, let $a_1, \cdots, a_r$ be generators of A. Then

$$H \bigcap (\cap\, a_i H a_i^{-1}) = \{1\},$$

as it is a normal subgroup of G in H. Hence, $\tilde{L}$ is the compositum of $r+1$ conjugates of L. Hence,

$$\log d_{\tilde{L}} \le |H|(r+1)\log d_L.$$

Putting these estimates together, we deduce that

$$\log d_{L_1} \le (r+1)\gamma(r)\log d_L.$$

Hence,

$$1-\frac{c}{(r+1)\log d_{L_1}} \;\le\; 1-\frac{c}{\delta(n)\log d_L}.$$

Now, as H_1 is subnormal in H, we can apply the Aramata-Brauer theorem in stages to deduce that $\zeta_L(s)$ divides $\zeta_{L_1}(s)$. In particular, $\zeta_{L_1}(s)$ vanishes at $s=\beta$. Moreover, as it lies in the Stark region (1), it is a simple zero.

Next, we wish to show that this is a simple zero of $\zeta_{\tilde{L}}(s)$. To do this, it suffices to show that it falls into the region considered in Proposition 3.1. Applying Proposition 3.2, we see that for any character χ of $\mathrm{Gal}(\tilde{L}/L_1)$,

$$\log A_\chi \;\le\; \frac{2}{[\tilde{L}:L_1]}\log d_{\tilde{L}} \;\le\; \frac{2}{[\tilde{L}:L_1]}(r+1)|H|\log d_L,$$

and so

$$\log A_\chi \;\le\; 2(r+1)\log d_{L_1}.$$

Thus, β is a simple real zero of $\zeta_{\tilde{L}}(s)$ and by Stark's theorem, there is an extension N with

$$K \subseteq N \subseteq \tilde{L},$$

with $[N:K]\le 2$ and $\zeta_N(\beta)=0$. Moreover, by the uniqueness part, and as $\zeta_L(\beta)=0$, it follows that

$$K \subseteq N \subseteq L.$$

As $[L:K]$ is assumed to be odd, $N=K$.

Now consider the case when H is not necessarily maximal. Suppose that

$$H \subset H_1 \subset G,$$

and let L_1 denote the fixed field of H_1. Then L/L_1 has solvable normal closure. Moreover, if $m=[L:L_1]$, then m is odd, $m<n$ and $\delta(m)\le\delta(n)$ so the induction hypothesis applies and we can conclude that $\zeta_{L_1}(\beta)=0$. Now we apply the theorem to L_1/K and the result follows by induction.

5. Application to the Brauer-Siegel theorem

We begin by proving a slight strengthening of Theorem 4.1. We will be considering the class $\mathfrak{S}$ of number fields M which have solvable normal closure over $\mathbb{Q}$. We define a subclass $\mathfrak{S}_R$ of fields for which there is a chain

$$\mathbb{Q} \;=\; M_0 \;\subseteq M_1 \subseteq \cdots \subseteq M_t \;=\; M,$$

such that for each $1\le i\le t$, the extension M_i/M_{i-1} has odd degree and solvable normal closure and

$$\max_i e([M_{i+1}:M_i]) \;\le\; R.$$

Proposition 5.1. *Let $M\in\mathfrak{S}_R$. Let $0<c<c_1/2$. Then, $\zeta_M(s)$ does not have any real zeros in the range*

$$1-\frac{c}{(R+1)^2\gamma(R)\log d_M}\le\sigma\;\le 1, \tag{16}$$

Proof. Let $K \subseteq L \subseteq M$ be fields with L/K and M/L of odd degree and having solvable normal closure. Suppose that $\zeta_M(\beta) = 0$ with β in the range (16). By Theorem 4.1, $\zeta_L(\beta) = 0$ Hence, we can shorten the sequence by one field and in doing this, we do not increase R. Moreover,

$$\log d_L \ \leq \ \log d_M.$$

Hence, we can proceed inductively to deduce that $\zeta_{\mathbb{Q}}(\beta) = 0$. But $\zeta_{\mathbb{Q}}$ has no real zeros in the critical strip. This proves the result.

Now using Proposition 5.1 and the estimate of Stark quoted in (7) we deduce the following.

Theorem 5.1. *There is an effective absolute constant $c > 0$ such that for any field K in $\mathfrak{S}_R$, we have*

$$\rho_K > \frac{c}{(R+1)^2 \gamma(R) \log d_K}.$$

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