Mathematical Research Letters **6**, 563–572 (1999)

SOME PROPERTIES AND EXAMPLES OF TRIANGULAR POINTED HOPF ALGEBRAS

Shlomo Gelaki

1. Introduction

A fundamental problem in the theory of Hopf algebras is the classification and construction of finite-dimensional (minimal) triangular Hopf algebras (A, R) introduced by Drinfeld [D]. Only recently this problem was completely solved for semisimple A over algebraically closed fields of characteristics 0 and $p \gg \dim(A)$ (any p if one assumes that A is also cosemisimple) [EG2,EG3].

In this paper we take the first step towards solving this problem for finite-dimensional pointed Hopf algebras over an algebraically closed field k of characteristic 0.

We first prove that the fourth power of the antipode of any triangular pointed Hopf algebra A is the identity. We do that by focusing on *minimal* triangular pointed Hopf algebras (A, R) (every triangular Hopf algebra contains a minimal triangular sub Hopf algebra) and proving that the group algebra of the group of grouplike elements of A (which must be abelian) admits a minimal triangular structure and consequently that A has the structure of a biproduct [R1]. We also generalize our result on the order of the antipode to any finite-dimensional quasitriangular Hopf algebra A whose Drinfeld element u acts as a scalar in any irreducible representation of A (e.g., when A^* is pointed).

Second, we describe a method of construction of finite-dimensional pointed Hopf algebras which admit a minimal triangular structure, and classify all their minimal triangular structures.

We conclude the paper by proving that *any* minimal triangular Hopf algebra which is generated as an algebra by grouplike elements and skew primitive elements is isomorphic to a minimal triangular pointed Hopf algebra constructed using our method.

Throughout the paper, unless otherwise specified, the ground field k will be assumed to be algebraically closed with characteristic 0.

Received October 7, 1999.

This research was supported by THE ISRAEL SCIENCE FOUNDATION founded by the Israel Academy of Sciences and Humanities.

2. Pointed Hopf algebras

The Hopf algebras which are studied in this paper are pointed. Recall that a Hopf algebra A is pointed if its simple subcoalgebras are all 1-dimensional or equivalently (when A is finite-dimensional) if the irreducible representations of A^* are all 1-dimensional. Let $G(A)$ denote the group of grouplike elements of A. For any $g, h \in G(A)$, we denote the vector space of $g:h$ skew primitives of A by $P_{g,h}(A) := \{x \in A | \Delta(x) = x \otimes g + h \otimes x\}$. Thus the classical primitive elements of A are $P(A) := P_{1,1}(A)$. The element $g - h$ is always $g:h$ skew primitive. Let $P'_{g,h}(A)$ denote a complement of $\mathrm{sp}_k\{g - h\}$ in $P_{g,h}(A)$. Taft-Wilson theorem [TW] states that the first term A_1 of the coradical filtration of A is given by:

$$A_1 = kG(A) \bigoplus \left(\bigoplus_{g,h \in G(A)} P'_{g,h}(A) \right). \tag{1}$$

In particular, if A is *not* cosemisimple then there exists $g \in G(A)$ such that $P'_{1,g}(A) \neq 0$.

If A is a Hopf algebra over the field k, which is generated as an algebra by a subset S of $G(A)$ and by $g:g'$ skew primitive elements, where g, g' run over S, then A is pointed and $G(A)$ is generated as a group by S (see e.g., [R4, Lemma 1]).

3. The antipode of triangular pointed Hopf algebras

In this section we recall some of the properties of finite-dimensional triangular Hopf algebras, and prove some new general results about finite-dimensional triangular pointed Hopf algebras.

Let (A, R) be a finite-dimensional triangular Hopf algebra over a field k of characteristic 0. Recall that the $R-$matrix R satisfies the relation $R^{-1} = R_{21}$ or equivalently, the Drinfeld element u of A is a grouplike element [D]. If A is semisimple then $u^2 = 1$ (see e.g., [EG1]). The associated map $f_R : A^{*cop} \to A$ defined by $f_R(p) = (p \otimes I)(R)$ is a Hopf algebra homomorphism which satisfies

$$\sum \langle p_{(1)}, a_{(2)} \rangle a_{(1)} f_R(p_{(2)}) = \sum \langle p_{(2)}, a_{(1)} \rangle f_R(p_{(1)}) a_{(2)}, \quad p \in A^*, \quad a \in A \tag{2}$$

and

$$\sum f_R(p_{(1)}) f_R^*(p_{(2)}) = \langle p, 1 \rangle 1, \quad p \in A^*. \tag{3}$$

Observe that (2) is equivalent to the condition $\Delta^{cop}(a)R = R\Delta(a)$ for all $a \in A$, and (3) is equivalent to the condition $RR_{21} = 1 \otimes 1$.

Let $A_R \subseteq A$ be the minimal sub Hopf algebra of A corresponding to R [R2]. It is straightforward to verify that the corresponding map $f_R : A_R^{*cop} \to A_R$ defined by $f_R(p) = (p \otimes I)(R)$ is a Hopf algebra isomorphism. This property of minimal triangular Hopf algebras will play a central role in this paper. It implies in particular that $G(A_R) \cong G(A_R^*)$, and hence that the group $G(A_R)$ is abelian (see e.g., [G2]). Thus, $G(A_R) \cong G(A_R)^*$ (where $G(A_R)^*$ denotes the character group of $G(A_R)$), and we can identify the Hopf algebras $k[G(A_R)^*]$

and $k[G(A_R)]^*$. Also, if (A,R) is minimal triangular and pointed then f_R being an isomorphism implies that A^* is pointed as well.

Note that if (A,R) is (quasi)triangular and $\pi : A \to A'$ is an onto morphism of Hopf algebras, then (A',R') is (quasi)triangular as well, where $R' = (\pi\otimes\pi)(R)$.

We are ready now to prove our first results.

Theorem 3.1. *Let (A,R) be a minimal triangular pointed Hopf algebra with Drinfeld element u over an algebraically closed field k of characteristic 0, and set $K := k[G(A)]$. Then there exists a projection of Hopf algebras $\pi : A \to K$, and consequently $A = B \times K$ is a biproduct where*

$$B = \{x \in A|(I\otimes\pi)\Delta(x) = x\otimes 1\} \subseteq A.$$

Moreover, K admits a minimal triangular structure with Drinfeld element $u_{_K} = u$.

Proof. We know that $k[G(A)]^* \cong k[G(A)]$ and $k[G(A)] \cong k[G(A^{*cop})]$ as Hopf algebras, and hence that $\dim(k[G(A)]^*) = \dim(k[G(A^*)])$. Consider the series of Hopf algebra homomorphisms

$$k[G(A)] \overset{i}{\hookrightarrow} A^{cop} \xrightarrow{f_R^{-1}} A^* \xrightarrow{i^*} k[G(A)]^*,$$

where i is the inclusion map. Since A^* is pointed it follows from [M, 5.3.5] and the above remarks that $i^*_{|k[G(A^*)]} : k[G(A^*)] \to k[G(A)]^*$ is an isomorphism of Hopf algebras, and hence that $i^*\circ f_R^{-1}\circ i$ determines a minimal quasitriangular structure on $k[G(A)]^*$. This structure is in fact triangular since f_R^{-1} determines a triangular structure on A^*. Clearly, $(i^*\circ f_R^{-1}\circ i)(u) = u_{_{K^*}}^{-1} = u_{_{K^*}}$ is the Drinfeld element of K^*. Since $k[G(A)]$ and $k[G(A)]^*$ are isomorphic as Hopf algebras we conclude that K admits a minimal triangular structure with Drinfeld element $u_{_K} = u$.

Finally, set $\varphi := i^*\circ f_R^{-1}\circ i$ and $\pi := \varphi^{-1}\circ i^*\circ f_R^{-1}$. Then $\pi : A \to k[G(A)]$ is onto, and moreover $\pi\circ i = \varphi^{-1}\circ i^*\circ f_R^{-1}\circ i = \varphi^{-1}\circ\varphi = id_{k[G(A)]}$. Hence π is a projection of Hopf algebras and by [R1], $A = B\times K$ is a biproduct where $B = \{x \in A|(I\otimes\pi)\Delta(x) = x\otimes 1\}$ as desired. This concludes the proof of the theorem. ∎

Theorem 3.2. *Let (A,R) be any triangular pointed Hopf algebra with antipode S and Drinfeld element u over a field k of characteristic 0. Then $S^4 = I$ is the identity map of A. If in addition A_R is not semisimple and A is finite-dimensional then $\dim(A)$ is divisible by 4.*

Proof. We may assume that k is algebraically closed. By [D], $S^2(x) = uxu^{-1}$ for all $x \in A$. Let $K := k[G(A_R)]$. Since $u \in A_R$, and by Theorem 3.1, $u = u_{_K}$ and $u_{_K}^2 = 1$, we have that $S^4 = I$.

In order to prove the second claim, we may assume that (A,R) is minimal (since by [NZ], $\dim(A_R)$ divides $\dim(A)$). Since A is not semisimple it follows from [LR] that $S^2 \neq I$, and hence that $u \neq 1$. In particular, $|G(A)|$ is even. Now, let B be as in Theorem 3.1. Since $S^2(B) = B$, B has a basis $\left\{a_i, b_j | S^2(a_i) = a_i, S^2(b_j) = -b_j, 1 \le i \le n, 1 \le j \le m\right\}$. Hence by Theorem 3.1,

$$\{a_i g, b_j g | g \in G(A), 1 \le i \le n, 1 \le j \le m\}$$

is a basis of A. Since by [R3], $\mathrm{tr}(S^2) = 0$, we have that $0 = \mathrm{tr}(S^2) = |G(A)|(n - m)$, which implies that $n = m$, and hence that $\dim(B)$ is even as well. $\square$

In fact, the first part of Theorem 3.2 can be generalized.

Theorem 3.3. *Let (A, R) be a finite-dimensional quasitriangular Hopf algebra with antipode S over a field k of characteristic* 0, *and suppose that the Drinfeld element u of A acts as a scalar in any irreducible representation of A (e.g., when A^* is pointed). Then $u = S(u)$ and in particular $S^4 = I$.*

Proof. We may assume that k is algebraically closed. In any irreducible representation V of A, $\mathrm{tr}_{|V}(u) = \mathrm{tr}_{|V}(S(u))$ (see [EG1]). Since $S(u)$ also acts as a scalar in V (the dual of $S(u)_{|V}$ equals $u_{|V^*}$) it follows that $u = S(u)$ in any irreducible representation of A. Therefore, there exists a basis of A in which the operators of left multiplication by u and $S(u)$ are represented by upper triangular matrices with the same main diagonal. Hence the special grouplike element $uS(u)^{-1}$ is unipotent. Since it has a finite order we conclude that $uS(u)^{-1} = 1$, and hence that $S^4 = I$. $\square$

Remark 3.4. *If (A, R) is a minimal triangular pointed Hopf algebra then all its irreducible representations are* $1-$*dimensional. Hence Theorem* 3.3 *is applicable, and the first part of Theorem* 3.2 *follows.*

Example 3.5. *Let A be Sweedler's* 4*-dimensional Hopf algebra. It is generated as an algebra by a grouplike element g and a $1 : g$ skew primitive element x satisfying the relations $g^2 = 1$, $x^2 = 0$ and $gx = -xg$. It is known* [R2] *that A admits minimal triangular structures all of which with g as the Drinfeld element. In this example, $K = k[\langle g \rangle]$ and $B = \mathrm{sp}\{1, x\}$. Note that g is central in K but is not central in A, so $(S_{|K})^2 = I_K$ but $S^2 \ne I$ in A. However, $S^4 = I$.*

Theorem 3.2 motivates the following two questions.

Question 3.6.

1) Is the fourth power of the antipode of any finite-dimensional triangular Hopf algebra equal to the identity?
2) Does the Drinfeld element u of any finite-dimensional triangular Hopf algebra satisfy $u^2 = 1$?

Note that a positive answer to Question 3.6 2) will imply that an odd-dimensional triangular Hopf algebra must be semisimple.

4. Construction of minimal triangular pointed Hopf algebras

In this section we give a method for the construction of minimal triangular pointed Hopf algebras which are *not* necessarily semisimple.

Let G be a finite abelian group, and $F : G\times G \to k^*$ be a non-degenerate skew symmetric bilinear form on G. That is, $F(xy,z) = F(x,z)F(y,z)$, $F(x,yz) = F(x,y)F(x,z)$, $F(1,x) = F(x,1) = 1$, $F(x,y) = F(y,x)^{-1}$ for all $x,y,z \in G$, and the map $f : G \to G^*$ defined by $< f(x), y >= F(x,y)$ for all $x,y \in G$ is an isomorphism. Let G_2 be the subgroup of involutions of G, and $U_F : G_2 \to \{-1,1\}$ be defined by $U_F(g) = F(g,g)$. Then U_F is a homomorphism of groups. Denote $U_F^{-1}(-1)$ by I_F.

Definition 4.1. *Let k be an algebraically closed field of characteristic zero. A datum $\mathcal{D} = (G,F,n)$ is a triple where G is a finite abelian group, $F : G\times G \to k^*$ is a non-degenerate skew symmetric bilinear form on G, and n is a non-negative integer function $I_F \to \mathbb{Z}^+$, $g \mapsto n_g$.*

Remark 4.2.

1) *The map $f : k[G] \to k[G^*]$ determined by $\langle f(g), h\rangle = F(g,h)$ for all g, $h \in G$ determines a minimal triangular structure on $k[G^*]$.*
2) *If I_F is not empty then G has an even order.*

To each datum $\mathcal{D}$ we associate a Hopf algebra $H(\mathcal{D})$ in the following way. For each $g \in I_F$, let V_g be a vector space of dimension n_g, and let $\mathcal{B} = \bigoplus_{g\in I_F} V_g$. Then $H(\mathcal{D})$ is generated as an algebra by $G \cup \mathcal{B}$ with the following additional relations (to those of the group G and the vector spaces V_g's):

$$xy = F(g,h)yx \quad \text{and} \quad xa = F(g,a)ax \tag{4}$$

for all $g,h \in I_F$, $x \in V_g$, $y \in V_h$ and $a \in G$.

The coalgebra structure of $H(\mathcal{D})$ is determined by letting $a \in G$ be a grouplike element and $x \in V_g$ be a $1 : g$ skew primitive element for all $g \in I_F$. In particular, $\varepsilon(a) = 1$ and $\varepsilon(x) = 0$ for all $a \in G$ and $x \in V_g$.

In the special case where $G = \mathbb{Z}_2 = \{1,g\}$, $F(g,g) = -1$ and $n := n_g$, the associated Hopf algebra will be denoted by $H(n)$. Clearly, $H(0) = k\mathbb{Z}_2$, $H(1)$ is Sweedler's 4-dimensional Hopf algebra, and $H(2)$ is the 8-dimensional Hopf algebra studied in [G1, Section 2.2] in connection with KRH invariants of knots and 3-manifolds. We remark that the Hopf algebras $H(n)$ are studied in [PO1,PO2] where they are denoted by $E(n)$.

For a finite-dimensional vector space V we let $\bigwedge V$ denote the exterior algebra of V. Set $B := \bigotimes_{g\in I_F} \bigwedge V_g$.

Proposition 4.3.

1) *The Hopf algebra $H(\mathcal{D})$ is pointed and $G(H(\mathcal{D})) = G$.*
2) *$H(\mathcal{D}) = B \times k[G]$ is a biproduct.*
3) *$H(\mathcal{D})_1 = k[G] \bigoplus (k[G]\mathcal{B})$, and $P_{a,b}(H(\mathcal{D})) = sp\{a - b\} \bigoplus aV_{a^{-1}b}$ for all $a,b \in G$ (here we agree that $V_{a^{-1}b} = 0$ if $a^{-1}b \notin I_F$).*

Proof. Part 1) follows since (by definition) $H(\mathcal{D})$ is generated as an algebra by grouplike elements and skew primitive elements. Now, it is straightforward to verify that the map $\pi : H(\mathcal{D}) \to k[G]$ determined by $\pi(a) = a$ and $\pi(x) = 0$ for all $a \in G$ and $x \in \mathcal{B}$ is a projection of Hopf algebras. Since $B = \{x \in H(\mathcal{D}) | (I \otimes \pi)\Delta(x) = x \otimes 1\}$, Part 2) follows from [R1]. Finally, by Part 2), B is a braided graded Hopf algebra in the Yetter-Drinfeld category ${}^{k[G]}_{k[G]}\mathcal{Y}D$ (see e.g., [AS]) with respect to the grading where the elements of $\mathcal{B}$ are homogeneous of degree 1. Write $B = \bigoplus_{n\geq 0} B(n)$, where $B(n)$ denotes the homogeneous component of degree n. Then, $B(0) = k1 = B_1$ (since $B \cong H(\mathcal{D})/H(\mathcal{D})k[G]^+$ as coalgebras, it is connected). Furthermore, by similar arguments used in the proof of [AS, Lemma 3.4], $P(B) = B(1) = \mathcal{B}$. But then by [AS, Lemma 2.5], $H(\mathcal{D})$ is coradically graded (where the *nth* component $H(\mathcal{D})(n)$ is just $B(n) \times k[G]$) which means by definition that $H(\mathcal{D})_1 = H(\mathcal{D})(0) \bigoplus H(\mathcal{D})(1) = k[G] \bigoplus (k[G]\mathcal{B})$ as desired. The second statement of Part 3) follows now, using (1), by counting dimensions. □

In the following we determine *all* the minimal triangular structures on $H(\mathcal{D})$. Let $f : k[G] \to k[G^*]$ be the isomorphism from Remark 4.2 1), and set $I'_F := \{g \in I_F | n_g \neq 0\}$. Let Φ be the set of all isomorphisms $\varphi : G^* \to G$ satisfying $\varphi^*(\alpha) = \varphi(\alpha^{-1})$ for all $\alpha \in G^*$, and $(\varphi \circ f)(g) = g$ for all $g \in I'_F$.

Extend any $\alpha \in G^*$ to an algebra homomorphism $H(\mathcal{D}) \to k$ by setting $\alpha(z) = 0$ for all $z \in \mathcal{B}$. Extend any $x \in V_g^*$ to $P_x \in H(\mathcal{D})^*$ by setting $\langle P_x, ay\rangle = 0$ for all $a \in G$ and $y \in \bigotimes_{g\in I_F} \bigwedge V_g$ of degree different from 1, and $\langle P_x, ay\rangle = \delta_{g,h}\langle x, y\rangle$ for all $a \in G$ and $y \in V_h$. We shall identify the vector spaces V_g^* and $\{P_x | x \in V_g^*\}$ via the map $x \mapsto P_x$.

Let $S_g(k)$ be the set of all symmetric isomorphisms $V_g^* \to V_g$. Note that $S_g(k)$ is in one to one correspondence with the set of all symmetric non-degenerate bilinear forms on V_g.

Theorem 4.4.

1) *For each* $T := (\varphi, (M_g)_{g\in I_F}) \in \Phi \times (\times_{g\in I_F} S_g(k))$, *there exists a unique Hopf algebra isomorphism* $f_T : H(\mathcal{D})^{*cop} \to H(\mathcal{D})$ *determined by* $\alpha \mapsto \varphi(\alpha)$ *and* $P_x \mapsto M_g(x)$ *for* $\alpha \in G^*$ *and* $x \in V_g^*$.
2) *There is a one to one correspondence between* $\Phi \times (\times_{g\in I_F} S_g(k))$ *and the set of minimal triangular structures on* $H(\mathcal{D})$ *given by* $T \mapsto f_T$.

Proof. We first show that f_T is a well defined isomorphism of Hopf algebras. Using Proposition 4.3 2), it is straightforward to verify that

$$\Delta(P_x) = \varepsilon \otimes P_x + P_x \otimes f(g), \quad P_x\alpha = \langle \alpha, g\rangle \alpha P_x, \quad \text{and} \quad P_xP_y = F(g,h)P_yP_x,$$

for all $\alpha \in G^*$, $g, h \in I_F$, $x \in V_g^*$ and $y \in V_h^*$. Let $\mathcal{B}^* := \{P_x | x \in V_g^*,\ g \in I_F\}$, and H be the sub Hopf algebra of $H(\mathcal{D})^{*cop}$ generated as an algebra by $G^* \bigcup \mathcal{B}^*$. Then, using (4) and our assumptions on T, it is straightforward to verify that the map $f_T^{-1} : H(\mathcal{D}) \to H$ determined by $a \mapsto \varphi^{-1}(a)$ and $z \mapsto M_g^{-1}(z)$ for $a \in G$ and $z \in V_g$, is a surjective homomorphism of Hopf algebras. Let us verify for instance that $f_T^{-1}(za) = F(g,a)f_T^{-1}(az)$. Indeed, this is equivalent to

$\langle\varphi^{-1}(a), g\rangle = \langle f(g), a\rangle$ which in turn holds by our assumptions on φ and the fact that g is an involution. Now, using Proposition 4.3 3), it is straightforward to verify that f_T^{-1} is injective on $P_{a,b}(H(\mathcal{D}))$ for all $a, b \in G$. Since $H(\mathcal{D})$ is pointed, f_T^{-1} is also injective (see e.g., [M, Corollary 5.4.7]). This implies that $H = H(\mathcal{D})^{*cop}$, and that $f_T : H(\mathcal{D})^{*cop} \to H(\mathcal{D})$ is an isomorphism of Hopf algebras as desired. Note that in particular, $G^* = G(H(\mathcal{D})^*)$.

The fact that f_T satisfies (2) follows from a straightforward computation (using (4)) since it is enough to verify it for algebra generators $p \in G^* \cup \mathcal{B}^*$ of $H(\mathcal{D})^{*cop}$, and $a \in G \cup \mathcal{B}$ of $H(\mathcal{D})$.

We have to show that f_T satisfies (3). Indeed, it is straightforward to verify that $f_T^* : H(\mathcal{D})^{*op} \to H(\mathcal{D})$ is determined by $\alpha \mapsto \varphi^*(\alpha)$ and $P_x \mapsto gM_g^*(x) = gM_g(x)$ for $\alpha \in G^*$ and $x \in V_g^*$. Hence, $f_T^* = f_T \circ S$, where S is the antipode of $H(\mathcal{D})$, as desired.

We now have to show that *any* minimal triangular structure on $H(\mathcal{D})$ comes from f_T for some T. Indeed, let $\mathbf{f} : H(\mathcal{D})^{*cop} \to H(\mathcal{D})$ be any Hopf isomorphism. Then $\mathbf{f}$ must map G^* onto G, $\{f(g)|g \in I'_F\}$ onto I'_F, and $P_{f(g),\varepsilon}(H(\mathcal{D})^{*cop})$ bijectively onto $P_{1,\varphi(f(g))}(H(\mathcal{D}))$. Therefore $\varphi := \mathbf{f}_{|G^*}$ preserves the function n, and for all $g \in I_F$ there exists an invertible operator $M_g : V_g^* \to V_{\varphi(f(g))}$ such that $\mathbf{f}$ is determined by $\alpha \mapsto \varphi(\alpha)$ and $P_x \mapsto M_g(x)$. Suppose $\mathbf{f}$ satisfies (2). Then letting $p = P_x$ and $a \in G$ in (2) yields that $a\mathbf{f}(P_x) = F(g, a)\mathbf{f}(P_x)a$ for all $a \in G$. But by (4), this is equivalent to $(\varphi \circ f)(g) = g$ for all $g \in I'_F$. Since by Theorem 3.1, $\varphi : k[G^*] \to k[G]$ determines a minimal triangular structure on $k[G]$ it follows that $\varphi \in \Phi$. Since $\mathbf{f} : H(\mathcal{D})^{*cop} \to H(\mathcal{D})$ satisfies (3) it follows that M_g is symmetric for all $g \in I_F$, and hence $\mathbf{f}$ is of the form f_T for some T as desired. □

For a triangular structure on $H(\mathcal{D})$ corresponding to the map f_T, we let R_T denote the corresponding R-matrix.

5. Classification of minimal triangular Hopf algebras generated as algebras by grouplike and skew primitive elements

In this section we use Theorems 3.1 and 3.2 to classify minimal triangular Hopf algebras which are generated as algebras by grouplike elements and skew primitive elements. Namely, we prove:

Theorem 5.1. *Let (A, R) be a minimal triangular Hopf algebra over an algebraically closed field k of characteristic* 0. *If A is generated as an algebra by grouplike elements and skew primitive elements, then there exist a datum $\mathcal{D}$ and $T \in \Phi \times (\times_{g\in I_F} S_g(k))$ such that $(A, R) \cong (H(\mathcal{D}), R_T)$ as triangular Hopf algebras.*

Before we prove Theorem 5.1 we need to fix some notation and prove a few lemmas.

In what follows, (A, R) will always be a minimal triangular pointed Hopf algebra over k, $G := G(A)$ and $K := k[G(A)]$. For any $g \in G$, $P_{1,g}(A)$ is a

$\langle g\rangle$-module under conjugation, and $\mathrm{sp}\{1-g\}$ is a submodule of $P_{1,g}(A)$. Let $V_g \subset P_{1,g}(A)$ be its complement, and set $n_g := \dim(V_g)$.

By Theorem 3.1, $A = B \times K$ where $B = \{x \in A | (I \otimes \pi)\Delta(x) = x \otimes 1\} \subseteq A$ is a left coideal subalgebra of A (equivalently, B is an object in the Yetter-Drinfeld category ${}^{k[G]}_{k[G]}\mathcal{YD}$). Note that $B \cap K = k1$. Let $\rho : B \to K \otimes B$ be the associated comodule structure and write $\rho(x) = \sum x^1 \otimes x^2$, $x \in B$. By [R1], $B \cong A/AK^+$ as coalgebras, hence B is a connected pointed coalgebra. Let $P(B) := \{x \in B | \Delta_B(x) = x \otimes 1 + 1 \otimes x\}$ be the space of primitive elements of B.

Lemma 5.2. *For any $g \in G$, $V_g = \{x \in P(B) | \rho(x) = g \otimes x\}$.*

Proof. Let $x \in V_g$. Since g acts on V_g by conjugation we may assume by [G1, Lemma 0.2], that $gx = \omega xg$ for some $1 \neq \omega \in k$. Since $\pi(x)$ and $\pi(g) = g$ commute we must have that $\pi(x) = 0$. But then $(I \otimes \pi)\Delta(x) = x \otimes 1$ and hence $x \in B$. Since $\Delta(x) = \sum x_1 \times x_2^1 \otimes x_2^2 \times 1$, applying the maps $\varepsilon \otimes I \otimes I \otimes \varepsilon$ and $I \otimes \varepsilon \otimes I \otimes \varepsilon$ to both sides of the equation $\sum x_1 \times x_2^1 \otimes x_2^2 \times 1 = x \times 1 \otimes 1 \times 1 + 1 \times g \otimes x \times 1$, yields that $x \in P(B)$ and $\rho(x) = g \otimes x$, as desired.

Suppose that $x \in P(B)$ satisfies $\rho(x) = g \otimes x$. Since $\Delta(x) = x \otimes 1 + \rho(x)$, it follows that $x \in V_g$ as desired. □

Lemma 5.3. *If $V_g \neq 0$ then the sub Hopf algebra of A generated as an algebra by $\{g\} \cup V_g$ is isomorphic to the Hopf algebra $H(n_g)$ from Section 4.*

Proof. Let $0 \neq x \in V_g$. Then $S^2(x) = g^{-1}xg$, $g^{-1}xg \neq x$ by [G1, Lemma 0.2], and $g^{-1}xg \in V_g$. Since by Theorem 3.2, $S^4 = I$ it follows that g^2 and x commute, and hence that $gx = -xg$ for all $x \in V_g$.

Second we wish to show that $g^2 = 1$ and $x^2 = 0$. By Lemma 5.2, $x \in B$ and hence $x^2 \in B$ (B is a subalgebra of A). Since $\Delta(x^2) = x^2 \otimes 1 + g^2 \otimes x^2$, and x^2 and g^2 commute, it follows from [G1, Lemma 0.2] that $x^2 = \alpha(1 - g^2) \in K$ for some $\alpha \in k$. We thus conclude that $x^2 = 0$. Now let N be the order of g. Then, using the notation of [G3, Section 1], the sub Hopf algebra H of A generated as an algebra by g, x equals $H_{2,-1,N,1}$. By [G3, Proposition 2.6], $H^* = H_{N,\omega,N,N/2}$ where $\omega \in k$ is a primitive Nth root of unity. Since H^* is triangular (as a homomorphic image of A^*), it follows from [G3, Theorem 2.9] that $N = 2$, and hence $g^2 = 1$ as desired. □

Recall that the map $f_R : A^{*cop} \to A$ is an isomorphism of Hopf algebras, and let $F : G \times G \to k^*$ be the associated non-degenerate skew symmetric bilinear form on G defined by $F(g,h) := \langle f_R^{-1}(g), h\rangle$ for all $g, h \in G$.

Lemma 5.4. *For any $x \in V_g$ and $y \in V_h$, $xy = F(g,h)yx$.*

Proof. If either $V_g = 0$ or $V_h = 0$, there is nothing to prove. Suppose $V_g, V_h \neq 0$, and let $0 \neq x \in V_g$ and $0 \neq y \in V_h$. Set $P := f_R^{-1}(x)$. Then $P \in P_{f_R^{-1}(g),\varepsilon}(A^{*cop})$. Substituting $p := P$ and $a := y$ in equation (2) yields that $yx - F(g,h)xy = \langle P, y\rangle(1 - gh)$. If $g = h$ then the result follows from Lemma 5.3, since $g^2 = 1$.

If $g \neq h$ then $\langle P, y\rangle(1-gh) \in B \cap K$ implies that $\langle P, y\rangle = 0$, and hence $yx = F(g,h)xy$. By Lemma 5.3, g, h are involutions, hence $F(g,h) = F(g,h)^{-1}$ and we are done. $\square$

Lemma 5.5. *For any $a \in G$ and $x \in V_g$, $xa = F(g,a)ax$.*

Proof. Set $P := f_R^{-1}(x)$. Then the result follows by letting $p := P$ and $a \in G$ in (2), and noting that $P \in P_{f_R^{-1}(g),\varepsilon}(A^{*cop})$. $\square$

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Let $n : I_F \to \mathbb{Z}^+$ be the non-negative integer function defined by $n(g) = n_g$, and let $\mathcal{D} := (G, F, n)$. Since by assumption, A is generated as an algebra by $G \cup (\bigoplus_{g \in I_F} V_g)$ it is pointed. By Lemmas 5.3-5.5, relations (4) are satisfied. Therefore there exists a surjection of Hopf algebras $\varphi : H(\mathcal{D}) \to A$. Using Proposition 4.3 3), it is straightforward to verify that φ is injective on $P_{a,b}(H(\mathcal{D}))$ for all $a, b \in G$. Since $H(\mathcal{D})$ is pointed, φ is also injective (see e.g., [M, Corollary 5.4.7]). Hence φ is an isomorphism of Hopf algebras. The rest of the theorem follows now from Theorem 4.4. $\square$

Theorem 5.1 raises the natural question whether any minimal triangular pointed Hopf algebra in characteristic 0 is generated as an algebra by group-like elements and skew primitive elements.

Acknowledgment

I am grateful to Pavel Etingof for numerous conversations, for reading the manuscript and for making suggestions which improved the presentation of the paper.

References

[AS] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), 658–691.

[D] V. Drinfeld, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990), 321-342.

[EG1] P. Etingof and S. Gelaki, *Some properties of finite-dimensional semisimple Hopf algebras*, Math. Res. Lett. **5** (1998), 191–197.

[EG2] ———, *A method of construction of finite-dimensional triangular semisimple Hopf algebras*, Math. Res. Lett. **5** (1998), 551–561.

[EG3] ———, *The classification of triangular semisimple and cosemisimple Hopf algebras over an algebraically closed field*, Internat. Math. Res. Notices, to appear, `math.QA/9905168`.

[G1] S. Gelaki, *On pointed ribbon Hopf algebras*, J. Algebra **181** (1996), 760–786.

[G2] ———, *Quantum groups of dimension pq^2*, Israel J. Math. **102** (1997), 227–267.

[G3] ———,*Pointed Hopf algebras and Kaplansky's 10th conjecture*, J. Algebra **209** (1998), 635–657.

[LR] R. Larson and D. Radford, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math. **110** (1988), 187–195.

[M] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, 82., American Mathematical Society, Providence, RI, 1993.

[NZ] W.D. Nichols and M.B. Zoeller, *A Hopf algebra freeness theorem*, Amer. J. Math. **111** (1989), 381–385.

[PO1] F. Panaite and F.V. Oystaeyen, *Quasitriangular structures for some pointed Hopf algebras of dimension* 2^n, Comm. Algebra, to appear.

[PO2] ———, *Clifford-type algebras as cleft extensions for some pointed Hopf algebras*, Comm. Algebra, to appear.

[R1] D.E. Radford, *The structure of Hopf algebras with a projection*, J. Algebra **2** (1985), 322–347.

[R2] ———, *Minimal quasitriangular Hopf algebras*, J. Algebra **157** (1993), 285–315.

[R3] ———, *The trace function and Hopf Algebras*, J. Algebra **163** (1994), 583-622.

[R4] ———, *On Kauffman's knot invariants arising from finite-dimensional Hopf algebras*, Advances in Hopf algebras (Chicago, IL, 1992), 205–266, Lecture Notes in Pure and Appl. Math., 158, Marcel Dekker, New York, 1994.

[TW] E.J. Taft and R.L. Wilson, *On antipodes in pointed Hopf algebras*, J. Algebra **29** (1974), 27–32.

Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720

E-mail address: shlomi@msri.org