

BOUNDS ON ARITHMETIC PROJECTIONS, AND APPLICATIONS TO THE KAKEYA CONJECTURE

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ABSTRACT. Let A, B , be finite subsets of a torsion-free abelian group, and let $G \subset A \times B$ be such that $\#A, \#B, \#\{a + b : (a, b) \in G\} \leq N$. We consider the question of estimating the quantity $\#\{a - b : (a, b) \in G\}$. In [2] Bourgain obtained the bound of $N^{2-\frac{1}{13}}$, and applied this to the Kakeya conjecture. We improve Bourgain's estimate to $N^{2-\frac{1}{6}}$, and obtain the further improvement of $N^{2-\frac{1}{4}}$ under the additional assumption $\#\{a + 2b : (a, b) \in G\} \leq N$. As an application we conclude that Besicovitch sets in \mathbb{R}^n have Minkowski dimension at least $\frac{4n}{7} + \frac{3}{7}$. This is new for $n > 8$.

1. Introduction

Let N be a positive integer, and let $(Z, +)$ be an abelian group. Let A, B , be finite subsets of Z with cardinality

$$(1) \quad \#A, \#B \leq N.$$

Let G be a subset of $A \times B$. We consider the question of bounding the quantity

$$(2) \quad \#\{a - b : (a, b) \in G\}.$$

Without any further assumptions on G one can only obtain the trivial bound of N^2 . However, in [2] Bourgain showed that under the additional assumption

$$(3) \quad \#C \leq N, \text{ where } C = \{a + b : (a, b) \in G\},$$

that one could improve the bound on (2) to $N^{2-\frac{1}{13}}$. The purpose of this paper is to obtain the following additional improvements.

Theorem 1.1. *Let the notation and assumptions be as above. Then we have*

$$(4) \quad (2) \leq N^{2-\frac{1}{6}}.$$

If we make the further additional assumption

$$(5) \quad \#D \leq N, \text{ where } D = \{a + 2b : (a, b) \in G\},$$

then we may improve this further to

$$(6) \quad (2) \leq N^{2-\frac{1}{4}}.$$

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It appears plausible that further improvements are possible by adding further assumptions of the type in (3), (5), but our methods do not seem to be able take advantage of such additional assumptions.

By the arguments in [2] this implies a bound on the Minkowski and Hausdorff dimensions of Besicovitch sets. Recall that a Besicovitch set in \mathbb{R}^n , $n > 1$ is a set which contains a unit line segment in every direction. The Kakeya conjecture states that such sets must have full dimension.

Corollary 1.2. [2] *Let E be a Besicovitch set in \mathbb{R}^n . Then the Minkowski dimension of E is at least $\frac{4n}{7} + \frac{3}{7}$, while the Hausdorff dimension of E is at least $\frac{6n}{11} + \frac{5}{11}$.*

The lower bound of $\frac{n+2}{2}$ for both types of dimension was obtained in [7]. Thus the Minkowski bound is new for $n > 8$ and the Hausdorff bound is new for $n > 12$. We remark that Theorem 1.1 can also be used to slightly improve some other recent work on the Kakeya problem in [2] and [6]. Roughly speaking, the connection between Besicovitch sets and Theorem 1.1 arises from taking A, B, C, D to essentially be the slices of the Besicovitch set at the hyperplanes $\{x_n = 0\}$, $\{x_n = 1\}$, $\{x_n = 1/2\}$, and $\{x_n = 2/3\}$ respectively, and G to be the set of pairs in $A \times B$ whose associated line segment is contained in the Besicovitch set. In principle the Hausdorff bound is improvable to match the Minkowski bound, but one must first prove an analogue of the results in [4], namely that that subsets of $\{1, \dots, N\}$ with density at least $1/(\log N)^\varepsilon$ contain four distinct elements which are affinely equivalent to $\{0, 1, 1/2, 2/3\}$ if N is sufficiently large and ε sufficiently small. For recent progress on this type of problem see Gowers [3].

Bourgain's bound of $N^{2-\frac{1}{13}}$ was obtained using some ideas of [3] in his work on the Balog-Szemerédi theorem [1]. Our arguments are more elementary, and will not yield any new results of Balog-Szemerédi type.

In the converse direction, a simple variant of an example in [5] shows that (2) can be as large as $N^{\log(6)/\log(3)} = N^{2-0.36907\dots}$ if one assumes (3), or as large as $N^{\log(8)/\log(4)} = N^{2-\frac{2}{3}}$ if one also assumes (5). To see the former claim, let n be a large integer and set

$$Z = \mathbb{Z}^n, \quad A = B = \{0, 1, 3\}^n, \quad C = \{1, 3, 4\}^n$$

and

$$G = \{(0, 1), (0, 3), (1, 0), (1, 3), (3, 0), (3, 1)\}^n;$$

we see that the hypotheses are satisfied with $N = 3^n$ and $(2) = \#G = 6^n$, hence the claim. The latter claim is similar but uses the sets

$$\begin{aligned} Z = \mathbb{Z}^n, \quad A = \{0, 2, 3, 4\}^n, \quad B = \{0, 1, 2, 3\}^n, \\ C = \{2, 3, 4, 5\}^n, \quad D = \{4, 5, 6, 8\}^n \end{aligned}$$

and

$$G = \{(4, 0), (2, 1), (3, 1), (4, 1), (0, 2), (2, 2), (0, 3), (2, 3)\}^n.$$

One can easily replace \mathbb{Z}^n in these examples by any other torsion free abelian group; see [5].

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2. A combinatorial lemma

We shall need the following combinatorial lemma.

Lemma 2.1. *Let X and A_1, \dots, A_n be finite sets for some $n \geq 0$, and for each $1 \leq i \leq n$ let $f_i : X \rightarrow A_i$ be a function. Then*

$$(7) \quad \{(x_0, \dots, x_n) \in X^{n+1} : f_i(x_{i-1}) = f_i(x_i) \text{ for all } 1 \leq i \leq n\} \geq \frac{(\#X)^{n+1}}{\prod_{i=1}^n \#A_i}.$$

The reader may verify from probabilistic methods that (7) is sharp.

Proof. We prove by induction on n . The claim is trivial for $n = 0$. Now suppose that $n \geq 1$, and the claim has been proven for $n - 1$.

Define an element $a \in A_n$ to be *popular* if

$$\#\{x \in X : f_n(x) = a\} \geq \frac{\#X}{2\#A_n},$$

and define X' to be those elements $x \in X$ such that $f_n(x)$ is popular. Since each unpopular element of A_n contributes at most $\#X/(2\#A_n)$ elements to X , we see that

$$\#(X \setminus X') \leq \frac{1}{2}\#X,$$

so

$$(8) \quad \#X' \geq \frac{1}{2}\#X.$$

By applying the induction hypothesis to X' we have

$$\{(x_0, \dots, x_{n-1}) \in (X')^n : f_i(x_{i-1}) = f_i(x_i) \text{ for all } 1 \leq i < n\} \geq \frac{(\#X')^n}{\prod_{i=1}^{n-1} \#A_i}.$$

Since $f_n(x_{n-1})$ is popular, we thus have

$$\begin{aligned} \{(x_0, \dots, x_{n-1}, x_n) \in (X')^n \times X : f_i(x_{i-1}) = f_i(x_i) \text{ for all } 1 \leq i \leq n\} \\ \geq \frac{(\#X')^n}{\prod_{i=1}^{n-1} \#A_i} \frac{\#X}{2\#A_n}. \end{aligned}$$

From (8) this implies

$$(9) \quad \text{LHS of (7)} \leq 2^{-n-1} \frac{(\#X)^{n+1}}{\prod_{i=1}^n \#A_i}.$$

To eliminate the factor of 2^{-n-1} , we let M be a large integer, and apply (9) with X, A_i replaced by X^M, A_i^M , and f_i replaced with the function $f_i^M : X^M \rightarrow A_i^M$ defined by

$$f_i^M(x^1, \dots, x^M) = (f_i(x^1), \dots, f_i(x^M)),$$

to obtain

$$(\text{LHS of (7)})^M \leq 2^{-n-1} \left(\frac{(\#X)^{n+1}}{\prod_{i=1}^n \#A_i} \right)^M.$$

The claim then follows by letting $M \rightarrow \infty$ (cf. [5]). \square

3. Proof of (4)

Fix A, B, C, N . By removing redundant elements of G , we may assume

(10) The map $(a, b) \mapsto a - b$ is one-to-one on G ,

in which case we need to show

$$(11) \quad \#G \leq N^{11/6}.$$

Define the set

$$(12) \quad V = \{(a, b, b') \in A \times B \times B : (a, b), (a, b') \in G\}.$$

By applying (7) with $n = 1$ and $f_1 : G \rightarrow A$ being the projection map, we see from (1) that

$$(13) \quad \#V \geq \frac{(\#G)^2}{N}.$$

Now consider the maps $f_1 : V \rightarrow C \times C$, $f_2 : V \rightarrow B \times B$, $f_3 : V \rightarrow C \times B$ by

$$\begin{aligned} f_1(a, b, b') &= (a + b, a + b'), \\ f_2(a, b, b') &= (b, b'), \\ f_3(a, b, b') &= (a + b, b'). \end{aligned}$$

Let S denote the set

$$S = \{(v_0, v_1, v_2, v_3) : f_1(v_0) = f_1(v_1), f_2(v_1) = f_2(v_2), f_3(v_2) = f_3(v_3)\};$$

from (7), (1) and (3) we have

$$(14) \quad \#S \geq \frac{(\#V)^3}{N^6}.$$

Write $v_i = (a_i, b_i, b'_i)$ for $i = 0, 1, 2, 3$, and consider the map $g : S \rightarrow V \times A \times B$ defined by

$$g(v_0, v_1, v_2, v_3) = (v_0, a_2, b_3).$$

We now observe that g is injective, or in other words that (v_0, a_2, b_3) determines (v_0, v_1, v_2, v_3) . To see this, we note from construction

$$(15) \quad a_0 + b_0 = a_1 + b_1, \quad a_0 + b'_0 = a_1 + b'_1, \quad b_1 = b_2, \quad b_2 = b'_2$$

and

$$(16) \quad a_2 + b_2 = a_3 + b_3, \quad b'_2 = b'_3.$$

From (15) we have

$$b_0 - b'_0 = b_1 - b'_1 = b_2 - b'_2$$

while from (16) we have

$$a_3 - b'_3 = a_3 + b_3 - b_3 - b'_2 = a_2 + b_2 - b'_2 - b_3.$$

Combining these equations, we obtain

$$a_3 - b'_3 = a_2 + b_0 - b'_0 - b_3.$$

Thus $a_3 - b'_3$ is determined by (v_0, a_2, b_3) . Since $(a_3, b'_3) \in G$, we see from (10) that (a_3, b'_3) , and thus v_3 , are determined by (v_0, a_2, b_3) . The injectivity of g then follows by using (16) to determine v_2 , and then (15) to determine v_1 .

Since g is injective, we have from (1) that

$$\#S \leq N^2 \#V.$$

Combining this with (14) we see that

$$\#V \leq N^{8/3},$$

and (11) follows from (13).

4. Proof of (6)

Fix A, B, C, D, N . We may assume (10) as before, so that we need to show that

$$(17) \quad \#G \leq N^{7/4}.$$

Define V by (12) as before, and define the function $f_1 : V \rightarrow D \times B$ by

$$f_1(a, b, b') = (a + 2b, b').$$

Let T denote the set

$$T = \{(v_0, v_1) \in V^2 : f_1(v_0) = f_1(v_1)\}.$$

From (7), (1), (5) we have

$$(18) \quad \#T \geq \frac{(\#V)^2}{N^2}.$$

Write $v_i = (a_i, b_i, b'_i)$ for $i = 0, 1$, and define the map $h : T \rightarrow C \times C \times B$ by

$$h(v_0, v_1) = (a_0 + b_0, a_0 + b'_0, b_1).$$

We claim that h is injective, so that $h(v_0, v_1)$ determines (v_0, v_1) . From construction we have

$$(19) \quad a_0 + 2b_0 = a_1 + 2b_1, \quad b'_0 = b'_1,$$

so that

$$a_1 - b'_1 = a_0 + a_1 + 2b_1 - 2b_1 - (a_0 + b'_1) = 2(a_0 + b_0) - 2b_1 - (a_0 + b'_0).$$

Thus $a_1 - b'_1$ is determined by $h(v_0, v_1)$. From (10) we thus see that (a_1, b'_1) is determined by $h(v_0, v_1)$. The injectivity of h then follows from (19).

Since h is injective, we see from (1) and (3) that

$$\#T \leq N^3.$$

From (18) we thus have

$$\#V \leq N^{5/2},$$

and (17) follows from (13).

References

- [1] A. Balog and E. Szemerédi, *A statistical theorem of set addition*, Combinatorica, **14** (1994), 263–268.
- [2] J. Bourgain, *On the dimension of Kakeya sets and related maximal inequalities*, preprint.
- [3] T. Gowers, *A new proof of Szemerédi’s theorem for arithmetic progressions of length four*, to appear in GAFA.
- [4] D. Heath-Brown, *Integer sets containing no arithmetic progressions*, J. London Math. Soc. (2) **35** (1987), 385–394.
- [5] I. Ruzsa, *Sums of finite sets*, Number Theory (New York, 1991–1995), pp. 281–293, Springer-Verlag, New York, 1996.
- [6] N. Katz, I. Łaba, T. Tao, *An improved bound on the Minkowski dimension of Besicovitch sets in \mathbb{R}^3* , submitted, Annals of Math.
- [7] T.H. Wolff, *An improved bound for Kakeya type maximal functions*, Rev. Mat. Iberoamericana **11** (1995), 651–674.

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