Mathematical Research Letters **6**, 697–710 (1999)

A SMALL UNSTABLE ACTION ON A TREE

M.J. Dunwoody

1. Introduction

The theory of groups acting on $\mathbb{R}$-trees was initiated by Tits [15] and by Alperin and Moss [1]. It was developed by Morgan and Shalen [10, 11] and others. A major advance in the theory was the classification by Rips (see [8]) of the finitely generated groups that have free actions on $\mathbb{R}$-trees. Every such group is a free product of free abelian groups and surface groups. This result had been conjectured by Morgan and Shalen.

Let T be an $\mathbb{R}$-tree on which the group G acts (on the left) by isometries. We say that that the action is *trivial* if for some point $x \in T$ the stabilizer $\mathrm{Fix}(x) = G$. We say that T is *minimal* if it has no proper G-invariant subtree. A nondegenerate (i.e., containing more than one point) subtree S of T is *stable* if for every nondegenerate subtree S' of S we have $\mathrm{Fix}(S) = \mathrm{Fix}(S')$. A nontrivial action is *stable* if every nondegenerate subtree of T contains a stable subtree. Bestvina and Feighn [3] have extended the classification of free actions to include all finitely presented groups that have stable minimal actions on $\mathbb{R}$-trees. In particular they show that if G is such a group, then there is an action of G on a simplicial tree S such that if $K < G$ and K fixes an edge of S then K contains a normal subgroup N such that N fixes an arc of T and K/N is abelian. Thus, if in the action of G on T arc stabilizers are small, i.e., they do not contain non-cyclic free subgroups, then edge stabilizers in S are small also.

This paper contains an example of a group H that has a non-trivial, unstable action on an $\mathbb{R}$-tree T with finite cyclic arc stabilizers. It is then shown that there is no non-trivial action of H on a simplicial $\mathbb{R}$-tree with small edge stabilizers. This provides a negative answer to Question D of Shalen [12].

The group H is finitely generated but not finitely presented.

The construction of T is by means of a folding sequence of simplicial trees. The basic folding operations were described by Bestvina and Feighn [2] following an earlier treatment by Stallings [14]. Earlier, Chiswell [4] used a folding operation for graphs of groups in his proof of Grushko's Theorem and Dicks [5] used the folding operation for trees to simplify Chiswell's argument. In [6] I introduced

Received June 12, 1995. Revised November 29, 1999.

another basic folding move which allows the group acting to be changed. We call this move a *vertex morphism*.

Let T, T' be a G-tree and a G'-tree respectively and let $u \in VT$. A vertex morphism is defined to be a surjective morphism (of graphs) $\phi : T \to T'$ and a surjective homomorphism (also denoted) $\phi : G \to G'$ such that $\phi(gx) = \phi(g)\phi(x)$ for every $g \in G$ and $x \in VT \cup ET$. Clearly ϕ induces homomorphisms $\phi_x : G_x \to G'_{\phi(x)}$ for each $x \in VT \cup ET$. We require that ϕ_x be an isomorphism for each $x \in ET \cup VT$ unless x is a vertex in the same orbit as u. We also require that ϕ induce an isomorphism $G\backslash T \to G'\backslash T'$.

Associated with the G-tree T is a graph of groups $(G(-), X)$ where $X = G\backslash T$, which is illustrated in a Bass Serre diagram. A spanning tree in $G\backslash T$ is lifted to T and each edge and vertex of $G\backslash T$ is labelled with the corresponding stabilizer in G. If the G-action is not free there is a lot of choice in the way one lifts the spanning tree. Let $v \in VX$, be such that $G(v) = H$ is the stabilizer of the vertex u of T. The graph of groups associated with the G'-tree T' is $(G'(-), X')$. Then ϕ induces an isomorphism $X \cong X'$, so that we can identify X with X'. We then have $G(x) = G'(x)$ unless $x = v$.

Conversely suppose we have a G-tree T with corresponding graph of groups $(G(-), X)$ and suppose that $(G'(-), X)$ is a graph of groups such that $G(x) = G'(x)$ for every $x \in X$ except for a particular $v \in VX$. If there is a surjective homomorphism $\phi_v : G(v) \to G'(v)$ (which has to restrict to the identity on each subgroup of the form $G(e)$, for $e \in EX$, $\iota e = v$), and G' is the fundamental group of this graph of groups (for some maximal subtree of X), then there is a G'-tree T' and a vertex morphism $\phi : T \to T'$, where $\phi : G \to G'$ is defined in the obvious way.

In our example we will use vertex morphisms as above, subdivision and type I and II Stallings folds. The effect of type I and type II folds is shown in the following Bass Serre diagrams. See [2] for full details. Type I and II folds are the only Stallings folds which apply to tree products of groups, i.e., for a group G with an action on a tree T such that $X = G\backslash T$ is also a tree. The Bass Serre theory gives one a way of describing these folding operations. For a Type I fold, since the edges folded together are in different G-orbits, one can arrange the lift so that that edges of the lifted spanning tree are folded together. This makes for a convenient way of describing the operation in terms of the Bass-Serre diagram.

A different choice of the lifted tree $\tilde{X}$ will change the labelling subgroups by conjugation, though of course the group G remains the same. This change of the lifted tree is used in our construction (illustrated in Figure 2) in which the labels of all the vertices and edges which are joined to a particular vertex u by a path passing through a particular edge e (incident with u) are conjugated by an element $x^{-1} \in G(u)$. If $\tilde{e}$ is the lift of e included in the original spanning tree $\tilde{X}$, then the new spanning tree will include $x\tilde{e}$. A more general account of

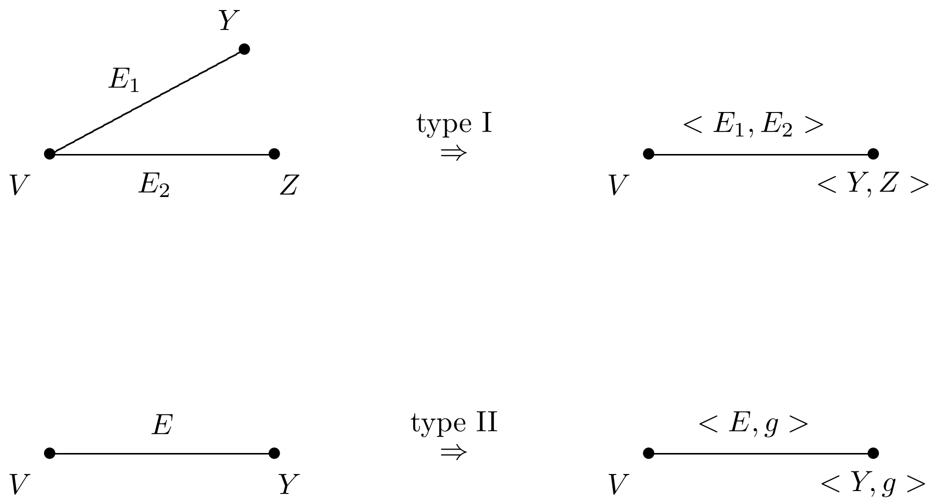

Figure 1

folding sequences is given in [7].

I am grateful to the referee for his care and patience.

2. Some group theory

In our construction the alternating groups A_n act as building blocks. They are used because if n is composite and odd then A_n can be generated by a conjugate pair of elements with order any proper divisor of n that is at least five, and each of the generators is a power of an element of order n.

Proposition 1. *Let $n = km$ be odd (so that k and m are both odd), and suppose $m \geq 5$. Let a be a k-th power of an n-cycle. Then there is an element b of A_n such that a and $b^{-1}ab$ generate A_n.*

Proof. We may assume that

$$a = (1, 2, \dots, m)(m+1, m+2, \dots, 2m) \cdots (m(k-1)+1, m(k-1)+2, \dots, km).$$

Let

$$b = (1, 2, 3)(m, m+1)(2m, 2m+1) \cdots ((k-1)m, (k-1)m+1).$$

Then

$$a^{-1}ba = (2, 3, 4)(1, m+2)(m+1, 2m+2) \cdots ((k-2)m+1, (k-1)m+2).$$

Put

$$\begin{aligned} c &= b^{-1}a^{-1}ba \\ &= (1,4,2,m+2)(m,2m+2,m+1)(2m,3m+2,2m+1)\cdots \\ &\qquad ((k-2)m,(k-1)m+2,(k-2)m+1)((k-1)m,(k-1)m+1). \end{aligned}$$

(We multiply permutations from left to right). Now $c \in G$, and hence

$$c^6 = (1,2)(4,m+2) \in G.$$

Conjugating by a, we have $(2,3)(5,m+3) \in G$. Hence $(1,2,3) \in G$, since $(1,2,3) = ((1,2)(4,m+2)(2,3)(5,m+3))^2$. Now

$$\begin{aligned} &d = a(1,3,2) = \\ &\quad (3,4,\ldots,m)(m+1,m+2,\ldots,2m)(m(k-1)+1,m(k-1)+2,\ldots,km). \end{aligned}$$

Conjugating $(1,2,3)$ by powers of d and c^4, we get $(1,2,j) \in G$ for $j = 3,4,\ldots,n$. It is not hard to show that these elements generate A_n. Since m is odd, a is an even permutation. Thus $G = A_n$. This completes the proof of the Proposition. $\square$

Note that, since k is odd, $b \in A_n$. There is an n-cycle x in A_n such that $x^k = a$. Thus we see that A_n is generated by conjugate n-cycles $x, y = b^{-1}xb$ such that x^k, y^k also generate G.

Let $p_1 < p_2 < p_3 < \ldots$ be a sequence of odd primes, $p_1 \neq 3$. Let

$$n_i = p_1 p_2 \ldots p_i.$$

Let K_i be the alternating group A_n, where $n = p_i p_{i+1} p_{i+2}$. Put $G_1 = K_1$. For $i > 1$, let G_i be the direct product of K_i with a cyclic group of order n_{i-1} generated by the element k_i. Let $w_i \in G_i$ be an element which projects to an n-cycle in K_i and for which $w_i^{p_i p_{i+1} p_{i+2}} = k_i$. Let $w_i^{p_{i+2}} = z_i$ and $z_i^{p_{i+1}} = u_i$ so that $u_i^{p_i} = k_i$. By Proposition 1 there exists $b_i \in G_i$ such that if $v_i = b_i^{-1} u_i b_i$, then $G_i = \langle u_i, v_i \rangle$ and $u_i^{p_i} = v_i^{p_i} = k_i$. Note that w_i has order n_{i+2}, z_i has order n_{i+1} and u_i has order n_i.

Suppose now we form the free product with amalgamation

$$G_{i,i+1} = G_i *_{w_i = z_{i+1}} G_{i+1}.$$

We show that $G_{i,i+1}$ is a direct product $G_{i,i+1} = \langle k_i \rangle \times V_{i,i+1}$, where $V_{i,i+1}$ is the subgroup generated by $K_i \cup K_{i+1}$. To see this note that k_i commutes with K_i and since $k_i = u_i^{p_i} = k_{i+1}^{p_i}$, it also commutes with K_{i+1}, and hence all of $V_{i,i+1}$. Also, in $G_{i,i+1}, k_i = k_{i+1}^{p_i}$ and, since p_i and n_i are coprime, $\langle k_{i+1} \rangle$ is the direct product of $\langle k_i \rangle$ and a cyclic group C_i of order p_i. Thus $G_{i+1} = \langle k_i \rangle \times C_i \times K_{i+1}$. The subgroup of G_{i+1} generated by z_{i+1} contains both k_i and k_{i+1}. Thus there is homomorphism from G_{i+1} on to $\langle k_i \rangle$ which takes z_{i+1} to k_i and has kernel $C_i \times K_{i+1}$. Since there is a homomorphism from G_i on to $\langle k_i \rangle$ which takes w_i to k_i, there is a homomorphism of $G_{i,i+1}$ on to $\langle k_i \rangle$ which contains $V_{i,i+1}$ in its kernel. Note that $C_i < \langle k_{i+1} \rangle < \langle w_i \rangle < G_i$, and in fact $C_i < K_i$, as it is generated by the n_{i-1}'th power of an element of G_i. It follows easily that $G_{i,i+1}$ is a direct product as required.

More generally if $i < j$, then

$$G_{i,j} = G_i *_{w_i = z_{i+1}} G_{i+1} *_{w_{i+1} = z_{i+2}} * \cdots *_{w_{j-1} = z_j} G_j$$

is the direct product of $\langle k_i \rangle$ and the subgroup $V_{i,j}$ generated by $K_i \cup K_{i+1} \cup \dots K_j$. For fixed i, this can be proved by induction on $j > i$. If $j = i+1$ this has already been proved. If $j > i+1$, then we can assume by the induction hypothesis that $G_{i,j-1}$ is a direct product as indicated. In particular there will be a homomorphism onto $\langle k_i \rangle$ taking w_{j-1} to a generator of the cyclic group. There is also a homomorphism of G_j onto $\langle k_i \rangle$ taking z_j to the same element. It follows easily that $G_{i,j}$ is a direct product as required.

Note that in $G_{i,j}, w_j$ has order n_{j+2} and the subgroup it generates contains the elements $w_i, w_{i+1}, \dots, w_{j-1}$ as well as the direct summand generated by k_i. If $j \geq i+3$ then $k_{i+3} = w_i$. We can write $w_j = k_i' h_{i,j}$, where k_i' is a generator of $\langle k_i \rangle$ and $h_{i,j} \in V_{i,j}$ has order $p_i p_{i+1} p_{i+2} \dots p_{j+2}$.

Let X be the restricted direct product

$$X = V_{1,2} \times V_{5,7} \times V_{10,13} \times V_{16,20} \times \cdots \ .$$

Thus each factor contains one more K_i than the previous one. Let W be the subgroup of X generated by $h_{1,2}, h_{5,7}, h_{10,13}, \dots$. Then W is a locally cyclic group containing a unique subgroup of order p_i for each $i > 0$.

Let i be an integer in the set

$$I = \{1, 2, 5, 6, 7, 10, 11, 12, 13, 16, 17, 18, 19, 20, \dots \}.$$

Thus i is one of the integers for which there is a monomorphism $K_i \to X$. Note that I contains arbitrarily long finite sequences of successive integers. This property will be important much later on. We now show inductively that we can define monomorphisms $\alpha_i : G_i \to X$ in such a way that $\alpha_i(w_i) = \alpha_{i+1}(z_{i+1})$ if $i, i+1 \in I$ and $\alpha_i(w_i) = \alpha_{i+3}(k_{i+3})$ if $i, i+3 \in I$. Suppose that $V_{i,j}$ is one of the summands of X and α_k has been defined if $k \in I$ and $k < i$. Define $\alpha_{i,j} : G_{i,j} \to X$ as follows: $\alpha_{1,2}$ is the natural inclusion of $V_{1,2} = G_{1,2}$ in X, if $i \geq 5$ set $\alpha_{i,j}(k_i) = \alpha_{i-3}(w_{i-3})$ and let $\alpha_{i,j} | V_{i,j}$ be the natural inclusion of $V_{i,j}$ in

X. Now define $\alpha_i, \dots, \alpha_j$ to be the restrictions of $\alpha_{i,j}$ to $G_i, \dots, G_j$ respectively. It can be seen that $\alpha_i(w_i) = \alpha_{i+1}(z_{i+1})$ if $i, i+1 \in I$ and $\alpha_i(w_i) = \alpha_{i+3}(k_{i+3})$ if $i, i+3 \in I$, as required. It is also the case that $\alpha_i(w_i) \in W$ if $i \in I$.

Let $C_{i,j}$ be a cyclic group of order $p_i p_{i+1} \dots p_j$. Let

$$Y = C_{1,2} \times V_{3,4} \times C_{7,7} \times V_{8,9} \times C_{12,13} \times V_{14,15} \times C_{18,20} \times \cdots .$$

Here the $V_{i,j}$ summands are chosen to contain precisely the K_i subgroups not contained in X. The extra cyclic summands are chosen so that Y contains an isomorphic copy of W. Thus $h_{3,4} \in V_{3,4}$ has order $p_3 p_4 p_5 p_6, h_{8,9} \in V_{8,9}$ has order $p_8 p_9 p_{10} p_{11}$, and so the summands $C_{1,2}, C_{7,7}, C_{12,13}$, and so on, fill in the gaps. Let W' be the subgroup of Y generated by the elements $h_{3,4}, h_{8,9}, h_{14,15}$ and all the cyclic summands. For each $i > 0, i \notin I$ we construct a monomorphism $\beta_i : G_i \to Y$. These maps are chosen so that $\beta_i(w_i) = \beta_{i+1}(z_{i+1})$ if both $i \notin I$, $i+1 \notin I$, also $\beta_i(w_i) = \beta_j(k_j)^{p_{j-1}p_{j-2}\dots p_{i+3}}$ if $i, j \notin I$, $j \geq i+4$, and $\beta_i(w_i) \in W'$ for each $i \notin I$. Again we use induction on i. We know that $G_{3,4} \cong C_{1,2} \times V_{3,4}$, and so we can define β_3, β_4. Suppose $j, j+1 \notin I$ and i is the largest integer $i < j$, $i \notin I$. Then Y contains a summand $C_{i+3,j-1}$. By induction, we assume that $\beta_i(w_i)$ has been defined and that it generates the unique subgroup W'_{i+2} of W' of order n_{i+2}. We know $G_{j,j+1} = \langle k_j \rangle \times V_{j,j+1}$. There is then a monomorphism $G_{j,j+1} \to X$ which maps k_j to a generator of $W'_{j-1} = W'_{i+2} \times C_{i+3,j-1}$ so that $k_j^{p_{i+3}p_{i+4}\dots p_{j-1}}$ maps to $\beta_i(w_i)$, and which is the inclusion on $V_{j,j+1}$. We can now define β_j, β_{j+1} with the required properties, by composing the inclusion maps $G_j \to G_{j,j+1}, G_{j+1} \to G_{j,j+1}$ with this monomorphism.

Note that the maps α_i, satisfy the relations $\alpha_i(w_i) = \alpha_j(w_j^{p_{j+2}p_{j+1}\dots p_{i+3}})$, $i, j \in I$, $i < j$; while the maps β_j satisfy $\beta_i(w_i) = \beta_j(w_j^{p_{j+2}p_{j+1}\dots p_{i+3}})$, $i, j \notin I$, $i < j$.

Let $L = X *_{W=W'} Y$. It follows from the relations stated in the last paragraph that the identification of W and W' can be carried out by identifying $\alpha_i(w_i)$ with $\beta_{i+1}(z_{i+1})$ if $i \in I$, $i+1 \notin I$.

Proposition 2. *The group L has the following properties:-*

(i) *For each i there is an injective homomorphism $\iota_i : G_i \to L$, such that $\iota_i(w_i) = \iota_{i+1}(z_{i+1})$.*

(ii) *X is generated by $\bigcup_{i \in I} \operatorname{Im} \iota_i$ and Y is generated by $\bigcup_{i \notin I} \operatorname{Im} \iota_i$, so that in particular L is generated by the set $\bigcup_{i=1}^{\infty} \operatorname{Im} \iota_i$,*

(iii) *In an action of L on an $\mathbb{R}$-tree with small arc stabilizers, there is a point x fixed by the subgroup X and a (not necessarily distinct) point y fixed by Y.*

Proof. Define $\iota_i : G_i \to L$, so that for $i \in I, \iota_i$ is α_i composed with the inclusion of X in L; while if $i \notin I, \iota_i$ is β_i composed with the inclusion of Y in L. Clearly conditions (i) and (ii) are satisfied.

To show condition (iii) is satisfied, let T be an $\mathbb{R}$-tree with small arc stabilizers. Consider the action of the subgroup X. The subgroups $V_{i,j}$ are direct summands of X. If one of these summands, say V, acted non-trivially on T, i.e., there is no vertex fixed by it, then there exists a hyperbolic element $a \in V$ with axis C_a. Let U be a $V_{i,j}$ summand distinct from V. Every element $b \in U$ commutes with a, and so $bC_a = C_a$. But U is a perfect group and has no non-trivial action on a line. Thus U must fix every point of C_a. But U is not small, since none of the $G_{i,j}$ and hence none of the $V_{i,j}$ is small. Hence we have a contradiction. Thus each $V_{i,j}$ acts trivially on T. In fact, each $V_{i,j}$ must fix a unique point of T, since $V_{i,j}$ is not small. However the points fixed by distinct $V_{i,j}$ must be the same, since otherwise the summands would not commute. Thus there is a unique point x fixed by X. Similarly there is a unique point y fixed by Y. This completes the proof of Proposition 2. $\square$

We write $\bar{G}_i$ for $\iota_i(G_i)$, and $\bar{k}_i$ for $\iota_i(k_i)$, $\bar{w}_i$ for $\iota_i(w_i)$ and so on. Let L_i be the subgroup generated by $\bar{G}_1 \cup \bar{G}_2 \cup \dots \bar{G}_{i-1}, i = 2, 3, \dots$. Note that there is a surjective homomorphism $L_i *_{\bar{w}_{i-1}=\bar{z}_i} \bar{G}_i \to L_{i+1}$, which restricts to the identity on the factors. In L we have the following identities:

$$\bar{w}_{i-3} = \bar{z}_{i-2} = \bar{u}_{i-1} = \bar{k}_i.$$

3. The folding sequence

The construction is by means of a folding sequence. We will define a sequence of simplicial trees T_n, actions of groups H_n on the T_n, surjective homomorphisms from H_n to H_{n+1} and morphisms $T_n \to T_{n+1}$, each of which is a product of elementary folding operations, and so is equivariant in terms of these homomorphisms. It will be shown that the sequence is strongly convergent in the sense of [9]. It follows that there is a limit $\mathbb{R}$-tree which is the limit of this sequence and there is an action of H on T where H is the direct limit of the sequence of homomorphisms $H_n \to H_{n+1}$. Since the homomorphisms are surjective and H_2 is finitely generated, H is finitely generated.

Consider then the folding sequence of simplicial trees for which the associated graphs of groups are as in Figure 2. The first tree T_2 in the sequence has one orbit of edges. The corresponding group H_2 is a free product with amalgamation $H_2 = G_2 *_{u_2=\bar{z}_1} L_2$, which acts on T_2, and in T_2 the edge e joining the vertices fixed by G_2 and L_2 has length one. One first subdivides each edge of T_2 by creating a vertex at the midpoint of each edge. Type II folds are then used to enlarge the group of the midpoint so that it becomes a free product with amalgamation of two cyclic groups. There is then a vertex morphism which changes the group associated with this vertex to G_3. The crucial point about this operation is that the two edge groups become conjugate in the vertex group. Thus we can change our choice of the lift of the spanning tree in the fourth

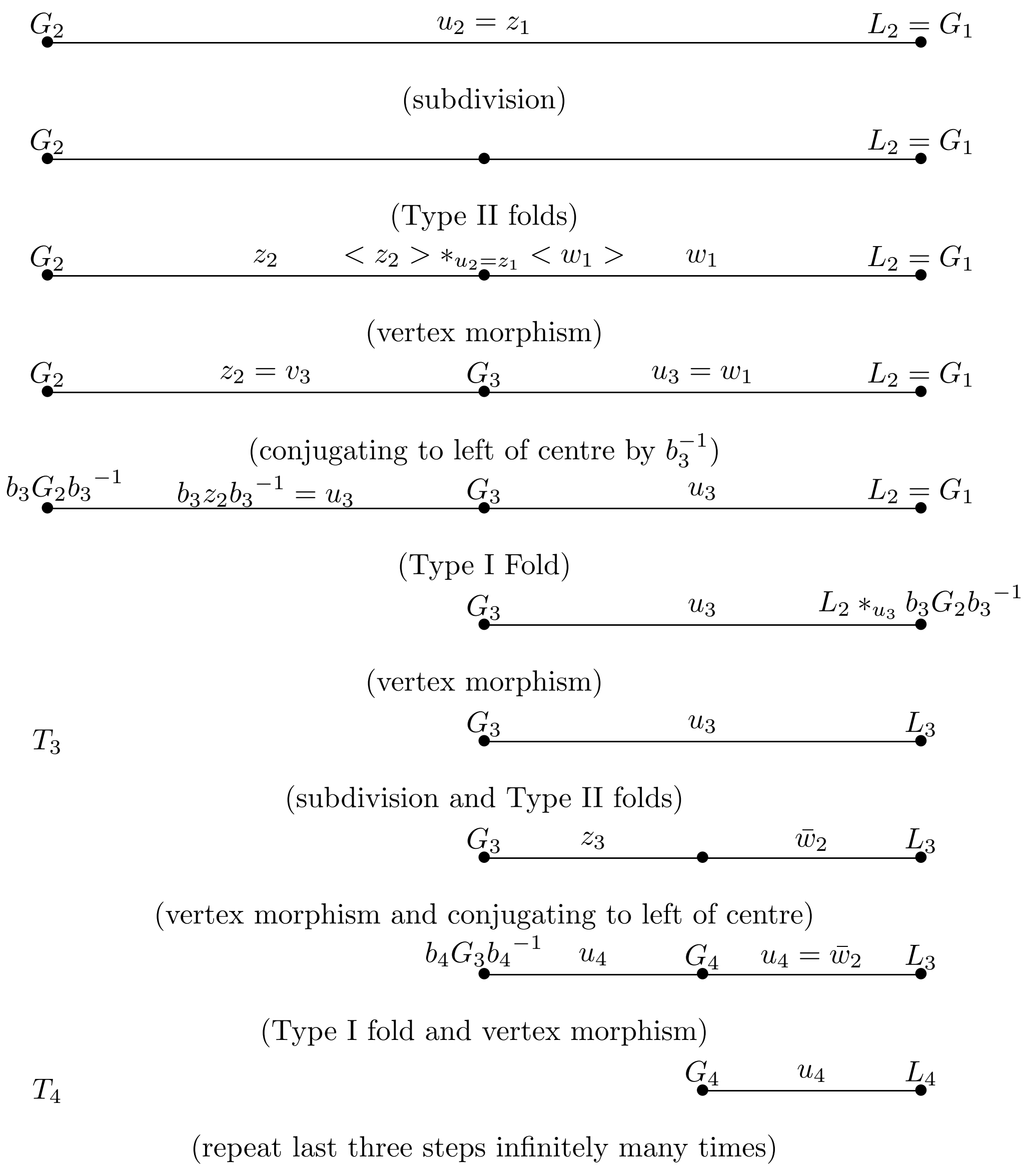

Figure 2

diagram of the sequence so that both edge stabilizers are generated by u_3. One can then carry out a Type I fold in which the edges folded together have the same group, and so the resulting edge also has the same group. The fold creates a vertex group on the right which is generated by $b_3G_2b_3^{-1}$ and L_2. In fact the vertex group on the right is the free product with amalgamation of these two groups in which the amalgamated subgroup is generated by u_3. This can be seen by examining the action of this new vertex group on the tree associated with the previous diagram. For this tree L_2 and $b_3G_2b_3^{-1}$ act on vertices in distinct orbits. These vertices are separated by two edges, each of which has $\langle u_3\rangle$ as its stabilizer. The subgroup generated by two such vertex groups must be a free product with amalgamation. This group is then changed to L_3 by a vertex morphism, in which the factor $b_3G_2b_3^{-1}$ is mapped to $\bar{G}_2$ by the obvious homomorphism. The tree resulting from these operations is denoted T_3. Thus T_3 is an H_3-tree where $H_3 = G_3 *_{u_3=\bar{z}_2} L_3$ and we can repeat the cycle described above. In general $T_n, n = 2, 3\ldots,$ is a subsequence of the sequence of simplicial $\mathbb{R}$-trees shown in Figure 2. Let H_i be the group acting on T_i. Let $\rho_n : T_n \to T_{n+1}, n \geq 2$ be the morphism which is the composite of the folds as shown in Figure 2, for $n = 2, 3, 4$. There are two vertex morphisms occurring in each ρ_n. The first involves the centre vertex. The induced homomorphism on the associated vertex group is

$$\langle z_2\rangle *_{u_2=\bar{z}_1} \langle\bar{w}_1\rangle \to G_3, z_2 \mapsto v_3, w_1 \mapsto u_3,$$

for $n = 2$, and

$$\langle z_n\rangle *_{u_n=\bar{z}_{n-1}} \langle\bar{w}_{n-1}\rangle \to G_{n+1}, z_n \mapsto v_{n+1}, \bar{w}_{n-1} \mapsto u_{n+1}$$

for $n > 2$. Note that the orders of elements are such that this homomorphism restricts to an injective map on edge groups. The second vertex morphism involves the right hand vertex. We have seen that there is a surjective homomorphism $L_n *_{\bar{w}_{n-1}=\bar{z}_n} \bar{G}_n \to L_{n+1}$, which restricts to the identity on the factors. Clearly there is a surjective homomorphism

$$L_n *_{\bar{w}_{n-1}=b_{n+1}z_nb_{n+1}^{-1}} b_{n+1}G_nb_{n+1}^{-1} \to L_{n+1},$$

in which the homomorphism restricts to the identity map on L_n and induces the obvious isomorphism $b_{n+1}G_nb_{n+1}^{-1} \to \bar{G}_n$ when restricted to $b_{n+1}G_nb_{n+1}^{-1}$.

Let $\theta_n = \rho_n\rho_{n-1}\ldots\rho_2 : T_2 \to T_n$. Let d_n be the metric in T_n. Let $x_2, y_2 \in T_2$ and put $x_n = \theta_{n-1}(x_2)$, $y_n = \theta_{n-1}(y_2)$. We show that the non-increasing sequence $\delta_n = d_n(x_n, y_n)$ is eventually constant. It will then follow that there is an $\mathbb{R}$-tree T on which H acts by isometries and the sequence T_n together with the composites of the ρ_n form a strongly convergent system of simplicial trees. The concept of strong convergence was introduced by Gillet and Shalen [9]. See also Shalen [13]. The definition of H as a direct limit gives rise to natural homomorphisms from the G_i and L_i to H, and these maps are injective. These homomorphisms are induced by the group homomorphisms associated with the morphisms ρ_n. Thus, for example, we can follow what happens to the group G_2 as we go down the sequence starting at the fourth diagram. Suppose G_2 is the stabilizer of the vertex x in the corresponding tree and L_2 is the stabilizer of the vertex y. The lift of the spanning tree is changed so that it includes b_3x

instead of x. In the subsequent fold b_3x and y are identified. This fold will also identify x with $b_3^{-1}y$ and so G_2 is mapped into the stabilizer of the identified vertex which is generated by $b_3^{-1}L_2b_3$ and G_2. The next vertex morphism takes this group to $b_3^{-1}L_3b_3$, in particular it takes G_2 to $b_3^{-1}\bar{G}_2b_3$.

We identify L_i with its image in H and we identify L with the union of the subgroups L_i in H. In an attempt to avoid confusion, we denote the image of G_n in H by $\hat{G}_n$ and we write $\hat{z}_i, \hat{u}_i, \hat{b}_i$ etc., for the images of the elements z_i, u_i, b_i etc. It follows that $\bar{G}_n = \hat{b}_{n+1}\hat{G}_n\hat{b}_{n+1}^{-1}$.

We show first that θ_n restricts to an isometry on each edge; it suffices to show that it restrict to an isometry on e, which is a representative of the single orbit of edges. Clearly for ρ_2 the only point of e at which folding can take place is at the midpoint. This will not occur in a vertex morphism since the two halves of e are in different H_2-orbits. The Type I fold, which is a factor of ρ_2, identifies edge pairs (of the appropriate image of T_2) which have the same stabilizer. The two parts of e are mapped to a pair of edges whose stabilizers are $\langle u_3\rangle$ and $\langle v_3\rangle$ and so they are not folded together. A similar argument applies for $\rho_n, n > 2$ applied to $\theta_{n-1}(e)$.

Consider now the effect of folding on the edges incident with a fixed vertex. In particular consider a vertex p in the tree T_n with stabilizer G_n. There is an incident edge e with stabilizer $\langle u_n\rangle$ for which the other vertex has stabilizer L_n. The edges of T_n incident with p form a single orbit under G_n, and so every such edge is ge for some $g \in G_n$. Let ge, he be two such edges. We assume they are distinct which means that $g\langle z_n\rangle = h\langle z_n\rangle$ They will be folded together in the Type II folds in passing from T_n to T_{n+1} and from T_{n+1} to T_{n+2} if and only if $g\langle w_n\rangle = h\langle w_n\rangle$. No further folding together of the edges incident with p in T_n occurs as a result of the $\rho_m, m > n+1$. This is because $\langle \bar{w}_m\rangle \cap \bar{G}_n = \langle \bar{w}_n\rangle$ in L and so if $g\langle w_n\rangle \neq h\langle w_n\rangle$, then $\bar{g}$ and $\bar{h}$ lie in different left cosets of $\langle \bar{w}_m\rangle$, for each $m \geq n+1$. A similar argument applies to the vertex with stabilizer L_n. Thus for any vertex of T_n any folding of the edges incident with that vertex in T_n which occurs in the whole subsequent folding process must occur for ρ_n or ρ_{n+1}.

Consider now the segment $[x_n, y_n]$. Let F_n be the set of points u of the segment at which folding occurs in the composite $\rho_m\rho_{m-1}\dots\rho_n$ for some $m > n$. In fact by the above if folding occurs at u then it will occur for $\rho_{n+1}\rho_n$. Now F_n is a subset of the set V_n of vertices of $[x_n, y_n]$, since no folding occurs in the interior of edges. The image of $[x_n, y_n]$ under ρ_n is a finite subtree of T_{n+1} which contains $[x_{n+1}, y_{n+1}]$. If $v \in V_n$ and there is a segment of $[x_n, y_n]$ containing v as an interior point which ρ_n maps isometrically into $[x_{n+1}, y_{n+1}]$, then $v \in F_n$ implies $\rho_n(v) \in F_{n+1}$. Let $u \in F_{n+1}, u \notin \rho_n(F_n)$. There are two edges e, f in $[x_{n+1}, y_{n+1}]$ incident with u and, by the above, there is no segment of $[x_n, y_n]$ which maps isometrically to a segment in $e \cup f$, which contains u

as an interior point. Thus we can find two disjoint segments s, t of $[x_n, y_n]$ containing interior points p, q which map to u such that $\rho_n(s) \cap [x_{n+1}, y_{n+1}]$ is a non-trivial segment of e and $\rho_n(t) \cap [x_{n+1}, y_{n+1}]$ is a non-trivial segment of f. We can also choose s, t so that they are the smallest distance apart and then $\rho_n([p, q]) \cap [x_{n+1}, y_{n+1}] = \{u\}$. Clearly $[p, q]$ will contain at least one point from F_n. If it contains exactly one point of F_n, then it will be the midpoint of $[p, q]$ which is folded in half by ρ_n. If $[p, q]$ contains more than one point of F_n, then F_{n+1} will have fewer elements than F_n. Thus either $F_n = \emptyset$ for some n, in which case $[x_n, y_n]$ is mapped isometrically by all $\rho_m, m > n$, or for some n there is a sequence $u_n, u_{n+1}, \ldots$, where for $m \geq n, u_m \in F_m$ and u_m is the midpoint of a segment folded in half with endpoints mapped to u_{m+1} by ρ_m.

In the composite operation ρ_n it is only the Type II folds which result in part of the subdivided edge being folded together. We show that if a Type II fold does fold parts of two edges together then no further folding occurs between the remaining parts of the edges. In particular suppose that a Type II fold in ρ_2 does not act as an isometry on an adjacent edge pair e, f. Thus e, f have a common vertex v. We assume that e has stabilizers as in the top line of Figure 2. Thus e has stabilizer $\langle u_2 \rangle$ and v has stabilizer G_1 or G_2. Suppose that v has stabilizer G_1. The argument for when the stabilizer is G_2 is similar. Under ρ_2 both e and f are subdivided, say $e = e_1 \cup e_2, f = f_1 \cup f_2$ and e_1, f_1 have common vertex v. Also there exists $w \in \langle w_1 \rangle$, $w \notin \langle z_1 \rangle$ which fixes v and $we = f$. After the Type II fold, e_1, f_1 are folded together. Also $we_2 = f_2$. If e_2 and f_2 were identified in the subsequent vertex morphism then the image of w under the homomorphism $\langle z_2 \rangle *_{u_2 = z_1} \langle w_1 \rangle \to G_3$ would be in $\langle z_2 \rangle$, which is not the case. Also e_2 and f_2 are not identified by any subsequent fold, since the image of w in G_3 is not in $\langle z_3 \rangle$ and the image of w in L_n lies in $\bar{G}_3$ but not in $\bar{G}_i$ for $i = 4, \ldots, n-1$. Similar arguments apply for Type II folds in any ρ_n. It follows that the folding at u_m in the above sequence cannot be the result of a Type II fold. The only other possibility is that the folding is the result of a vertex morphism. However vertex morphisms fold complete edges together and so there is a bound on the number of vertex morphisms involved in the $\rho_m, m > n$ which do not act as an isometry on $[x_n, y_n]$. It follows that the sequence (δ_n) is eventually constant. This means that the sequence T_n together with the morphisms which are composites of the ρ_n form a strongly convergent system of simplicial trees (see [13], pp. 605). For such a system there is an $\mathbb{R}$-tree T which is the *limit* of the system. It is easy to define T directly as follows.

Let $T = T_2/\sim$, where $x_2 \sim y_2$ if $\theta_n(x_2) = \theta_n(y_2)$ for n large. From the above, we can make T into a metric space in which

$$d([x_2], [y_2]) = \lim_{n \to \infty} d_n(\theta_{n-1}(x_2), \theta_{n-1}(y_2)).$$

It is clear that T is an $\mathbb{R}$-tree, since the convex closure of any finite set of points is the isometric image of a closed, connected subset of T_n for n sufficiently large. There is a morphism $\omega_n : T_n \to T$, and $\omega_n = \omega_{n+1}\rho_n$. Recall that H is the direct limit of the sequence of surjective homomorphisms $H_2 \to H_3 \to \dots$. We see that H acts by isometries on T. There are surjective homomorphisms (also denoted) $\omega_n : H_n \to H$. Let $(x_n), x_n \in T_n$ define a point $x \in T$. Thus $\rho_n(x_n) = x_{n+1}$ and $\omega_n(x_n) = x$. The homomorphisms $\omega_n : H_n \to H$ are injective when restricted to stabilizers. The stabilizer of x is the union of the images of the stabilizers of the $x_n, n = 2, 3, \dots$. This follows because if $h \in H$ has defining sequence (h_n), then $hx = x$ if and only if $h_n x_n = x_n$ for n large. Each vertex of T_n is mapped by ω_n into a single H-orbit of points in T. This is because the two orbits of vertices of T_n are mapped by ρ_n into the same orbit of vertices of T_{n+1}. Every other point of T has a locally cyclic stabilizer. Thus a finite non-cyclic subgroup of H fixes a unique point of T.

We now show that arc stabilizers in T are finite cyclic. It suffices to consider arcs which are contained in the image of the edge e of T_2, since every arc is covered by finitely many translates of this arc. We identify the image of e in T with the unit interval $[0,1]$. Let $x, y \in e$. Let m be the smallest positive integer for which the open interval (x, y) contains a rational number of the form $k/2^m$. Then $[x, y]$ is stabilized by a conjugate of u_{m-1}, since it is contained in the image of an edge of T_{m-1}. It is not contained in the image of an edge of T_m, and from the folding sequence the two halves of an edge of T_{m-1} are stabilized by different conjugates of u_m. Hence the stabilizer of the arc $[x, y]$ is cyclic of order n_{m-1}.

Recall that we identified $L_i, i = 2, 3, \dots$ and L with subgroups of H. The L_i form an ascending chain of subgroups of H. Let p be the unique point of T stabilized by all of these subgroups. By the above the stabilizer p is in fact the union $L = \cup_{i=2}^{\infty} L_i$. Each point of T has stabilizer a conjugate of L or an infinite locally cyclic group. In particular the action of H on T is non-trivial. Since arc stabilizers are finite cyclic, the action is small.

We show now that there is no non-trivial action of H on a simplicial $\mathbb{R}$-tree with small edge stabilizers. Recall that $\omega_n(G_n)$ is denoted $\hat{G}_n$, and that $\bar{G}_n$, which is a subgroup of L, is equal to $\hat{b}_{n+1}\hat{G}_n\hat{b}_{n+1}^{-1}$. Let S be a simplicial H-tree with small edge stabilizers. By Proposition 2, in the action of the subgroup L on S, there is a unique vertex x fixed by X, and a unique vertex y fixed by Y. For each $n, \hat{G}_n$ fixes some vertex q_n. Note that $\hat{G}_{n+1} = \langle \hat{z}_n, \hat{b}_n\hat{w}_{n-1}\hat{b}_n^{-1} \rangle < \langle \hat{G}_{n-1}, \hat{G}_n \rangle$. Thus if a vertex is fixed by both $\hat{G}_{n-1}$ and $\hat{G}_n$, then it is fixed by all $\hat{G}_m, m > n$. Suppose first that $x = y$, so that L fixes x. If for any $n, q_n = x$ then $H = \langle \hat{G}_n, L_n \rangle$ fixes that vertex and the action is trivial. Suppose $q_n \neq x$ for all n. Choose n and q_n so that the geodesic $[q_n, x]$ joining q_n and x has the smallest number of edges. Here we use the fact that the action is simplicial. Suppose firstly that this distance is non-zero. Let y_n be the vertex of $[q_n, x]$ that is closest to q_{n+1}. Note that y_n lies in both $[q_n, q_{n+1}]$ and $[x, q_{n+1}]$. Now $\hat{v}_{n+1} \in \hat{G}_n \cap \hat{G}_{n+1}$ and $\hat{u}_{n+1} \in L \cap \hat{G}_{n+1}$. Thus y_n is stabilized by $\langle \hat{u}_{n+1}, \hat{v}_{n+1} \rangle = \hat{G}_{n+1}$ and so we can assume $y_n = q_{n+1}$. We have a contradiction, unless $q_n = q_{n+1}$, since if this is not the case, then $[q_{n+1}, x]$ has fewer edges than $[q_n, x]$. Thus we may suppose $q_n = q_{n+1} = p$ say. In fact we obtain a contradiction unless $q_m = p, m \geq n$. Thus p is fixed by every $\hat{G}_m, m \geq n$. Hence p is fixed by $\langle \hat{b}_{n+1} \hat{G}_n \hat{b}_{n+1}^{-1}, \hat{b}_{n+2} \hat{G}_{n+1} \hat{b}_{n+2}^{-1} \rangle = \langle \bar{G}_n, \bar{G}_{n+1} \rangle$, which is a subgroup of L, and this subgroup is not small. It follows that $p = x$, and we have a contradiction. Suppose then that $x \neq y$. For each n, there is a vertex q_n fixed by $\hat{G}_n$. Again using the fact that the action is simplicial, choose n and q_n so that q_n has the smallest distance from $[x, y]$. Let S_n be the smallest subtree of S containing the vertices x, y and q_n. Suppose $q_{n+1} \notin S_n$. Let y_n the vertex of S_n that is closest to q_{n+1}. Note that y_n lies in $[q_n, q_{n+1}], [x, q_{n+1}]$ and $[y, q_{n+1}]$. Now $\hat{v}_{n+1} = \hat{z}_n \in \hat{G}_n \cap \hat{G}_{n+1}$. Also either x or y is stabilized by $\bar{G}_n$. But $\hat{u}_{n+1} \in \bar{G}_n \cap \hat{G}_{n+1}$ and so it fixes y_n. Thus y_n is stabilized by $\langle \hat{u}_{n+1}, \hat{v}_{n+1} \rangle = \hat{G}_{n+1}$ and so we can assume $y_n = q_{n+1}$. We have a contradiction, unless $q_n = q_{n+1}$, since if this is not the case, then q_{n+1} is closer to $[x, y]$ than q_n is. Thus we may suppose $q_n = q_{n+1} = q_m, m \geq n$. But we can choose $m \geq n$ so that $m, m+1 \in I$. Then $\langle \hat{b}_{m+1} \hat{G}_m \hat{b}_{m+1}^{-1}, \hat{b}_{m+2} \hat{G}_{m+1} \hat{b}_{m+2}^{-1} \rangle = \langle \bar{G}_m, \bar{G}_{m+1} \rangle$ is a subgroup of X which also fixes q_n, and this subgroup is not small. This contradiction means that we can assume $q_n \in [x, y]$ for all n.

Let $n \in I$. A similar argument to the above shows that there is a vertex of $[q_n, x]$ fixed by $\hat{G}_{n+1}$, i.e., we can choose $q_{n+1} \in [q_n, x]$. Since I contains arbitrarily long sequences of consecutive integers, $q_n = q_{n+1}$ for some n, so that q_n is fixed by $\hat{G}_m$ for all $m > n$. It follows easily that neither $[q_n, x]$ nor $[q_n, y]$ can have a small stabilizer. Since at least one of $[q_n, x], [q_n, y]$ is non-trivial, we have a contradiction.

We summarize the above in a theorem.

Theorem. *There is a finitely generated group H which acts on an $\mathbb{R}$-tree T so that the action is minimal non-trivial and so that all arc stabilizers are finite cyclic. There is no non-trivial action of H on a simplicial tree with small edge stabilizers.*

The group H does have a non-trivial simplicial action on a tree, since

$$H \cong L *_J M,$$

where $M = \langle \hat{G}_2, \hat{G}_3, \dots \rangle$, and $J = L \cap M = \langle \hat{b}_3 \hat{G}_2 \hat{b}_3^{-1}, \hat{b}_4 \hat{G}_3 \hat{b}_4^{-1}, \dots \rangle = \langle \bar{G}_2, \bar{G}_3, \dots \rangle$. To see this let $K = L *_J M$. Clearly there is a surjective homomorphism $\theta : K \to H$. However the homomorphism $\omega_n : H_n \to H$ factors through θ, since $H_n \cong L_n *_{u_n} G_n$. It follows that θ is an isomorphism.

I think it may be possible to find a similar example which has no non-trivial action on a simplicial tree.

References

[1] R.C. Alperin and K.N. Moss, *Complete trees for groups with a real-valued length funcition*, J. London Math. Soc. (2) **31** (1985), 55–68.

[2] M. Bestvina and M. Feighn, *Bounding the complexity of simplicial group actions on trees*, Invent. Math. **103** (1991), 449–469.

[3] ______, *Stable actions of groups on real trees*, Invent. Math. **121** (1995), 287–321.

[4] I.M. Chiswell, *The Grushko-Neumann theorem*, Proc. London Math. Soc. **33** (1976), 385–400.

[5] W. Dicks, *Groups, trees and projective modules*, Lecture Notes in Mathematics, 790, Springer-Verlag, Berlin, 1980.

[6] M.J. Dunwoody, *Groups acting on protrees*, J. London Math. Soc. **56** (1997), 125–136.

[7] ______, *Folding sequences*, Geometry and Topology Monographs 1 (I. Rivin, C. Rourke and C. Series, ed.), The Epstein Birthday Schrift, International Press, 1998, pp. 139–158.

[8] D. Gaboriau, G. Levitt, and F. Paulin, *Pseudogroups of isometries of* $\mathbb{R}$ *and Rips' theorem*, Israel J. Math. **87** (1994), 403–428.

[9] H. Gillet and P.B. Shalen, *Dendrology of groups in low* $\mathbf{Q}$*-ranks*, J. Differential Geom. **32** (1990), 605–712.

[10] J.W. Morgan and P.B. Shalen, *Valuations, trees, and degenerations of hyperbolic structures, I*, Ann. of Math. (2) **120** (1984), 401–476.

[11] ______, *Valuations, trees, and degenerations of hyperbolic structures, II*, Ann. of Math. (2) **127** (1988), 403–465.

[12] P.B. Shalen, *Dendrology of groups: an introduction*, Essays in Group Theory, MSRI Publ., Math. Sci. Res. Inst. Publ., 8, Springer-Verlag, New York-Berlin, 1987, pp. 265–319.

[13] ______, *Dendrology and its applications*, Group Theory from a Geometrical Viewpoint (Trieste, 1990), World Sci. Publishing, River Edge, NJ, 1991, pp. 543–616.

[14] J.R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), 551–565.

[15] J. Tits, *A theorem of Lie-Kolchin for trees*, Contributions to algebra (a collection of papers dedicated to Ellis Kolchin), Academic Press, New York, 1977, pp. 377–388.

Faculty of Mathematical Studies, University of Southampton, Southampton, SO17 1BJ.

E-mail address: `mjd@maths.soton.ac.uk`