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EISENSTEIN SERIES ON ARITHMETIC QUOTIENTS OF LOOP GROUPS

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Abstract. We construct Eisenstein series on arithmetic quotients of loop groups, give a convergence criterion, and compute their constant term in terms of the Riemann zeta function. We also give a description of certain measures which will provide in infinite dimensions, the convolution operators needed to obtain an analytic continuation. In the last two sections we discuss the question of volumes of arithmetic quotients, and we discuss various generalizations of our results.

1. Introduction

Some years ago we had developed the beginnings of a theory such as discussed here ([ES] and also the remarks in the introduction to [CD]). However the methods were ad hoc, and it was not clear how to proceed toward more substantial results, such as Eisenstein series from cusp forms on the reductive parts of parabolic subgroups, meromorphic continuation and volume computations. Over the last couple of years, however, we have made progress toward rectifying this situation. For one thing, we have now obtained a more natural argument (using the measure μ_U on $\mathcal{I}_U/\Gamma_U$, of §4, below) for proving Theorem 4.1 below. These arguments extend to more general Eisenstein series (we discuss this briefly in §7). But more importantly, we have now obtained new analytic results (see §5 below) which should lead to the meromorphic continuation.

To set the stage, let C be the $l \times l$ Cartan matrix corresponding to a complex, simple Lie algebra $\mathfrak{g}$ of rank l, and let $\hat{C}$ denote the $(l+1)\times(l+1)$ affine extension of C (so $\hat{C}$ corresponds to the extended Dynkin diagram of $\mathfrak{g}$). We let $\mathfrak{g}(\hat{C})$ be the complex, affine Kac-Moody Lie algebra associated to $\hat{C}$ and let $\mathfrak{h}(\hat{C}) \subseteqq \mathfrak{g}(\hat{C})$ denote a Cartan subalgebra, constructed from the standard presentation of $\mathfrak{g}(\hat{C})$ by generators $e_i, f_i, h_i, i = 1, \dots, l+1$ (so $\mathfrak{h}(\hat{C})$ is the linear span of the h_i). For a dominant, integral linear functional $\lambda \in \mathfrak{h}(\hat{C})^*$, the dual space of $\mathfrak{h}(\hat{C})$, we have the highest weight module V^λ and we fix a Chevalley form $V_{\mathbb{Z}}^\lambda$ of V^λ, as in [LA]. For any field k, we set $V_k^\lambda = k \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda$, and we consider the ("complete") Chevalley group $\hat{G}_k^\lambda \subseteqq \mathrm{Aut}(V_k^\lambda)$ (see [LG] and §2, below). We now consider the

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case when $k = \mathbb{R}$, the real numbers, and set $\hat{G} = \hat{G}^\lambda = \hat{G}^\lambda_{\mathbb{R}}$. We let $\hat{\Gamma} \subsetneqq \hat{G}$ denote the arithmetic subgroup

$$\hat{\Gamma} = \{\gamma \in \hat{G} \mid \gamma \cdot V^\lambda_{\mathbb{Z}} = V^\lambda_{\mathbb{Z}}\}.$$

We have the degree operator D operating on $V^\lambda_{\mathbb{Z}}$ and hence on V^λ_k for any field k. $[D, e_i] = 0, i = 1, \ldots, l; [D, e_{l+1}] = e_{l+1}$, where we can assume $e_1, \ldots, e_l$; $f_1, \ldots, f_l$ generate $\mathfrak{g}$ - see §2. For $r > 0$, we have constructed in [LG] a fundamental domain from Siegel sets, for the right action of $\hat{\Gamma}$ on the space $\hat{G}e^{-rD} \subseteq \mathrm{Aut}(V^\lambda_{\mathbb{R}})$. In [LG2] we extended this construction to the Hilbert modular case, where we replaced $\mathbb{Z}$ by the ring of integers in a totally real, algebraic number field. In the present paper we will describe our results concerning Eisenstein series on the arithmetic quotients $\hat{G}e^{-rD}/\hat{\Gamma}$.

2. Some basic constructions and definitions of groups and subgroups

We let $\mathfrak{h}^e(\hat{C}) = \mathfrak{h}(\hat{C}) \oplus \mathbb{C}D$, the extended Cartan subalgebra, and we let $\hat{\Delta} \subseteq \mathfrak{h}^e(\hat{C})^*$, the dual space of $\mathfrak{h}^e(\hat{C})$, be the set of affine roots. Then the extended, affine, Lie algebra $\mathfrak{g}^e(\hat{C}) = \mathfrak{g}(\hat{C}) \oplus \mathbb{C}D$ has a root space decomposition

$$\mathfrak{g}^e(\hat{C}) = \mathfrak{h}^e(\hat{C}) \oplus \coprod_{\alpha \in \hat{\Delta}} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g}(\hat{C}) \mid [h, x] = \alpha(h)x$, for all h in $\mathfrak{h}^e(\hat{C})\}$. The Kac-Moody generator e_i spans a root space, which corresponds to a simple root α_i. The set $\{\alpha_i\}_{i=1,\ldots,l+1}$ is a set of simple roots. We let $\hat{\Delta}\pm$ denote the corresponding set of $\pm$ roots. We let $\hat{\Delta}_W \subseteq \hat{\Delta}$ denote the subset of Weyl roots, and set $\hat{\Delta}_{W,\pm} = \hat{\Delta} \cap \hat{\Delta}_\pm$. We recall that for $\alpha \in \hat{\Delta}_W$, the space $\mathfrak{g}^\alpha$ is one dimensional. The Lie algebra $\mathfrak{g}(\hat{C})$ admits a Chevalley form $\mathfrak{g}_{\mathbb{Z}}(\hat{C}) = \mathfrak{h}_{\mathbb{Z}}(\hat{C}) \oplus \coprod_{\alpha \in \hat{\Delta}} \mathfrak{g}^\alpha_{\mathbb{Z}}$, where $\mathfrak{g}^\alpha_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{Z}}(\hat{C}) \cap \hat{\mathfrak{g}}^\alpha$, and $\mathfrak{h}_{\mathbb{Z}}(\hat{C}) = \mathfrak{g}_{\mathbb{Z}}(\hat{C}) \cap \mathfrak{h}(\hat{C})$. For each $\alpha \in \hat{\Delta}_W$, we fix a generator ξ_α of $\mathfrak{g}^\alpha_Z$. Then the Chevalley form $V^\lambda_{\mathbb{Z}}$ is invariant under the divided powers $\xi^n_\alpha/n!$, and we set

$$\chi_\alpha(s) = \sum_{n \geq 0} s^n(\xi^n_\alpha/n!), \quad s \in k.$$

The sum on the right is well defined when applied to an element v of V^λ_k, since in fact only finitely many terms $(\xi^n_\alpha/n!) \cdot v$ are nonzero. Moreover, $\xi_\alpha(s + s') = \xi_\alpha(s)\xi_\alpha(s')$, $s, s' \in k$. In particular the elements $\xi_\alpha(s)$, $s \in k$, lie in $\mathrm{Aut}(V^\lambda_k)$. Among the elements of $\hat{\Delta}$ we have the subset of classical roots Δ, which may be identified with the roots of the complex, simple Lie algebra $\mathfrak{g}$. We assume the simple roots are ordered so that Δ is the intersection of $\hat{\Delta}$ with the span of $\alpha_1, \ldots, \alpha_l$. Also the elements of the set of imaginary roots $\hat{\Delta}_I = \hat{\Delta} - \hat{\Delta}_W$ are the integral multiples of a single positive, imaginary root ι. Then the Weyl roots may be described as

$$\hat{\Delta}_W = \{\alpha + n\iota\}_{\alpha\in\Delta, n\in\mathbb{Z}}.$$

For a commutative ring R with unit we let $\mathcal{L}_R$ be the ring of all formal Laurent series $\sigma = \sum_{j\geq j_0} a_j t^j, a_j \in R$, and we let $\mathcal{O}_R \subseteq \mathcal{L}_R$ be the subring of all formal power series $\sigma = \sum_{j\geq 0} a_j t^j, a_j \in R$, and then set $\mathcal{P}_R = t\mathcal{O}_R$. For any commutative ring R with unit we let R^* denote the units in R. For $\alpha \in \Delta$, $\sigma = \sum_{j\geq j_0} a_j t^j$, we set

$$\chi_\alpha(\sigma) = \prod_{j\geq j_0} \chi_{\alpha+j\iota}(a_j),$$

again noting that when applied to an element v of $V_R^\lambda = R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda$, this infinite product makes sense, since all but finitely many factors leave the element v invariant. We then let $\hat{G}_k^\lambda$ be the subgroup of $\mathrm{Aut}(V_k^\lambda)$ generated by the elements $\chi_\alpha(\sigma)$, $\alpha \in \Delta, \sigma \in \mathcal{L}_k$. Continuing with the case of $R = k$, a field, we set

$$\begin{aligned} w_\alpha(\sigma) &= \chi_\alpha(\sigma)\chi_{-\alpha}(-\sigma^{-1})\chi_\alpha(\sigma),\\ h_\alpha(\sigma) &= w_\alpha(\sigma)w_\alpha(1)^{-1}, \qquad \alpha\in\Delta, \sigma\in\mathcal{L}_k^*. \end{aligned}$$

We also have elements:

$$\begin{aligned} w_\alpha(s) &= \chi_\alpha(s)\chi_\alpha(-s^{-1})\chi_\alpha(s),\\ h_\alpha(s) &= w_\alpha(s)w_\alpha(1)^{-1}, \quad \alpha\in\hat{\Delta}_W, s\in k^*. \end{aligned}$$

We let $\mathcal{I}_k \subseteq \hat{G}_k^\lambda$, the Iwahori subgroup, be the subgroup generated by the elements $\chi_\alpha(\sigma)$, with either $\alpha \in \Delta_+$ $(= \Delta\cap\hat{\Delta}_+)$ and $\sigma \in \mathcal{O}_k$, or $\alpha \in \Delta_-(= \Delta\cap\hat{\Delta}_-)$ and $\sigma \in \mathcal{P}_k$, and by the elements $h_\alpha(\sigma), \alpha \in \Delta, \sigma \in \mathcal{O}_k^*, h_\alpha(s), \alpha \in \hat{\Delta}_W, s \in k^*$. We let $\mathcal{I}_{U,k} \subseteq \mathcal{I}_k$ be the subgroup generated by the elements $\chi_\alpha(\sigma)$, with either $\alpha \in \Delta_+$ and $\sigma \in \mathcal{O}_k$, or $\alpha \in \Delta_-$ and $\sigma \subset \mathcal{P}_k$, and by the elements $h_\alpha(\sigma), \alpha \in \Delta, \sigma \in \mathcal{O}_k^*$ with constant term 1. We let A be the subgroup of $\hat{G}_{\mathbb{R}}^\lambda$ generated by the elements $h_\alpha(s), s > 0, \alpha \in \hat{\Delta}_W$.

We let $\hat{N}$ be the subgroup of $\hat{G}_{\mathbb{R}}^\lambda$ generated by the elements $w_\alpha(s)$ and $\hat{H}$ the subgroup generated by the elements $h_\alpha(s)$, $\alpha \in \hat{\Delta}_W$, $s \in \mathbb{R}^*$. Then $\hat{H}$ is a normal subgroup of $\hat{N}$, and the quotient group $\hat{W}$ is isomorphic to the Weyl group of $\mathfrak{g}(\hat{C})$, and hence acts on each of the spaces $\mathfrak{h}^e(\hat{C}), \mathfrak{h}^e(\hat{C})^*, \mathfrak{h}(\hat{C}), \mathfrak{h}(\hat{C})^*$. The set $\hat{\Delta} \subseteq \mathfrak{h}^e(\hat{C})^*$ is left invariant under this action of $\hat{W}$. In the case when $k = \mathbb{R}$ or $\mathbb{C}$, then in fact this action is induced by an adjoint action of the group $\hat{N}$, which normalizes each of these abelian subalgebras, and for which $\hat{H}$ acts trivially. For $w \in \hat{W}$, we set $\Psi_w = \hat{\Delta}_+ \cap w(\hat{\Delta}_-)$. For $\alpha \in \hat{\Delta}_W$, we let $h_\alpha \in \mathfrak{h}(\hat{C})$ denote the corresponding coroot.

It is a result of Kac that the Lie algebra $\mathfrak{g}^e(\hat{C})$ admits an invariant, symmetric, bilinear form $(\, ,)$, (see [K]) whose restriction to $\mathfrak{h}^e(\hat{C})$ is nondegenerate. To each $\alpha \in \hat{\Delta}$ corresponds a unique element h'_α in $\mathfrak{h}(\hat{C})$ such that

$$(h'_\alpha, h) = \alpha(h), \qquad h \in \mathfrak{h}^e(\hat{C}).$$

The non-degenerate inner product $(\, ,)$ on $\mathfrak{h}^e(\hat{C})$ induces then, an inner product (again denoted $(\, ,)$) on $\mathfrak{h}^e(\hat{C})^*$. We set $h_\iota = 2h'_\iota/(\alpha_0, \alpha_0)$, where $\alpha_0 \in \hat{\Delta}$ is the highest root of $\Delta \subseteq \hat{\Delta}$.

3. The Iwasawa decomposition and fundamental domain

We first recall that the group $\hat{G}$ has an Iwasawa decomposition. In order to describe this, we first recall that the Lie algebra $\mathfrak{g}(\hat{C})$ has a compact form $\mathfrak{k}(\hat{C})$ – see e.g., [LA] (so $\mathfrak{g}(\hat{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}(\hat{C})$ and the natural invariant inner product on $\mathfrak{g}(\hat{C})$ is negative, semi-definite when restricted to $\mathfrak{k}(\hat{C})$), and that the representation space V^λ admits a positive-definite, Hermitian inner product $\{ , \}$, so that the elements of $\mathfrak{k}(\hat{C})$ act as skew-Hermitian operators with respect to $\{ , \}$ (see [LA]). We let $\hat{K} \subseteq \hat{G}$ be the subgroup

$$\hat{K} = \{k \in \hat{G} \mid k \text{ is unitary with respect to } \{ \, , \}\}.$$

Now the group $\hat{G}$ has an Iwasawa decomposition (see [LG])

$$\hat{G} = \hat{K} A \mathcal{I}_U, \text{ with uniqueness of expression.} \tag{3.1}$$

(where we set $\mathcal{I}_U = \mathcal{I}_{U,\mathbb{R}}$). Moreover, the automorphism $\mu(r) = e^{-rD}, r \in \mathbb{R}$, of $V^\lambda_{\mathbb{R}}$, normalizes the group $\mathcal{I}_U$, and so (3.1) implies

$$\hat{G}\mu(r) = \hat{K} A \mu(r) \mathcal{I}_U, \quad r \in \mathbb{R}. \tag{3.1$'$}$$

We set $A^e =$ the set of all $a\mu(r), a \in A, r \in \mathbb{R}$. Each element α of $\hat{\Delta}$ defines a character

$$a \to a^\alpha, \quad a \in A^e,$$

of A^e, with values in $\mathbb{R}_{>0}$. For a real number $s > 0$, and for $r > 0$, we set $(A\mu(r))_s = \{a \in A\mu(r) \mid a^{\alpha_i} < s, i = 1, \dots, l+1\}$. On the other hand the quotient $\mathcal{I}_U/(\hat{\Gamma} \cap \mathcal{I}_U)$ is naturally the inverse limit of compact nilmanifolds, and hence inherits the structure of a compact Hausdorff space (a point which will be very useful as we'll see later on). In [LG] and [LG2], we constructed a fundamental domain $\mathcal{I}_{U,\mathcal{D}}$ for the right action, say, of $\hat{\Gamma} \cap \mathcal{I}_U$ on $\mathcal{I}_U$. Let $r > 0$. By a Siegel set, we mean a set $\mathfrak{S}_s \subset \hat{G}\mu(r), s > 0$, of the form

$$\mathfrak{S}_s = \hat{K}(A\mu(r))_s \mathcal{I}_{U,\mathcal{D}}.$$

In [LG], we had proved:

Theorem 3.1. *If* $s_0 = 2/\sqrt{3}$*, then*

$$\hat{G}\mu(r) = \mathfrak{S}_{s_0}\hat{\Gamma}.$$

This theorem is of course in direct analogy with the classical results in the finite-dimensional case (see [Bo]). Moreover, in [LG2] we extended this theorem to the Hilbert modular case, by constructing in that case a fundamental domain from Siegel sets, with finitely many cusps. In addition, in [LG], Theorem 21.16, we proved for the situation of Theorem 3.1, that one has a theorem of parabolic transformations (again this parallels the finite-dimensional case). Those arguments go over to the Hilbert modular case as well. We also have, at least for r sufficiently large, that the double coset space $\hat{K}\backslash\hat{G}\mu(r)/\hat{\Gamma}$ may be given, a natural locally compact, Hausdorff topology, such that further projecting modulo the center of $\hat{G}$, one obtains a compact, Hausdorff space.

4. Eisenstein series

An element ν of $\mathfrak{h}(\hat{C})^*$ defines a quasi-character (still denoted ν) $a \to a^\nu$ from A to $\mathbb{C}^*$. We can then define a complex valued function Φ_ν on $\hat{G}\mu(r)$ by

$$\Phi_\nu(ka\mu(r)i) = a^\nu, \qquad k \in \hat{K}, a \in A, i \in \mathcal{I}_U.$$

Of course, from the definition, we have

$$\Phi_\nu(kg\mu(r)i) = \Phi_\nu(g\mu(r)), \qquad k \in \hat{K}, g \in \hat{G}, i \in \mathcal{I}_U,$$

and setting $\Gamma_U = \hat{\Gamma} \cap \mathcal{I}_U$, it follows that Φ_ν may be regarded as a function on the double coset space $\hat{K}\backslash\hat{K}A\mu(r)\mathcal{I}_U/\Gamma_U$, which given the compact topology on $\mathcal{I}_U/\Gamma_U$ discussed in §3, has a natural locally compact, Hausdorff topology. We let $\rho \in \mathfrak{h}(\hat{C})^*$ be defined by $\rho(h_i) = 1, i = 1, \ldots, l+1$. We let Γ_M denote the subgroup of elements in $\hat{\Gamma}$, which lie in the intersection of $\hat{K}$ and the centralizer of A (the group Γ_M is then a finite, abelian group). We let $\pi : \hat{K}A\mu(r)\mathcal{I}_U \to \hat{K}\backslash\hat{K}A\mu(r)\mathcal{I}_U/\Gamma_U$ be the natural projection. We consider the infinite sum

$$\sum_{\gamma\in\hat{\Gamma}/\Gamma_M\Gamma_U} \Phi_\nu(g\mu(r)\gamma), \quad g \in \hat{G}. \tag{4.1}$$

We then have the following

Theorem 4.1. (Godement's criterion) *The sum* (4.1) *converges absolutely and uniformly for* $g\mu(r)$ *varying over a set* S *with* $\pi(S)$ *compact, provided*

$$Re\ \nu(h_i) < -2, i = 1, \ldots, l+1. \tag{4.2}$$

In the event that (4.2) is satisfied and hence the series (4.1) does converge, we denote that infinite sum by $E_\nu(g\mu(r))$ and call E_ν an Eisenstein series.

As we said in §3, the space $\mathcal{I}_U/\Gamma_U$ is the inverse limit of an inverse family of compact nilmanifolds N_i/Γ_i, with Γ_i a discrete, cocompact subgroup of the simply connected, (finite-dimensional) nilpotent, Lie group N_i. Each of these compact nilmanifolds has an N_i invariant, probability measure induced from a suitably normalized Haar measure on N_i, and these fit together to define an $\mathcal{I}_U$-invariant probability measure μ_U on $\mathcal{I}_U/\Gamma_U$. Thanks to the uniform convergence of Theorem 4.1, it follows that the Eisenstein series $E_\nu(g\mu(r))$, when regarded as a function on the compact Hausdorff space $\hat{K}\backslash\hat{G}\mu(r)/\hat{\Gamma}$, is continuous, and the function

$$i \mapsto E_\nu(g\mu(r)i), \quad i \in \mathcal{I}_U$$

which is right invariant with respect to the group Γ_U may be integrated to give a "constant term"

$$E_\nu^\#(g\mu(r)) = \int\limits_{\mathcal{I}_U/\Gamma_U} E_\nu(g\mu(r)i)d\mu_U(i).$$

This constant term is then invariant with respect to left translation by elements of $\hat{K}$ and also invariant with respect to right translation by elements of $\mathcal{I}_U$. Hence $E_\nu^\#$ is completely determined by its values on elements of A (thanks to the Iwasawa decomposition (3.1)). Moreover one can compute this constant term explicitly: First we let $\xi(s)$ be the normalized, Riemann zeta function

$$\xi(s) = \left[\prod_{p \text{ prime}} \left(\frac{1}{1-\frac{1}{p^s}}\right)\right] \times \Gamma(s/2) \times \pi^{-s/2}.$$

Then we have

Theorem 4.2. *For an element a in A, we have*

$$E_\nu^\#(a\mu(r)) = \sum_{w\in\hat{W}} (a\mu(r))^{w(\nu+\rho)-\rho} \prod_{\alpha\in\Psi_{w^{-1}}} \frac{\xi(-(\nu+\rho)(h_\alpha))}{\xi(-(\nu+\rho)(h_\alpha)+1)}. \tag{4.3}$$

5. Toward the analytic continuation

The expression (4.3) suggests that the Eisenstein series $E_\nu(g)$ derived from Theorem 4.1 has a meromorphic continuation to the region

$$\text{Re } \nu(h_\iota) < -\rho(h_\iota). \tag{5.1}$$

The point is that when (5.1) holds, then the constant term $E_\nu^\#$ has such a continuation, and one expects in analogy with the finite-dimensional case, that the same should be true for E_ν.

When one compares with the finite-dimensional case, the first missing component for obtaining this continuation is that of suitable bi - $\hat{K}$ - invariant convolution operators, which yield for example, approximate identities. In the present situation we can obtain such operators as "measures", which provide integrals of sections of certain line bundles. We proceed to give a precise description of what we mean.

The subgroup A has a one-dimensional subgroup A_{cen} in the center of $\hat{G}$ (in fact $A_{cen} = A \cap (\text{center of } \hat{G})$, and we consider the spaces

$$X = \hat{K}\backslash\hat{K}A\mathcal{I}_U, \bar{X} = \hat{K}\backslash\hat{K}A\mathcal{I}_U/A_{cen} \cong \hat{K}\backslash\hat{K}(A/A_{cen})\mathcal{I}_U.$$

Then we have a principal A_{cen} fibration

$$\begin{array}{ccc} A_{cen} & \longrightarrow & X \\ & & \Big\downarrow \\ & & \bar{X} \end{array} . \tag{5.2}$$

Then too, for each quasi-character σ of A_{cen}, we have an associated line bundle $\mathcal{L}_\sigma \to \bar{X}$ to the principal fibration (5.2). Then for example, taking $\sigma = \nu\mid_{A_{cen}}$, the Eisenstein series E_ν may be identified with a section of the line bundle L_σ, which has some invariance with respect to $\hat{\Gamma}$. We can easily give a precise description of this invariance property: for $g \in \hat{G}$, and $h \in \hat{G}$, we define $g \star_r h$ by ($g = k_g a_g i_g, k_g \in \hat{K}, a_g \in A, i_g \in \mathcal{I}_U$):

$$\begin{aligned} k_g a_g \mu(r) i_g h &= ka\mu(r)i, \qquad k \in \hat{K}, a \in A, i \in \mathcal{I}_U, \\ g \star_r h &= kai; \end{aligned}$$

i.e., we identify E_ν with the function F_ν on $\hat{G}$, defined by

$$F_\nu(kai) = E_\nu(ka\mu(r)i), \qquad k \in \hat{K}, a \in A, i \in \mathcal{I}_U.$$

This indeed defines a section of $\mathcal{L}_\nu$, as advertised, and

$$F_\nu(g \star_r \gamma) = F_\nu(g), \qquad g \in \hat{G}, \gamma \in \hat{\Gamma}.$$

For a normal subgroup $\mathcal{I}' \subseteq \mathcal{I}_U$ of finite codimension, we let

$$\tilde{\omega}_{\mathcal{I}'} : \hat{K}\backslash\hat{K}(A/A_{cen})\mathcal{I}_U \to \hat{K}\backslash\hat{K}(A/A_{cen})\mathcal{I}_U/\mathcal{I}'.$$

denote the projection. Each of the spaces $\bar{X}' = \hat{K}\backslash\hat{K}(A/A_{cen})\mathcal{I}_U/\mathcal{I}'$ is a locally compact Hausdorff space. We let $\mathcal{B}'$ be the σ-algebra of Borel subsets of $\bar{X}'$. On the space $\bar{X}$, we let $\mathcal{B}$ be the smallest σ-algebra of subsets such that each of the projections $\tilde{\omega}_{\mathcal{I}'}$ is a Borel measurable map ($\tilde{\omega}_{\mathcal{I}'}^{-1}(S)$ is in $\mathcal{B}$, for each S in $\mathcal{B}'$).

We let $\prod = \{\alpha_i\}_{i=1,\dots l+1}$ denote the simple roots of $\mathfrak{g}(\hat{C})$, as above. For each $j = 1, \dots, l+1$, we set $\prod_j = \prod - \{\alpha_j\}$. Then for each $j = 1, \dots, l+1$, we have a direct product decomposition

$$A = A_{cen} \times A_j, \tag{5.3}$$

where the Lie algebra of A_j is the real span of the h_i, i not equal to j. A_{cen} has Lie algebra $\mathbb{R}h_\iota$, and a quasi-character σ on A_{cen} is determined by the value $u = \sigma(h_\iota)$.

We have for each $j = 1, \dots, l+1$, that the projection $A \to A/A_{cen}$ induces an isomorphism $A_j \cong A/A_{cen}$. We may then identify $\bar{X}$ with $\bar{X}_j = \hat{K}\backslash\hat{K}A_j\mathcal{I}_U \subseteq X$. For a quasi-character $\sigma : A_{cen} \to \mathbb{C}^*$, we let L_σ be the space of all $\mathbb{C}$-valued functions on X such that

$$f(xa) = f(x)a^\sigma, \qquad x \in X, a \in A_{cen},$$

and such that for $j = 1, \dots, l+1, f_j =_{df} f \mid_{\bar{X}_j}$ ($\bar{X}_j$ being identified with $\bar{X}$) is $\mathcal{B}$-measurable.

Theorem 5.1. *We assume that if we write the highest root α_0 as $\alpha_0 = n_1\alpha_1 + \dots + n_l\alpha_l$, then at least one n_i is equal to one. There is a family of probability measures $m_{u,r,j}, r > 0, u < -\rho(h_\iota)$, $j = 1, \dots, l+1$ defined on the σ-algebra $\mathcal{B}$, satisfying the following conditions:*

(i). *For each projection $\tilde{\omega}_{\mathcal{I}'}$, the direct image measure $(\tilde{\omega}_{\mathcal{I}'})_\star(m_{u,r,j})$ is a regular, Borel measure on $\bar{X}'$, which in fact is given by a C^∞ function on $\bar{X}'$, this space being naturally identified with Euclidean space.*

(ii). *Let σ be the quasi-character of A_{cen} defined by the condition that $\sigma(h_\iota) = u$. For a given j_0 such that $j_0 = l+1$ or $n_{j_0} = 1$, and for $f \in L_\sigma$ such that f_{j_0} is m_{u,r,j_0}-integrable, we have that f_j is $m_{u,r,j}$-integrable for all j such that $j = l+1$ or $n_j = 1$, and*

$$\int f_{j_0} dm_{u,r,j_0} = \int f_j dm_{u,r,j}.$$

We set the common value of all these integrals equal to

$$\int f(x) dm_u(x), \tag{5.4}$$

and say that the section f is m_u-integrable.

(iii). *We have*

$$\int f(x \star_r k) dm_u(x) = \int f(x) dm_u(x), \quad \textit{for all } k \in \hat{K}.$$

(iv). *For* $j = 1, \ldots, l+1$, *and* U *an open neighborhood of the identity coset in* $\bar{X}'$, *we have*

$$\lim_{r \to 0} m_{u,r,j}(\tilde{\omega}_{\mathcal{I}'}^{-1}(U)) = 1.$$

Remark 5.1. The regular Borel measures $(\tilde{\omega}_{\mathcal{I}'})_\star(m_{u,r,j})$ are defined using a slight modification of the heat kernel. The assumption on α_0 is probably not necessary, but to date we have only checked the result of Theorem 5.1, (iii), with this restriction (which holds in most cases). Each measure $m_{u,r,j}$ gives (iii) for k in $K_j = K \cap P_j$, where P_j is the maximal parabolic subgroup of $\hat{G}$, corresponding to j; i.e., each measure $m_{u,r,j}$ is invariant invariant with respect to K_j acting on the right by means of $*_r$. Finally, using Theorem 21.16 of [LG], one can construct analogues of the integral (5.4) on an arithmetic quotient of the type constructed in [LG]. This construction is an analogue of averaging a heat kernel on a Riemannian manifold over a discrete group of isometries.

6. Remarks on the volume of arithmetic quotients

We have defined in §4, the $\mathcal{I}_U$ -invariant probability measure μ_U on the compact Hausdorff space $\mathcal{I}_U/\Gamma_U$. We now denote this measure by di. On the group A we have Lebesgue measure da. More precisely, every element a of A has a unique expression

$$a = h_{\alpha_1}(s_1) \ldots h_{\alpha_{l+1}}(s_{l+1}), \qquad s_1, \ldots, s_{l+1} \in \mathbb{R}_{>0}.$$

We then let

$$da = (s_1 \ldots s_{l+1})^{-1} ds_1 \ldots ds_{l+1},$$

with ds being (additive) Lebesgue measure. Then one can show that $a^{2\rho} dadi$ defines a measure on the space

$$Y = \hat{K} \backslash \hat{K} A \mu(r) \mathcal{I}_U / \hat{\Gamma}, \tag{6.1}$$

but because of the presence of A_{cen}, Y does not have finite volume with respect to $a^{2\rho} dadi$. However, as we observed in §3, the projection of the space (6.1) modulo A_{cen} is compact. We set $\bar{Y}$ = projection of Y modulo A_{cen}. We then have a principal fibration

$$\begin{array}{ccc} A_{cen} & \longrightarrow & Y \\ & & \big\downarrow \\ & & \bar{Y} \end{array},$$

and as in §5, we may consider line bundles $\mathcal{L}(\hat{\Gamma})_\sigma$ associated to quasi-characters σ of A_{cen}. We let σ_0 be the restriction of -2ρ to A_{cen}, and let f be a continuous section of $\mathcal{L}(\hat{\Gamma})_{\sigma_0}$. Then $fa^{2\rho}dadi$ can be shown to define a measure on $\bar{Y}$, which has finite volume. The problem then is to choose a natural f. However, this natural choice emerges, once we obtain the analytic continuation of the Eisenstein series. Indeed an examination of the expression (4.3) shows that the constant term $E_\nu^{\#}(a\mu(r))$ has a pole at $\nu = -2\rho$, coming from the simple pole of the Riemann zeta function at 1. The residue of the analytic continuation of the Eisenstein series E_ν at $\nu = -2\rho$ would then provide a natural choice for f! In the finite-dimensional case this residue is in fact the constant function, and thereby makes possible the volume computations in Langlands [L]. What we have here is a cancellation of anomalies strikingly analogous to the cancellation discussed for example in [FGZ], in connection with the no-ghost theorem!

7. Further generalizations and speculations

The results described here can be generalized in various ways. Theorem 4.1 can be generalized to Eisenstein series corresponding to more general "$\hat{K}$-types", and to Eisenstein series defined from nonminimal parabolic subgroups and corresponding to cusp forms on the reductive parts of those parabolics. One can also extend the theory of Eisenstein series to Hilbert modular loop groups as described in [GL2]. One should also be able to extend the theory to other Kac-Moody and generalized Kac-Moody groups. In particular, the $\mathbb{Z}$-form of Borcherds [B], for the universal enveloping algebra of the fake monster Lie algebra provides an analogue in that case to our $\mathbb{Z}$-form for the universal enveloping algebra of a loop algebra (see [LA]), and so suggests that one could treat that case as well. However, the form of the constant term in (4.3) suggests that one will be constrained by the convergence properties of Weyl characters, in whatever context one is working. Also, the theory of the fundamental domain will only make sense for part of the arithmetic quotient in general, as one must prove that appropriate functionals achieve maximal values over $\hat{\Gamma}$.

References

[B] R.E. Borcherds, *The fake monster formal group*, preprint.

[Bo] A. Borel, *Introduction aux groupes arithmétiques*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341, Hermann, Paris, 1969.

[EFK] P.I. Etingoff, I.B. Frenkel, and A.A. Kirillov, Jr., *Spherical functions on affine Lie groups*, Duke Math. J. **80** (1995), 59–90.

[FGZ] I.B. Frenkel, H. Garland, and G.J. Zuckerman, *Semi-infinite cohomology and string theory*, Proc. Natl. Acad. Sci. USA **83** (1986), 8442–8446.

[LA] H. Garland, *The arithmetic theory of loop algebras*, J. Algebra **53** (1978), 480–551.

[LG] ———, *The arithmetic theory of loop groups*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 181–312.

[CD] ———, *A Cartan decomposition for p-adic loop groups*, Math. Ann. **302** (1995), 151–175.

[LG2] ———, *The arithmetic theory of loop groups II. The Hilbert-modular case*, J. Algebra **209** (1998), 446–532.

[ES] ———, *Eisenstein series on arithmetic quotients of loop groups*, preprint (1992).

[K] V.G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.

[KP] V.G. Kac and D.H. Peterson, *Unitary structure in representations of infinite-dimensional groups and a convexity theorem*, Invent. Math. **76** (1984), 1–14.

[L] R.P. Langlands, *The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups*, Algebraic groups and discontinuous subgroups (A. Borel and G.D. Mostow, eds.), Proc. Sympos. Pure Math., (Boulder, Colo., 1965), vol. 9, Amer. Math. Soc., Providence, R.I., 1966.

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