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CARATHÉODORY-FEJÉR INTERPOLATION ON POLYDISKS

Jörg Eschmeier, Linda Patton and Mihai Putinar

1. Introduction

The question of whether a polynomial can be extended with higher degree terms to a convergent series from the unit disk **D** into itself is known as the Carathéodory-Fejér problem. The solvability of this problem is equivalent to the positive semi-definiteness of an associated Toeplitz type matrix which depends only on the coefficients of the original polynomial. Due to a fundamental observation of Sarason [12], the Carathéodory-Fejér problem, and its relative, the Nevanlinna-Pick problem, are now accessible via Hilbert space methods; more precisely these problems were reduced to a dilation scheme of linear operators. The commutant lifting theorem of Sz. Nagy and Foiaş is the abstract version of this dilation scheme; its numerous applications to function theory and engineering are well illustrated by the monograph [8].

Comparatively little is known in several complex variables. The integral representation formula for bounded analytic functions in classical domains of $\mathbf{C}^n$, known after its classical one variable version as the Herglotz-Nevanlinna formula, is now well understood, cf. [3], [10] and [13]. Via this formula, a characterization in positivity terms of the coefficients of the power series of an analytic function from $\mathbf{D}^n$ to $\mathbf{D}$ was derived by Korányi and Pukánszky [10], see also [3] and [13]. Other partial results towards a solution of the Carathéodory-Fejér problem in the polydisk are contained in Pfister's thesis [11] and, for low degree or special polynomials, in the works of the Russian school, see [3], [6] and the references contained there.

In what follows we discuss separately the case $n = 2$, which is special due to the dilation theorem for commuting contractions of Ando; see [8]. Agler recently developed a successful Hilbert space theory which complements the function theory in the bidisk, with the special aim at solving the Nevanlinna-Pick interpolation problem; see [1], [2], and the references therein. A different approach to the same problem is due to Cole and Wermer [5].

In the present note we propose a solution to the Carathéodory-Fejér problem in the bidisk via Agler's version of the Herglotz-Nevanlinna formula. In vague terms, our main result asserts that the necessary and sufficient condition for solving this problem in degree d is to truncate the Agler-Herglotz-Nevanlinna

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formula modulo terms of degree $d+1$ or higher. We parametrize all solutions of the Carathéodory-Fejér problem in terms of all unitary extensions of an isometry between finite dimensional spaces (a recurrent theme in the theory of commutant liftings). We prove that a rational function always exists among the extensions. As in all other similar results in dimension two or higher, the statements and their proofs contain a non-constructive part which makes all applications considerably more difficult than in the case of a single complex variable.

In higher dimensions Ando's theorem fails and we can rely only on the Herglotz-Nevanlinna formula. By slightly improving the main result of [10], we prove below that if the interpolation data are given on a large enough set, then the necessary positivity condition deduced from the Herglotz-Nevanlinna formula is also sufficient.

Section 2 below contains the main results and a few of their ramifications. Section 3 is devoted to proofs.

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2. Main Results

Let $n > 1$ be a fixed integer, and let $\mathbf{T}^n$ denote the n-dimensional torus, regarded as the distinguished boundary of the polydisk $\mathbf{D}^n$. Let f be an analytic function on $\mathbf{D}^n$. We denote by H a separable, complex Hilbert space. We will encounter two conditions on the real part of f:

(HN) There exists a vector valued analytic function $g : \mathbf{D}^n \longrightarrow H$ such that

$$f(z) + \overline{f(z)} = \prod_{j=1}^{n} (1 - |z_j|^2) \|g(z)\|^2, \qquad z \in \mathbf{D}^n,$$

and,

(AHN) There are vector valued analytic functions $g_j : \mathbf{D}^n \longrightarrow H, 1 \leq j \leq n$, such that

$$f(z) + \overline{f(z)} = \sum_{j=1}^{n} (1 - |z_j|^2) \|g_j(z)\|^2, \qquad z \in \mathbf{D}^n.$$

Condition (HN) is equivalent to the fact that $Ref(z) \geq 0$ for $z \in \mathbf{D}^n$; in this case the non-tangential boundary values of $Ref(z)$ exist at points of $\mathbf{T}^n$ and give there a positive measure μ with non-zero Fourier coefficients only in the positive octant and its opposite. The converse also holds and

$$\|g(z)\|^2 = \int_{\mathbf{T}^n} \frac{d\mu(\zeta)}{\prod_{j=1}^{n} |1 - \zeta_j \overline{z_j}|^2},$$

cf. [10] and [13].

On the other hand, condition (AHN) is equivalent to the fact that $Ref(T) \geq 0$ for every commutative n-tuple of strictly contractive operators. Due to Ando's theorem, for $n = 2$ the two conditions are equivalent, see [1]. By replacing $f(z) + \overline{f(z)}$ in the two conditions by $1 - |f(z)|^2$ one obtains the corresponding representations of analytic maps from $\mathbf{D}^n$ into $\mathbf{D}$.

For a positive integer d and two polynomials $P(z, \overline{z}), Q(z, \overline{z})$ we denote in the sequel $P \equiv Q \mod(z^{d+1}, \overline{z}^{d+1})$ if the difference $P(z, \overline{z}) - Q(z, \overline{z})$ contains only monomials of degree $d + 1$ or higher in at least one set of variables z or $\overline{z}$. The same definition can be applied to holomorphic functions in the bidisk by using power series expansions about zero.

Theorem 1. *Let d be a positive integer and let $P(z)$ be a polynomial of degree less than or equal to d in two complex variables. There exists an analytic function $F : \mathbf{D}^2 \longrightarrow \mathbf{D}$ such that $P \equiv F \mod(z^{d+1})$ if and only if there are Hilbert spaces H_1 and H_2 and a pair of vector valued polynomial functions of degree less than or equal to d, $A_k : \mathbf{D}^2 \longrightarrow H_k$, $k = 1, 2$, such that:*

$$(1)\quad 1 - P(z)\overline{P(z)} \equiv (1 - |z_1|^2)\|A_1(z)\|_1^2 + (1 - |z_2|^2)\|A_2(z)\|_2^2 \mod(z^{d+1}, \overline{z}^{d+1}).$$

The Hilbert spaces H_1 and H_2 in the statement may not be identical and can each be chosen of finite dimension less than or equal to $(d + 1)(d + 2)/2$. Note that $\| \ \|_k$ and $\langle\, ,\rangle_k$ denote the norm and inner product, respectively, on H_k for $k = 1, 2$.

The proof of Theorem 1, which will be given in the next section, provides also a matricial parametrization of all extensions F. Such a parametrization has already been discussed in [2] for the Nevanlinna-Pick problem on the bidisk. A direct consequence of the proof of Theorem 1 is the next corollary.

Corollary 1. *In the conditions of Theorem 1, there always exists an extension F which is a rational function of degree less than or equal to $d^2 + 3d + 2$.*

As a matter of fact, Theorem 1 holds in any dimension $n \geq 2$, but only for analytic functions F in the polydisk which satisfy $\|F(T)\| \leq 1$ for all commutative n-tuples of strict contractions T, cf. [1].

Let (d_1, d_2) be a pair of positive integers, and assume that the given polynomial $P(z)$ has degree d_1 in z_1 and d_2 in z_2, respectively. Then we say that the bidegree of P is less than or equal to (d_1, d_2). Pfister's method [11] gives necessary and sufficient conditions for the existence of a rational extension F, modulo a fixed bidegree, of the form $F = Q^\sharp/Q$, where Q is a polynomial of bidegree less than or equal to (d_1, d_2) which has no zeroes in the bidisk, and

$$Q^\sharp(z) = {z_1}^{d_1}{z_2}^{d_2}\overline{Q\left(\frac{1}{\overline{z}}\right)}.$$

See also [6]. Unfortunately, as Pfister points out, such a special rational solution does not always exist.

For the next interpolation result in higher dimensions we need to define a uniqueness point for a subset of $\mathbf{D}^n$ where n is an arbitrary positive integer. If

$p \in \mathbf{C}^n$ then let $B(p,r)$ denote the ball with center p and radius r. Let K be a subset of $\mathbf{D}^n$ and let $p \in K$. We say p is a uniqueness point of K if the following implication holds for all $r \in (0,1)$: If g and h are analytic functions on $B(p,r)$ and

$$h(z) = g(z) \quad \text{for all } z \in B(p,r) \cap K,$$

then

$$h(z) = g(z) \quad \text{for all } z \in B(p,r).$$

Recall that a function h defined on $\mathbf{D}^n \times \mathbf{D}^n$ is non-negative definite if for any positive integer m and points $z_1, \dots, z_m$ in $\mathbf{D}^n$, the expression

$$\sum_{j=1}^{m} \sum_{i=1}^{m} h(z_i, z_j) v_i \overline{v}_j,$$

is non-negative for all vectors $v \in \mathbf{C}^m$.

Theorem 2. *Let K be a subset of $\mathbf{D}^n$ which contains at least one uniqueness point. If f is a function on K with the property that the kernel*

$$\frac{1 - f(z)\overline{f(w)}}{\prod_{k=1}^{n}(1 - z_k \overline{w_k})}, \qquad z, w \in K,$$

is non-negative definite, then there exists an analytic function $F : \mathbf{D}^n \longrightarrow \mathbf{D}$ which coincides with f on K.

This is a refinement of [10], where the set K has the following property: there exists a point $\omega \in \mathbf{D}^n$ such that $z \in K$ implies that the point $\tilde{z}$ is also in K, where $\tilde{z}$ is obtained by replacing any coordinate of z with the corresponding coordinate of ω. Otherwise, we shall see that the proof is very similar. Note that there are simple examples of subsets of $\mathbf{D}^n$ from which bounded analytic functions do not have analytic extensions to $\mathbf{D}^n$ which are still bounded. To be more specific, such an example is the analytic set $\{(z,w); w^2 = z(z-1)^2\}$ in the unit polydisk in the two variables z and w; see [7] for details.

3. Proofs

Proof of Theorem 1. The necessity part of the proof follows from the (AHN) formula above, by taking its reduction modulo $|z|^{d+1}$.

For the sufficiency part of the proof, we need the following terminology: let $a = \{a_\alpha\}_{|\alpha|=0}^{d}$ be a bi-indexed finite sequence of vectors in a Hilbert space H with $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$. Define the shift operators S_1^* and S_2^* on the sequence a by

$$(S_1^* a)_\alpha = \begin{cases} a_{\alpha-(1,0)} & \text{if } \alpha_1 > 0, \\ 0 & \text{if } \alpha_1 = 0, \end{cases}$$

and

$$(S_2^* a)_\alpha = \begin{cases} a_{\alpha-(0,1)} & \text{if } \alpha_2 > 0, \\ 0 & \text{if } \alpha_2 = 0, \end{cases}$$

for all α with $|\alpha| \leq d$.

Now let $P(z) = \sum_{|\alpha|=0}^{d} c_\alpha z^\alpha$ and assume there exist A_1 and A_2 satisfying (1) with expansions

$$A_k(z) = \sum_{|\alpha|=0}^{d} a_\alpha^{(k)} z^\alpha \quad \text{for } k = 1, 2.$$

We will be applying S_k^* to the finite sequence $a^{(k)} \equiv \{a_\alpha^{(k)}\}_{|\alpha|=0}^{d}$ of coefficients of A_k for $k = 1, 2$. Note that $S_k^* a^{(k)}$ is a bi-indexed finite sequence of vectors in H_k, while for each bi-index α, $(S_k^* a^{(k)})_\alpha \in H_k$.

By expanding (1) and equating coefficients of like terms, we obtain

$$1 - |c_{00}|^2 = \|a_{00}^{(1)}\|_1^2 + \|a_{00}^{(2)}\|_2^2, \tag{2}$$

and

$$-c_\alpha \overline{c_\beta} = \Big(\sum_{k=1}^{2} \langle a_\alpha^{(k)}, a_\beta^{(k)} \rangle_k\Big) - \Big(\sum_{k=1}^{2} \langle (S_k^* a^{(k)})_\alpha, (S_k^* a^{(k)})_\beta \rangle_k\Big), \tag{3}$$

where (3) holds for all nonzero (α, β) where $|\alpha|$ and $|\beta|$ are each less than or equal to d.

Equations (2) and (3) show that the map V defined on a subspace of $\mathbf{C} \oplus H_1 \oplus H_2$ by

$$V(1 \oplus 0 \oplus 0) = c_{00} \oplus a_{00}^{(1)} \oplus a_{00}^{(2)}, \tag{4}$$

and

$$V(0 \oplus (S_1^* a^{(1)})_\alpha \oplus (S_2^* a^{(2)})_\alpha) = c_\alpha \oplus a_\alpha^{(1)} \oplus a_\alpha^{(2)}, \qquad 0 < |\alpha| \leq d, \tag{5}$$

is an isometry.

Pick a unitary extension U of V, still acting on $\mathbf{C} \oplus [H_1 \oplus H_2]$, and let us denote:

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

with respect to the decomposition into $\mathbf{C}$ and $[H_1 \oplus H_2]$, respectively.

Equations (4) and (5) can be expressed in terms of U as follows:

$$\begin{cases} U_{11} &= c_{00}, \\ U_{21} &= a_{00}^{(1)} \oplus a_{00}^{(2)}, \\ U_{12}((S_1^* a^{(1)})_\alpha \oplus (S_2^* a^{(2)})_\alpha) &= c_\alpha, \qquad 0 < |\alpha| \leq d, \\ U_{22}((S_1^* a^{(1)})_\alpha \oplus (S_2^* a^{(2)})_\alpha) &= a_\alpha^{(1)} \oplus a_\alpha^{(2)}, \qquad 0 < |\alpha| \leq d. \end{cases} \tag{6}$$

Let P_k be the orthogonal projection of $H_1 \oplus H_2$ onto H_k and introduce the operator-valued map:

$$D(z) = z_1 P_1 \oplus z_2 P_2, \qquad (z_1, z_2) \in \mathbf{D}^2.$$

Now we can define the function F on the bidisk which will satisfy the Carathéodory-Fejér conditions. Let

$$F(z) = U_{11} + U_{12} D(z)(I - U_{22} D(z))^{-1} U_{21}. \tag{7}$$

Obviously, F is analytic in the bidisk; less obvious, but elementary, is the observation that $|F(z)| \leq 1$ for z in the bidisk. For a proof see [2] or let $v(z) = (I - U_{22}D(z))^{-1}U_{21}$ and consider the linear system:

$$U \begin{pmatrix} 1 \\ D(z)v(z) \end{pmatrix} = \begin{pmatrix} F(z) \\ v(z) \end{pmatrix}.$$

This means (since U is unitary)

$$|F(z)|^2 + \|v(z)\|^2_{H_1 \oplus H_2} = 1 + \|D(z)v(z)\|^2_{H_1 \oplus H_2}.$$

Multiplication by $D(z)$ in $H_1 \oplus H_2$ is a contraction for each $z \in \mathbf{D}^2$. Thus

$$|F(z)|^2 + \|v(z)\|^2_{H_1 \oplus H_2} \leq 1 + \|v(z)\|^2_{H_1 \oplus H_2},$$

which implies that $|F(z)| \leq 1$ for $z \in \mathbf{D}^2$.

It remains to show that $F \equiv P \mod(z^{d+1})$. Note that equality $mod(z^{d+1})$ of polynomials with coefficients in $H_1 \oplus H_2$ is preserved under multiplication by $D(z)$ or any operator independent of z. A calculation using this fact and (6) shows that

$$(I - U_{22}D(z))(A_1(z) \oplus A_2(z)) \equiv a_{00}^{(1)} \oplus a_{00}^{(2)} \quad mod(z^{d+1}),$$

which implies that

$$(I - U_{22}D(z))^{-1}(a_{00}^{(1)} \oplus a_{00}^{(2)}) \equiv A_1(z) \oplus A_2(z) \quad mod(z^{d+1}).$$

Thus

$$\begin{aligned} F(z) &\equiv c_{00} + U_{12}D(z)(A_1(z) \oplus A_2(z)) \\ &\equiv c_{00} + U_{12}((\sum_{|\alpha|=1}^{d} (S_1^* a^{(1)})_\alpha z^\alpha) \oplus (\sum_{|\alpha|=1}^{d} (S_2^* a^{(2)})_\alpha z^\alpha)) \\ &\equiv c_{00} + \sum_{|\alpha|=1}^{d} c_\alpha z^\alpha \qquad mod(z^{d+1}). \end{aligned}$$

The last equivalence follows from (6) and concludes the proof of the theorem.

□

For the proof of Corollary 1, it remains to remark that we can always choose the Hilbert spaces H_1 and H_2 of finite dimension D, less than or equal to the number of coefficients c_α, where $|\alpha| \leq d$. That is, $D \leq (d+1)(d+2)/2$. Thus, the operator $D(z)(I - U_{22}D(z))^{-1}$ in (7) can be expressed as a $(d^2+3d+2) \times (d^2+3d+2)$ matrix with entries which are rational functions of z. These rational functions have identical denominators and each rational function has degree at most $d^2 + 3d + 2$. Therefore, the function F will be rational of degree at most this integer.

The above proof gives a parametrization of all solutions F to the Carathéodory-Fejér problem. Namely, pick a decomposition (1), with auxiliary Hilbert spaces H_1, H_2; then define by formulas (4) and (5) the isometric map V on a subspace of $\mathbf{C} \oplus [H_1 \oplus H_2]$ and then choose an extension of V to a unitary map

U, on the same space. As a result, it turns out that, for the fixed decomposition (1), formula (7) gives all possible solutions F. If it happens that the map V is already defined on the whole space, then the solution to the Carathéodory-Fejér problem is unique within the specific minimal choice of H_1, H_2. Compare with the one-variable case [8].

These techniques could be used to combine the Nevanlinna-Pick and Carathéodory-Fejér problems on the bidisk by finding conditions that are equivalent to the existence of a bounded analytic function whose Taylor expansion is specified to a certain degree at several points in the bidisk. For more details about such results, and their relation with the commutant lifting theorem, see [4].

Proof of Theorem 2. The proof is similar, at least in principle, to the proof of the main result in [10]. We merely indicate the points of the proof which are different from those in [10]. By standard Möbius transforms we can assume that $p = 0$ is the uniqueness point of K and that f maps K onto the right half plane $\mathbf{C}_+$ rather than the unit disk.

We introduce some notation involving the set $\mathbf{N}^n$ of multi-indices of length n. Define

$$O(n) = \{\alpha \in \mathbf{N}^n | \ \ |\alpha| \le n, \alpha_i = 0 \text{ or } \alpha_i = 1 \ \ \forall \ \ 1 \le i \le n, \ \ \text{and } |\alpha| \text{ is odd}\}.$$

$E(n)$ is defined similarly but only contains α such that $|\alpha|$ is even and strictly positive. Let N denote the cardinality of $O(n)$.

Let H be a complex separable Hilbert space which carries the factorization of the non-negative kernel in the statement of the theorem :

$$\frac{f(z) + \overline{f(w)}}{\prod_{k=1}^n (1 - z_k \overline{w_k})} = \langle e(z), e(w) \rangle, \quad z, w \in K. \tag{8}$$

The function e maps K into H and we can assume that H is spanned by the values of e.

Alternately setting z and w equal to 0 in equation (8) above results in the identity:

$$\prod_{k=1}^n (1 - z_k \overline{w_k}) \langle e(z), e(w) \rangle = \langle e(z), e(0) \rangle + \langle e(0), e(w) \rangle - \langle e(0), e(0) \rangle, \tag{9}$$

which holds for all z and w in K and which in turn (after expanding the product) implies

$$\sum_{\alpha \in O(n)} \langle z^\alpha e(z), w^\alpha e(w) \rangle = \left(\sum_{\alpha \in E(n)} \langle z^\alpha e(z), w^\alpha e(w) \rangle \right) + \langle e(z) - e(0), e(w) - e(0) \rangle, \tag{10}$$

for all z and w in K.

We will use equation (10) to define an isometry which will be used to create an analytic function that extends e to $\mathbf{D}^n$.

Let H^N denote the direct sum of N copies of H. Let S be the subset of H^N consisting of all elements of the form $\bigoplus_{\alpha\in O(n)} z^\alpha e(z)$ for $z \in K$. Finally let H_o^N be the vector subspace of H^N consisting of all finite linear combinations of elements of S.

Define an operator $\tilde{V}$ from H_o^N into H^N by specifying $\tilde{V}$ on S as

$$\tilde{V}\left(\oplus_{\alpha\in O(n)} z^\alpha e(z)\right) = \left(\oplus_{\alpha\in E(n)} z^\alpha e(z)\right) \oplus (e(z) - e(0)). \tag{11}$$

and extending $\tilde{V}$ linearly to elements of H_o^N.

Note that the right side of equation (11) is in H^N because the cardinality of $E(n)$ is $N-1$; therefore equation (10) shows that $\tilde{V}$ is an isometry from H_o^N to a subspace of H^N. We can thus extend $\tilde{V}$ to a contraction V on H^N.

Let $\alpha_1, \ldots, \alpha_N$ be a list of the elements of $O(n)$. Let $V = [V_{i,j}]_{i,j=1}^N$ be the block decomposition of V such that each $V_{i,j}$ is a contraction on H.

When equation (11) is stated for the last row of the block decomposition, we obtain

$$\sum_{i=1}^N V_{N,i} z^{\alpha_i} e(z) = e(z) - e(0), \quad z \in K.$$

Thus

$$\left(I - \sum_{i=1}^N z^{\alpha_i} V_{N,i}\right) e(z) = e(0), \quad z \in K. \tag{12}$$

Since each $V_{N,i}$ is a contraction, there exists $r \in (0,1)$ such that the operator $\sum_{i=1}^N z^{\alpha_i} V_{N,i}$ is a contraction for all $z \in B(0,r)$. Therefore

$$e(z) = \left(I - \sum_{i=1}^N z^{\alpha_i} V_{N,i}\right)^{-1} e(0), \tag{13}$$

for all $z \in K \cap B(0,r)$, and the right side of equation (13) defines a function E which is analytic for all z in $B(0,r)$. If $\sum_{\alpha\in\mathbf{N}^n} c_\alpha z^\alpha$ is the Taylor expansion of E at 0, then this series converges to E in $B(0,r)$.

Since $e(z) = E(z)$ for $z \in K \cap B(0,r)$, equation (9) shows that

$$\prod_{k=1}^n (1 - z_k \overline{w_k}) \langle E(z), E(w)\rangle = \langle E(z), e(0)\rangle + \langle e(0), E(w)\rangle - \langle e(0), e(0)\rangle, \tag{14}$$

for all $z \in K \cap B(0,r)$.

Fix $w \in K \cap B(0,r)$. If the uniqueness point property is applied to the analytic function of z defining each side of equation (14), we obtain equation (14) for all $z \in B(0,r)$; likewise the identity holds for all $w \in B(0,r)$. We can thus substitute the Taylor series of E into both sides of equation (14). Identifying the coefficients of $z^\alpha \overline{w}^\alpha$ that arise on each side of this equation yields

$$\|c_0\| = \|c_\alpha\| \quad \text{for all } \alpha \in N^n.$$

Therefore the multiradius of convergence for the Taylor series representation of the function E is $(1, \ldots, 1)$, and analytic continuation in z and w yields equation (14) for all z and w in $\mathbf{D}^n$.

Now define

$$F(z) = \langle E(z), e(0)\rangle - \overline{f(0)}, \qquad z \in \mathbf{D}^n.$$

F is clearly a complex-valued analytic function on $\mathbf{D}^n$ since E is a vector-valued analytic function on $\mathbf{D}^n$. Furthermore, for $z \in K$ and $w = 0$, equation (8) implies

$$f(z) = \langle e(z), e(0)\rangle - \overline{f(0)}, \qquad z \in K.$$

Since $e(z) = E(z)$ for $z \in K$, it follows that $f(z) = F(z)$ for $z \in K$.

Finally,

$$\begin{aligned} ReF(z) &= \langle E(z), e(0)\rangle + \langle e(0), E(z)\rangle - \overline{f(0)} - f(0) \\ &= \langle E(z), e(0)\rangle + \langle e(0), E(z)\rangle - \langle e(0), e(0)\rangle \\ &= \prod_{k=1}^{n} (1 - |z_k|^2)\langle E(z), E(z)\rangle \geq 0. \end{aligned}$$

The last equality follows from equation (14). This completes the proof of Theorem 2. □

A simple example contained in [10] shows that the uniqueness condition in Theorem 2 cannot be dropped from the hypothesis.

References

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, pp. 47–66, Oper. Theory Adv. Appl., 48, Birkhäuser Verlag, Basel, 1990.

[2] J. Agler and J. McCarthy, *Nevanlinna-Pick interpolation on the bidisk*, J. Reine Angew. Math **506** (1999), 191–204.

[3] L.A. Aĭzenberg and Š.A. Dautov, *Holomorphic functions of several complex variables with nonnegative real part*, Math. USSR Sb. **28** (1976), 301–312.

[4] J. Ball, W. Li, D. Timotin, and T. Trent, *A commutant lifting theorem on the polydisc*, preprint 1998.

[5] B. Cole and J. Wermer, *Pick interpolation, von Neumann inequalities, and hyperconvex sets*, preprint 1996.

[6] Sh.A. Dautov and G. Khudaiberganov, *On the Carathéodory-Fejér problem in multidimensional complex analysis*, Sb. Mat. Zh. **23** (1982), 58–64.

[7] A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier **16** (1966), 1–95.

[8] C. Foiaş and A.E. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44, Birkhäuser Verlag, Basel, 1990.

[9] J.W. Helton, *An application of several complex variables theory to linear model varying systems*, preprint 1998.

[10] A. Korányi and L. Pukánszky, *Holomorphic functions with positive real part on polycylinders*, Trans. Amer. Math. Soc. **108** (1963), 449–456.

[11] A. Pfister, *Über das Koeffizientenproblem der beschränkten Funktionen von zwei Veränderlichen*, Math Ann. **146** (1962), 249–262.

[12] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

[13] V.S. Vladimirov, *Holomorphic functions with positive imaginary part in the future tube IV*, Mat. Sb. (N.S.) **104 (146)** (1977), 341–370.

Fachbreich Mathematik, Universität des Saarlandes, Postfach 15 11 50, D-66041, Saarbrücken, GERMANY
E-mail address: eschmei@math.uni-sb.de

Department of Mathematics, California Polytechnical State University, San Luis Obispo, CA 93407
E-mail address: lpatton@calpoly.edu

Department of Mathematics, University of California, Santa Barbara, CA 93106
E-mail address: mputinar@math.ucsb.edu