Mathematical Research Letters **7**, 263–277 (2000)

NON-VANISHING OF THE CENTRAL DERIVATIVE OF CANONICAL HECKE L-FUNCTIONS

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Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D < -4$, $\mathcal{O}$ its ring of integers, and h its ideal class number. A Hecke character χ of K of conductor $\mathfrak{f}$ is a called "canonical" ([Ro1]) if

(1.1) $\chi(\bar{\mathfrak{a}}) = \overline{\chi(\mathfrak{a})}$ for each ideal $\mathfrak{a}$ relatively prime to $\mathfrak{f}$.

(1.2) $\chi(\alpha\mathcal{O}) = \pm\alpha$ for principal ideals $\alpha\mathcal{O}$ relatively prime to $\mathfrak{f}$.

(1.3) The conductor $\mathfrak{f}$ is divisible only by primes dividing D.

Every Hecke character of K satisfying (1.1) and (1.2) is actually a quadratic twist of a canonical Hecke character (see Section 2 for a precise description of these characters and which fields have them).

Let $L(s,\chi)$ denote the Hecke L-function of χ, and $\Lambda(s,\chi)$ its completion; $\Lambda(s,\chi)$ satisfies the functional equation $\Lambda(s,\chi) = W(\chi)\ \Lambda(2-s,\chi)$, where $W(\chi) = \pm 1$ is the root number. If χ is a canonical Hecke character with $W(\chi) = 1$, then the central value $\Lambda(1,\chi) \neq 0$ by a theorem of Montgomery and Rohrlich [MR]. Of course, it automatically vanishes when $W(\chi) = -1$ by the functional equation. The main result of this paper is

Theorem 1.1. *Let χ be a canonical Hecke character whose root number $W(\chi) = -1$. Then the central derivative $\Lambda'(1,\chi) \neq 0$.*

In Theorem 2.2 we also prove that $\Lambda'(1,\chi) \neq 0$ when χ is a small quadratic twist of a canonical character with $W(\chi) = -1$.

When $D = p$ is a prime, canonical Hecke characters are closely connected with the elliptic curves $A(p)$ extensively studied by Gross [Gr]. These curves are defined over $F = \mathbb{Q}(j(\frac{1+\sqrt{-p}}{2}))$, where j is the usual modular j-function, and have complex multiplication by $\mathcal{O}$. Combining Theorem 1.1 and the above result of [MR] with Gross-Zagier [GZ] and Kolyvagin-Logachev [KL], one has

Corollary 1.2. *Let $p > 3$ be a prime congruent to 3 modulo 4. Then*

(a) *The Mordell-Weil rank of $A(p)$ is*

$$\operatorname{rank}_{\mathbb{Z}} A(p)(F) = \begin{cases} h, & p \equiv 3 \pmod 8, \\ 0, & p \equiv 7 \pmod 8. \end{cases}$$

(b) *The Shafarevich-Tate group $Ш(A(p)/F)$ is finite.*

Received March 7, 2000.

In [Gr], Gross proved part (a) when $p \equiv 7 \pmod 8$ using a 2-descent.

In the next section we will outline the proof of Theorem 1.1 and an analog for quadratic twists (Theorem 2.2). Sections 3, 4, and 5 are devoted to analytic estimates used in the proofs of the theorems. We conclude in Section 6 with the proof of Corollary 1.2 and other arithmetic applications.

2. Notation and strategy

We first recall some facts about canonical Hecke characters from [Ro2]. They exist if and only if $D \equiv 3 \pmod 4$ or is a multiple of 8. Multiplying a canonical character by an ideal class character always yields another canonical character. This operation preserves the root number and defines natural families of canonical Hecke characters. When $D \equiv 3 \pmod 4$, there is exactly one family and it has root number $\left(\frac{2}{D}\right)$; when D is a multiple of 8, there are two families — one has root number 1 and the other has root number -1.

To avoid confusion, we will sometimes write χ_{can} for a canonical Hecke character of K. In this paper we consider Hecke characters χ of K satisfying conditions (1.1) and (1.2), which are always of the form

$$\chi_{D,d} = \chi_{\text{can}} \cdot (\epsilon_d \circ N_{K/\mathbb{Q}}). \tag{2.1}$$

Here d is a fundamental discriminant and $\epsilon_d = \left(\frac{d}{}\right)$ is the quadratic dirichlet character with conductor d, prime to D. The root number $W(\chi)$ is explicitly computed in [Ro2]. In particular, when D is odd

$$W(\chi_{D,d}) = \left(\frac{2}{D}\right) \operatorname{sign}(d). \tag{2.2}$$

From now on we will assume that $W(\chi) = -1$. Set

$$B = \sqrt{DN\mathfrak{f}} = \begin{cases} D|d|, & D \text{ odd} \\ 2D|d|, & 8 \mid D. \end{cases} \tag{2.3}$$

The Hecke L-function is defined as

$$\begin{aligned} L(s,\chi) &= \sum_{\mathfrak{a} \text{ integral}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s} \\ &= \sum_{\text{ideal classes } C} L(s,\chi,C), \end{aligned}$$

where $L(s,\chi,C)$ is the partial L-series summed over integral ideals in C. Their completed L-functions are defined by

$$\Lambda(s,\chi) = \left(\frac{B}{2\pi}\right)^s \Gamma(s) L(s,\chi)$$

and

$$\Lambda(s,\chi,C) = \left(\frac{B}{2\pi}\right)^s \Gamma(s) L(s,\chi,C).$$

Lemma 2.1. *When $W(\chi) = -1$, $\Lambda'(1,\chi) = 0$ if and only if $\Lambda'(1,\chi,C) = 0$ for each ideal class C of K.*

Proof. Associated to χ is a cuspidal new form f of weight 2 and level B^2 such that $L(s,f) = L(s,\chi)$. So Corollary 2 of [GZ] implies that $\Lambda'(1,\chi) = 0$ if and only if $\Lambda'(1,\chi^\sigma) = 0$ for every $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. On the other hand, by Theorem 1 of [Ro3],

$$\{\chi^\sigma : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)\} = \{\chi\phi : \phi \text{ is an ideal class character of } K\},$$

and $L(s,\chi) = L(s,\bar{\chi})$ by (1.1). Thus $\Lambda'(1,\chi) = 0$ if and only if $\Lambda'(1,\chi\phi) = 0$ for all ideal class characters ϕ of K. The ideal class characters are linearly independent and

$$L(s,\chi\phi) = \sum_C \phi(C) L(s,\chi,C),$$

so the lemma follows. □

To prove $\Lambda'(1,\chi) \neq 0$, it now suffices to show $\Lambda'(1,\chi,c_1) \neq 0$ for the trivial class c_1 (i.e., the class of principal ideals). Since the root number $W(\chi) = -1$, we have the functional equation

$$\Lambda(s,\chi,c_1) = -\Lambda(2-s,\chi,c_1). \tag{2.4}$$

By Cauchy's theorem

$$\Lambda'(1,\chi,c_1) = \frac{1}{2\pi i}\left(\int_{2-i\infty}^{2+i\infty} \Lambda(s,\chi,c_1)\frac{ds}{(s-1)^2} - \int_{-i\infty}^{+i\infty} \Lambda(s,\chi,c_1)\frac{ds}{(s-1)^2}\right).$$

Applying (2.4) we arrive at the formula

$$\frac{1}{2}\Lambda'(1,\chi,c_1) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Lambda(s,\chi,c_1)\frac{ds}{(s-1)^2}. \tag{2.5}$$

It is clear from property (1.2) of χ that there is a quadratic character ϵ of $(\mathcal{O}/\mathfrak{f})^*$ such that

$$\chi(\alpha\mathcal{O}) = \epsilon(\alpha)\alpha. \tag{2.6}$$

We can express $L(s,\chi,c_1)$ as a sum over real and complex ideals:

$$L(s,\chi,c_1) = \sum_{n=1}^{\infty} \epsilon(n) n^{1-2s} + \sum_{n=1}^{\infty} a_n n^{-s}, \tag{2.7}$$

where

$$a_n = \sum_{\substack{N\mathfrak{a}=n,\ \mathfrak{a}\neq\bar{\mathfrak{a}} \\ \text{principal,} \\ \text{integral}}} \chi(\mathfrak{a}) = \sum_{\substack{u^2+Dv^2=4n \\ u,v>0}} \epsilon\left(\frac{u+\sqrt{-D}v}{2}\right) u. \tag{2.8}$$

Let

$$f(x) = \frac{\Gamma(0,x)}{x} = \frac{1}{x}\int_x^\infty e^{-t}\frac{dt}{t} \tag{2.9}$$

be the inverse Mellin transform of $\frac{\Gamma(s)}{(s-1)^2}$. Indeed

$$\begin{aligned}\int_0^\infty f(x)x^s\frac{dx}{x} &= \textstyle\int_0^\infty\int_x^\infty x^{s-1}e^{-t}\frac{dt}{t}\frac{dx}{x}\\ &= \textstyle\int_0^\infty\int_0^t \frac{dx}{x}x^{s-1}e^{-t}\frac{dt}{t}\\ &= \textstyle\frac{1}{s-1}\int_0^\infty t^{s-1}e^{-t}\frac{dt}{t} = \displaystyle\frac{\Gamma(s-1)}{(s-1)} = \frac{\Gamma(s)}{(s-1)^2},\end{aligned}$$

so

$$f(y) = \frac{1}{2\pi i}\int_{Re(s)=2}\frac{\Gamma(s)}{(s-1)^2}y^{-s}ds. \tag{2.10}$$

Combining (2.5), (2.7), and (2.10) we obtain

$$\frac{1}{2}\Lambda'(1,\chi,c_1) = \overbrace{\sum_{n=1}^\infty \epsilon(n)n\cdot f(2\pi n^2/B)}^{R} + \overbrace{\sum_{n=1}^\infty a_n f\left(\frac{2\pi n}{B}\right)}^{C}. \tag{2.11}$$

Formula (2.11) is essentially due to Rohrlich ([Ro4]), except that he expressed R in terms of dirichlet L-functions.

Examples: $D = 8$ and 11.

We will now illustrate (2.11) with the first two discriminants which occur. Since both $\mathbb{Q}(\sqrt{-8})$ and $\mathbb{Q}(\sqrt{-11})$ have class number 1,

$$\Lambda'(1,\chi,c_1) = \Lambda'(1,\chi).$$

In order to compute it using (2.11), we must first describe the character $\epsilon : (\mathcal{O}/\mathfrak{f})^* \to \{\pm 1\}$. When $D = 8$, $\mathfrak{f} = 2\sqrt{-D}\mathcal{O} = \mathbb{Z}8 \oplus \mathbb{Z}\sqrt{-32}$, and $(\mathcal{O}/\mathfrak{f})^*$ is generated by $(\mathbb{Z}/8)^*$ and $1+\sqrt{-2}$. The character $\epsilon(n)$ must restrict to $\left(\frac{-8}{n}\right)$ for $n \in (\mathbb{Z}/8)^*$, and is thus determined by its value on $1+\sqrt{-2}$. In fact, $W(\chi) = \epsilon(1+\sqrt{-2})$, so in our case the values of ϵ on the relatively-prime residue classes are given in the following chart:

$\epsilon(u+v\sqrt{-2})$	u	1	3	5	7
v					
0		1	1	-1	-1
1		-1	-1	1	1
2		-1	-1	1	1
3		-1	-1	1	1

When $D = 11$, $\epsilon\left(\frac{u+\sqrt{-11}v}{2}\right) = \left(\frac{2u}{11}\right)$. We now compute $L'(1,\chi)$ for these canonical characters, and check (2.11) by comparing the known values of $L'(1,E)$ for the associated elliptic curves E.

	$D=8$	$D=11$
Term R with $n^2 \leq 50$	1.82582357875147	0.81497705252487
Term C with $n \leq 50$	-0.28596530872740	-0.0600975766040368
$L'(1,\chi) = \left(\frac{2\pi}{B}\right)\Gamma(1)\Lambda'(1,\chi)$ $= \frac{4\pi}{B}(R+C)$	1.209401857169272	0.862372296690396
Associated curve E	$y^2 = x^3+4x^2+2x$ ([Cr], curve 256A)	$y^2+y =$ $x^3-x^2-7x+10$ ([Cr], curve 121B)
$L'(1,E)$ from [Cr]	1.2094018572	.8623722967

Proof of Theorem 1.1. By Lemma 2.1 and (2.11), it suffices to prove $R > |C|$. In the next section, we will prove that R is bounded below by

$$\begin{aligned} R &> \sum_{n=1}^{\infty} \lambda(n) n \cdot f\left(\frac{2\pi n^2}{B}\right) \\ &> .5235B - .8458B^{3/4} - .3951B^{1/2}. \end{aligned} \tag{2.12}$$

Here $\lambda(n)$ is Liouville's function – the completely multiplicative function which is -1 at each prime.

In Section 4 we consider the special case $d=1$, and bound term C by Proposition 4.1:

$$|C| < \begin{cases} .2369D, & D \text{ even} \\ .0269D, & D \text{ odd.} \end{cases} \tag{2.13}$$

Having collected these estimates, proving $R > |C|$ is a simple calculation. Indeed, if $D \geq 24$ is even, then $B = 2D$, and

$$R > .5235(2D) - .8458(2D)(48)^{-1/4} - .3951(2D)(48)^{-1/2} = .2902D,$$

so

$$R > .2369D > |C|.$$

If $D \geq 19$ is odd then

$$R > .5235D - .8458 \cdot D \cdot 19^{-1/4} - .3951 \cdot D \cdot 19^{-1/2} > .0277D,$$

$$R > .0269D > |C|.$$

There are only two values of D not covered by this argument: $D=8$ and 11, which were dealt with in the examples. □

Quadratic twists.

To prove non-vanishing for quadratic twists of canonical characters, the bound (2.13) is not useful. In Section 5 we apply Rohrlich's method to obtain the following bound on C for $\chi_{D,d}$ (Proposition 5.1):[1]

$$|C| \ll D^{15/16+\delta}|d|^{51/16+\delta}, \tag{2.14}$$

where $\delta > 0$ is arbitrary and the implied constant depends only on it. Combining (2.11), (2.12), and (2.14) we conclude

Theorem 2.2. *For any fixed $\delta > 0$,*

$$\Lambda'(1, \chi_{D,d}) \neq 0$$

for $|d| \ll D^{1/35-\delta}$ and $W(\chi_{D,d}) = -1$.

Remarks. When the root number $W(\chi) = 1$, similar non-vanishing results for twists were obtained in [Ro1], [RVY], and [**?**] for the central L-value.

For canonical Hecke characters, Rohrlich ([Ro4]) computed R as a contour integral of dirichlet L-functions. By shifting contours, R can be expressed as the sum of a residue and a remainder integral. He showed the residue is of size $\gg D$, and used Burgess' sub-convexity estimate to bound the remainder integral by $\ll D^\alpha$, $\alpha < 1$. Also, he used the method in Section 5 to show $C \ll D^\alpha$. The power of the main term is larger than that of the other two terms, and positivity follows for large D. However, the implied constants one gets for these estimates are quite unfavorable. In our proof of Theorem 1.1 we sacrifice the gain in the powers of D for a tie — in favor of better constants.

3. The main term R

The purpose of this section is to prove (2.12). We will show term R is large and positive by eventually bounding it from below by the following sum.

Proposition 3.1.

$$\sum_{n=1}^{\infty} \lambda(n) n \cdot f\left(\frac{2\pi n^2}{x}\right)$$

is always positive for $x > 0$ and in fact

$$\sum_{n=1}^{\infty} \lambda(n) n \cdot f\left(\frac{2\pi n^2}{x}\right) > .5235x - .8458x^{3/4} - .3951x^{1/2} \tag{3.1}$$

for $x > 1$.

Proof. Using (2.10) and the identity

$$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)},$$

[1] The notation $A \ll B$ means $A = O(B)$, i.e., there exists a positive constant C such that $|A| \leq CB$.

we write

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda(n) n \cdot f\left(\frac{2\pi n^2}{x}\right) &= \frac{1}{2\pi i}\int_{\mathrm{Re}\ s=2} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)} ds \\ &= \frac{1}{2\pi i}\int_{\gamma} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)} ds \\ &\quad + \mathrm{Res}_{s=1} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)} \\ &\quad + \mathrm{Res}_{s=3/4} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)}.\end{aligned}$$

Here $\gamma = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ is the contour consisting of the union of the following five line segments:

$$\begin{gathered} C_1 \text{ from } 1 - i\infty \text{ to } 1 - 7i, \\ C_2 \text{ from } 1 - 7i \text{ to } \frac{1}{2} - 7i, \\ C_3 \text{ from } \frac{1}{2} - 7i \text{ to } \frac{1}{2} + 7i, \\ C_4 \text{ from } \frac{1}{2} + 7i \text{ to } 1 + 7i, \\ C_5 \text{ from } 1 + 7i \text{ to } 1 + i\infty \end{gathered}$$

(7 is chosen because the first critical zeroes of $\zeta(s)$ are approximately $\frac{1}{2} \pm 14.13472i$). The residue at $s = 1$ is $\frac{\pi x}{6} \approx .523599x$ and the residue at $s = 3/4$ is

$$\mathrm{Res}_{s=3/4} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)} = \frac{2^{5/4} x^{3/4} \Gamma(3/4)}{\pi^{3/4}\zeta(1/2)} \approx -.845767 x^{3/4}.$$

One can easily estimate the integrals over γ as follows.[2] First,

$$\begin{aligned}\left|\int_{C_1} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)} ds\right| &= \left|\int_{C_5} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)} ds\right| \\ &\le \frac{x}{2\pi}\int_{t=7}^{\infty} \frac{|\Gamma(1+it)|}{t^2}\frac{|\zeta(2+4it)|}{|\zeta(1+2it)|} dt \\ &\le x(5 \cdot 10^{-7}).\end{aligned}$$

[2]All computations were done using Mathematica v4.0 on an Intel Celeron processor under Windows 98.

Next

$$\begin{aligned}\left|\int_{C_2}\left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\,\frac{\zeta(4s-2)}{\zeta(2s-1)}ds\right| \\ &= \left|\int_{C_4}\left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)}ds\right| \\ &\leq x\int_{\sigma=1/2}^{1}(2\pi)^{-\sigma}\frac{|\Gamma(\sigma+7i)|}{(\sigma-1)^2+49}\frac{|\zeta(4\sigma-2+28i)|}{|\zeta(2\sigma-1+14i)|}d\sigma \\ &\leq x(2\cdot 10^{-6}).\end{aligned}$$

Finally,

$$\begin{aligned}\left|\int_{C_3}\left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{(s-1)^2}\frac{\zeta(4s-2)}{\zeta(2s-1)}ds\right| \\ &\leq \sqrt{\frac{x}{2\pi}}\int_{t=-7}^{7}\frac{|\Gamma(1/2+it)|}{1/4+t^2}\frac{|\zeta(4it)|}{|\zeta(2it)|}dt \\ &\leq 2.48218\sqrt{x}.\end{aligned}$$

Combining these estimates proves (3.1). For $x \geq 20$,

$$\begin{aligned}.5235x - .8458x^{3/4} - .3951x^{1/2} \\ &\geq (.5235 - .8458\cdot 20^{-1/4} - .3951\cdot 20^{-1/2})x \\ &\geq .0351x > 0.\end{aligned}$$

The positivity for $x < 20$ is handled by the next lemma. □

Lemma 3.2. *For* $0 < x < 20$*, one has*

$$f\left(\frac{2\pi}{x}\right) > \sum_{n=2}^{\infty} n\cdot f\left(\frac{2\pi n^2}{x}\right).$$

Proof. It is easy to see that for any $0 < a < 1$ and $t > \frac{a}{1-a}$

$$f(t) > ae^{-t}/t^2.$$

Take $a = \frac{\pi}{10+\pi} > .23$. Then for $0 < x < 20$, one has

$$f\left(\frac{2\pi}{x}\right) > .23\frac{x^2}{4\pi^2}e^{-\frac{2\pi}{x}}.$$

One the other hand, clearly, $f(x) < e^{-x}/x^2$, and so

$$\begin{aligned}\sum_{n=2}^{\infty} n\cdot f\left(\frac{2\pi n^2}{x}\right) &\leq \sum_{n=2}^{\infty} n\frac{x^2}{4\pi^2 n^4}e^{-2\pi n^2/x} \\ &\leq \frac{x^2}{4\pi^2}\sum_{n=2}^{\infty} n^{-3}e^{-2\pi n^2/x}.\end{aligned} \tag{3.2}$$

Since $n^2 \geq n+2$ for $n \geq 2$, this is

$$\leq \frac{x^2}{4\pi^2}\frac{e^{-8\pi/x}}{8}\sum_{n=0}^{\infty} e^{-2\pi n/x} = \frac{x^2 e^{-2\pi/x}}{4\pi^2}\frac{1}{8}\frac{e^{-6\pi/x}}{(1-e^{-2\pi/x})}. \tag{3.3}$$

Since $\frac{1}{8}\frac{e^{-6\pi/x}}{(1-e^{-2\pi/x})}$ is clearly increasing, it is thus bounded above in $0 < x < 20$ by its value $\approx .181$ at $x = 20$. Therefore (3.3) is bounded above by

$$.19\frac{x^2}{4\pi^2}e^{-2\pi/x} < f(2\pi/x).$$

□

Proposition 3.3.

(1) *If m is any completely multiplicative function with values $-1, 0$, or 1, then*

$$\sum_{n=1}^{\infty} m(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) > 0\ ,\ \ x > 0. \tag{3.4}$$

(2) *If m_1 and m_2 are two distinct such functions with $m_1(p) \geq m_2(p)$ for every prime p, then*

$$\sum_{n=1}^{\infty} m_1(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) > \sum_{n=1}^{\infty} m_2(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) \tag{3.5}$$

for all $x > 0$.

Proof. We first assume that the functions m, m_1, and m_2 differ from λ at only finitely many primes. Therefore, by Proposition 3.1 and induction, it suffices to prove (3.5) under the following conditions:

(a) m_2 satisfies (3.4) for all $x > 0$.

(b) m_1 and m_2 differ at exactly one prime, say p, and $m_1(p) = m_2(p) + 1$.

Under assumption (b), the difference is

$$\sum_{n=1}^{\infty} m_1(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) - \sum_{n=1}^{\infty} m_2(n)n \cdot f\left(\frac{2\pi n^2}{x}\right)$$
$$= \sum_{p|n}(m_1(n) - m_2(n))n \cdot f\left(\frac{2\pi n^2}{x}\right).$$

When $m_2(p) = -1$, $m_1(p) = 0$, the difference is then

$$-\sum_{p|n} m_2(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) = p\sum_{n=1}^{\infty} m_2(n)n \cdot f\left(\frac{2\pi n^2}{x/p^2}\right) > 0$$

by assumption (a). When $m_2(p) = 0$, $m_1(p) = 1$ and $m_1(p^k n) = m_2(n)$ for $p \nmid n$. So the difference is

$$\sum_{p|n} m_1(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) = \sum_{k=1}^{\infty} p^k \sum_{n=1}^{\infty} m_2(n)n \cdot f\left(\frac{2\pi n^2}{x/p^{2k}}\right) > 0$$

by assumption (a) again.

In the general case, define $m^N(n)$ to be the completely multiplicative function derived from m by

$$m^N(p) = \begin{cases} m(p), & p \le N, \\ \lambda(p), & p > N. \end{cases} \tag{3.6}$$

Then m^N differs from λ at only a finite number of primes, and thus

$$\sum_{n=1}^{\infty} m^N(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) \ge \sum_{n=1}^{\infty} \lambda(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) > 0.$$

Taking the limit as $N \to \infty$,

$$\sum_{n=1}^{\infty} m(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) \ge \sum_{n=1}^{\infty} \lambda(n)n \cdot f\left(\frac{2\pi n^2}{x}\right) > 0.$$

This completes the proof of part (1); part (2) can be handled similarly. □

Corollary 3.4. *One has*

$$R \ge \sum_{n=1}^{\infty} \lambda(n)n \cdot f\left(\frac{2\pi n^2}{B}\right).$$

Combining Proposition 3.1 with Corollary 3.4, one obtains (2.12).

4. The trivial bound on remainder term C

In this section, we will only treat canonical characters, and prove (2.13):

Proposition 4.1. *When $d = 1$ and $D \ge 7$, term C is bounded by*

$$|C| < \begin{cases} .0269D, & D \text{ odd}, \\ .2369D, & D \text{ even}. \end{cases} \tag{4.1}$$

Proof. We first assume D is odd, so $B = D$. From (2.8) we can bound C term-wise, without appealing to cancellation from the character. To wit,

$$|C| \le \sum_{\substack{u,v>0 \\ u \equiv v \pmod 2}} uf\left(\frac{\pi}{2}(v^2 + u^2/D)\right).$$

Since $f(x) < e^{-x}/x^2$,

$$\begin{aligned} |C| &< \sum_{\substack{u,v>0 \\ v \equiv u \pmod 2}} ue^{-\frac{\pi u^2}{2D}} \frac{e^{-\pi v^2/2}}{\left(\frac{\pi}{2}(v^2 + u^2/D)\right)^2} \\ &< \sum_{u=1}^{\infty} ue^{-\frac{\pi u^2}{2D}} \frac{4}{\pi^2} \sum_{\substack{v=1 \\ v \equiv u \pmod 2}}^{\infty} v^{-4} e^{-\pi v^2/2}. \end{aligned}$$

The inside sum is bounded by

$$\sum_{v=1,\ odd}^{\infty} v^{-4}e^{-\pi v^2/2} \approx .20788,$$

so

$$|C| \leq .0843 \sum_{u=1}^{\infty} ue^{-\frac{\pi u^2}{2D}}.$$

If D is even, then $B = 2D$,and the same argument shows that

$$\begin{aligned}|C| &< 2\left(\frac{16}{\pi^2}\sum_{v=1}^{\infty} v^{-4}e^{-\pi v^2/4}\right)\left(\sum_{u=1}^{\infty} ue^{-\pi u^2/D}\right)\\ &\leq 1.488\sum_{u=1}^{\infty} ue^{-\pi u^2/D}.\end{aligned}$$

The proof of (4.1) now follows from bound in the Lemma below. □

Lemma 4.2. *For $a \geq 1$,*

$$\sum_{n=1}^{\infty} ne^{-n^2/a} < a/2. \tag{4.2}$$

This conclusion actually holds for all $a > 0$.

Proof. Using the Poisson summation formula applied to $|n|e^{-n^2/a}$,

$$\sum_{n=1}^{\infty} ne^{-n^2/a} = \int_0^{\infty} ne^{-n^2/a}dn + 2\sum_{r=1}^{\infty}\int_0^{\infty} ne^{-n^2/a}\cos(2\pi rn)dn.$$

The first integral is $a/2$ and the others are actually negative. This is because

$$\int_0^{\infty} ne^{-n^2/a}\cos(2\pi rn)dn = \frac{a}{2}\left(1 - e^{-a\pi^2r^2}2\pi r\sqrt{a}\int_0^{\pi r\sqrt{a}} e^{t^2}dt\right)$$

(cf. [GR], 17.13.27) and

$$\int_0^{\pi r\sqrt{a}} e^{t^2}dt > 1 + \int_1^{a\pi^2r^2} \frac{e^t}{2\sqrt{t}}dt = 1 + \frac{1}{2\pi r\sqrt{a}}\left(e^{a\pi^2r^2} - e\right) > \frac{e^{a\pi^2r^2}}{2\pi r\sqrt{a}}.$$

□

5. Rohrlich's bound on remainder term C

We now return to the general case of a canonical character twisted by $\epsilon_d = \left(\frac{d}{\cdot}\right)$. The method here is adapted from [Ro1] and [Ro4].

Proposition 5.1. *For any $\delta > 0$, term C is bounded by*

$$|C| \ll D^{15/16+\delta}|d|^{51/16+\delta},$$

where the implied constant depends only on δ.

Proof. Set $A(t) = \sum_{n<t} a_n$. Integration by parts gives

$$C = \int_{D/4}^{\infty} f\left(\frac{2\pi t}{B}\right)\frac{d}{dt}A(t)dt = -\int_{D/4}^{\infty} A(t)\frac{d}{dt}f\left(\frac{2\pi t}{B}\right)dt, \tag{5.1}$$

because there are no complex ideals of norm $< D/4$. By [Ro1], p. 553(27), $A(t)$ is bounded above by

$$|A(t)| \ll t^{5/4}D^{-5/16+\delta}|d|^{19/16+\delta}$$

for $D > 8$ and $t > 0$ (the implied constant again depends only on δ). Along with the inequalities

$$0 < -\frac{d}{dt}f\left(\frac{2\pi t}{B}\right) < \frac{B}{2\pi t^2}e^{-2\pi t/B}\left(1+\frac{B}{2\pi t}\right),$$

(5.1) implies

$$\begin{aligned}|C| &\ll D^{-5/16+\delta}|d|^{19/16+\delta}\int_{D/4}^{\infty}\left[t^{5/4}\frac{B}{2\pi t^2}e^{-2\pi t/B}\left(\frac{4}{3}+\frac{B}{2\pi t}\right)\right]dt \\ &\ll D^{-5/16+\delta}|d|^{19/16+\delta}\int_{D/4}^{\infty}\frac{d}{dt}\left[-\frac{B^2}{2\pi^2}t^{-3/4}e^{-2\pi t/B}\right]dt \\ &\ll D^{-17/16+\delta}|d|^{19/16+\delta}B^2.\end{aligned}$$

Since either $B = D|d|$ or $2D|d|$, this completes the proof. □

6. Arithmetic applications

Having completed their proofs, we will now give some arithmetic applications of Theorems 1.1 and 2.2, including Corollary 1.2.

Let j be the j-invariant of a fixed isomorphism class of elliptic curves with complex multiplication (CM) by $\mathcal{O}$. Then $H = K(j)$ is the Hilbert class field of K. We can extend any Hecke character χ of K to one on H by

$$\psi = \chi \circ N_{H/K}.$$

When χ satisfies (1.1) and (1.2),

$$\psi(\mathfrak{A}^\sigma) = \psi(\mathfrak{A})^\sigma,\ \sigma \in \mathrm{Gal}(H/\mathbb{Q})$$

for every ideal $\mathfrak{A}$ of H relatively prime to the conductor of ψ. By Theorem 9.1.3 and Lemma 11.1.1 of [Gr], there is a unique elliptic $\mathbb{Q}$-curve A over H with

$$j(A) = j \text{ and } L(s, A/H) = L(s,\psi)L(s,\bar{\psi}).$$

(Here we recall that a "$\mathbb{Q}$-curve" is an elliptic curve over a number field which is isogenous to all of its Galois conjugates.) Furthermore, A descends to two isogenous elliptic curves over the subfield $F = \mathbb{Q}(j)$ ([Gr], Theorem 10.2.1). By abuse of notation we will also refer to these curves as A. Let $B = \mathrm{Res}_{F/\mathbb{Q}} A$ be the abelian variety over $\mathbb{Q}$ obtained from A by restriction of scalars. When $D = p$ is prime, Gross proved ([Gr], Theorem 15.2.5) that $T = \mathrm{End}_K B \otimes \mathbb{Q}$ is a CM number field of degree $2h$; thus B is also a CM abelian variety. This result actually extends to composite D via a different argument:

Lemma 6.1. (a) *Let T be the subfield of $\mathbb{C}$ generated by $\chi(\mathfrak{a})$, where $\mathfrak{a}$ runs over all ideals of K prime to χ's conductor. Then T is a CM number field of degree $2h$, and $\Phi = \{\sigma : T \to \mathbb{C} \mid \sigma \text{ trivial on } K\}$ is a CM type of T.*

(b) *B is a CM abelian variety of type (T, Φ).*

Proof. For each embedding $\sigma : T \to \mathbb{C}$ fixing K, $\sigma \circ \chi$ is another canonical Hecke character of K, and thus it is of the form $\chi\phi$, where ϕ is an ideal class character of K. By Theorem 1 of [Ro3], $\sigma \mapsto \phi$ actually gives a one-to-one correspondence between the complex embeddings of T into $\mathbb{C}$ fixing K, and the ideal class characters of K. Thus $[T : K] = h$ and $[T : \mathbb{Q}] = 2h$. It is a general fact that T is a CM number field; in this case it can easily be verified using property (1.1) of χ.

By [Sh], Theorem 10, there is a CM abelian variety $B'/\mathbb{Q}$ of type (T, Φ) associated to χ, and it is unique up to isogeny. In particular

$$L(s, B') = \prod_{\sigma : T^+ \longrightarrow \mathbb{C}} L(s, \chi^\sigma) = \prod_{\phi} L(s, \chi\phi), \tag{6.1}$$

where T^+ is the maximal totally-real subfield of T. On the other hand,

$$L(s, B) = L(s, A/F) = L(s, \psi) = \prod_{\phi} L(s, \chi\phi).$$

This shows $L(s, B) = L(s, B')$, so a theorem of Faltings [Fa] guarantees B and B' are isogenous, proving (b). □

Lemma 6.2. *Let χ be a Hecke character of K of the form* (2.1). *Let A be an associated $\mathbb{Q}$-curve over $F = \mathbb{Q}(j)$ with j-invariant j, and let $B = \operatorname{Res}_{F/\mathbb{Q}} A$. If* $\operatorname{ord}_{s=1} L(s, \chi) \le 1$ *then*

(a) *The Mordell-Weil ranks of A and B are given by*

$$\operatorname{rank}_{\mathbb{Z}} A(F) = \operatorname{rank}_{\mathbb{Z}} B(\mathbb{Q}) = h \cdot \operatorname{ord}_{s=1} L(s, \chi).$$

(b) *The Shafarevich-Tate groups $Ш(A/F)$ and $Ш(B/\mathbb{Q})$ are finite.*

Proof. Since the Mordell-Weil and Shafarevich-Tate groups of A over F are identical to those of B over $\mathbb{Q}$, it is sufficient to prove the Lemma for B. Let f be the normalized weight 2 new-form associated to χ as in the proof of Lemma 2.1. The field generated by f's Fourier coefficients is generated by $\chi(\mathfrak{a}) + \overline{\chi(\mathfrak{a})}$, and is thus T^+. Equation (6.1) implies

$$L(s, B) = \prod_{\sigma : T^+ \longrightarrow \mathbb{C}} L(s, f^\sigma).$$

Now the Lemma follows from a result of Kolyvagin and Logachev ([KL]). □

Combining Lemma 6.2 with the non-vanishing theorems above (Theorems 1.1 and 2.2) and in [MR], one gets the following two corollaries.

Corollary 6.3. *Let $\chi = \chi_{\text{can}}$ be a canonical Hecke character of K, and let A and $B = \operatorname{Res}_{F/\mathbb{Q}} A$ respectively be associated $\mathbb{Q}$-curves and CM abelian varieties as above. Then*

(a) *The Mordell-Weil ranks of A and B are given in terms of the root number $W(\chi)$ by*

$$\operatorname{rank}_{\mathbb{Z}} A(F) = \operatorname{rank}_{\mathbb{Z}} B(\mathbb{Q}) = \begin{cases} h, & W(\chi) = -1 \\ 0, & W(\chi) = 1. \end{cases}$$

In particular, when D is odd, these ranks are h or zero depending on whether $D \equiv 3$ or $7 \mod 8$.

(b) *The Shafarevich-Tate groups $\text{Ш}(A/F)$ and $\text{Ш}(B/\mathbb{Q})$ are finite.*

Proof of Corollary 1.2. Take $D = p$, $j = j\left(\frac{1+\sqrt{-p}}{2}\right)$, $A = A(p)$, and apply Corollary 6.3. □

Corollary 6.4. *Let $\chi = \chi_{D,d}$ be a Hecke character of K of the form (2.1). Let A and $B = \operatorname{Res}_{F/\mathbb{Q}} A$ be as above, and fix any $\delta > 0$. If $|d| \ll D^{1/35-\delta}$ (the implied constant depending on δ) and $W(\chi_{D,d}) = -1$, then*

(a) *The Mordell-Weil ranks of A and B are*

$$\operatorname{rank}_{\mathbb{Z}} A(F) = \operatorname{rank}_{\mathbb{Z}} B(\mathbb{Q}) = h.$$

(b) *The Shafarevich-Tate groups $\text{Ш}(A/F)$ and $\text{Ш}(B/\mathbb{Q})$ are finite.*

Finally, we wish to point out that when D is prime, all $\mathbb{Q}$-curves over F are associated to Hecke characters of the form (2.1), though this is not true for every composite D. See [Na] for a more-precise description.

Acknowledgements

For very large D, Theorem 1.1 was originally obtained by D. Rohrlich [Ro4]. We are indebted to him for sharing his method with us, as well as for allowing us to mention several of his results here. Also, we thank him for his inspiration and for suggesting this problem to us. We are grateful to M. Baker, B. Conrad, N. Elkies, B. Gross, K. Rubin, Z. Rudnick, D. Zagier, and S. Zhang for discussions, and to Harvard University for their hospitality. S.M. was partially supported by an NSF post-doctoral fellowship as well as a Yale Hellman fellowship. T.Y. was partially supported by an AMS Centennial fellowship and NSF grant DMS-9700777.

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