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THE WEINSTEIN CONJECTURE IN THE UNIRULED MANIFOLDS

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Abstract. In this note we prove the Weinstein conjecture for a class of symplectic manifolds including the uniruled manifolds based on Liu-Tian's result.

In 1978, A. Weinstein proposed his famous conjecture that *every hypersurface of contact type in the symplectic manifolds carries a closed characteristic* [We]. Many results were obtained (cf. [C][FHV][H][HV1][HV2][LiuT][Lu1][Lu2][V1][V2][V3]) after C.Viterbo first proved it in $(\mathbb{R}^{2n}, \omega_0)$ in 1986 [V1]. Not long ago Gang Liu and Gang Tian established a deep relation between this conjecture and the Gromov-Witten invariants and got several general results as corollaries [LiuT].

Assume S to be a hypersurface of contact type in a closed connected symplectic manifold (V, ω) separating V in the sense of [LiuT], i.e., there exist submanifolds V_+ and V_- with common boundary S such that $V = V_+ \cup V_-$ and $S = V_+ \cap V_-$, then the following result holds.

Theorem 1 ([LiuT])**.** *If there exist $A \in H_2(V;\mathbb{Z})$ and $\alpha_+, \alpha_- \in H_*(V;\mathbb{Q})$, such that*

(i) *$supp(\alpha_+) \hookrightarrow int(V_+)$ and $supp(\alpha_-) \hookrightarrow int(V_-)$,*
(ii) *the Gromov-Witten invariant $\Psi_{A,g,m+2}(C; \alpha_-, \alpha_+, \beta_1, \cdots, \beta_m) \neq 0$ for some $\beta_1, \cdots, \beta_m \in H_*(V;\mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q})$,*

then S carries at least one closed characteristic.

Recall that for a given $A \in H_2(V;\mathbb{Z})$ the Gromov-Witten invariant of genus g and with m+2 marked points is a homomorphism

$$\Psi_{A,g,m+2} : H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q}) \times H_*(V;\mathbb{Q})^{m+2} \to \mathbb{Q},$$

(see [FO][LiT][R][Si]). Though one so far does not yet know whether the GW invariants defined in the four papers agree or not, we believe that they have the same vanishing or nonvanishing properties, i.e., for any given classes $C \in H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q})$ and $\beta_1, \cdots, \beta_{m+2} \in H_*(V;\mathbb{Q})$ one of this four versions vanishes on $(C; \beta_1, \cdots, \beta_{m+2})$ if and only if any other three ones vanish on them. In addition, the version of [R] is actually a homomorphism from $H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{R}) \times$

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$H_*(V;\mathbb{R})^{m+2}$ to $\mathbb{R}$. However, using the facts that $H_*(M;\mathbb{Q})$ is dense $H_*(M;\mathbb{R})$ for $M = V, \overline{\mathcal{M}}_{g,k}$ and $\Psi_{A,g,m+2}$ is always a homomorphism one can naturally extend the other three versions to the homomorphisms from $H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{R}) \times H_*(V;\mathbb{R})^{m+2}$ to $\mathbb{R}$. Below we always mean the extended versions when they can not clearly explained in the original versions. Our main result is

Theorem 2. *For a connected closed symplectic manifold (V,ω), if there exist $A \in H_2(V;\mathbb{Z})$, $C \in H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q})$, and $\beta_1, \cdots, \beta_{m+1} \in H_*(V;\mathbb{Q})$ such that*

$$\Psi_{A,g,m+2}(C;[pt],\beta_1,\cdots,\beta_{m+1}) \neq 0$$

for $(g,m) \neq (0,0)$ and the single point class $[pt]$, then every hypersurface of contact type S in the symplectic manifold V separating V carries a closed characteristic. Specially, if $g=0$ we can also guarantee that S carries one contractible (in V) closed characteristic.

In case $g=0$ it is not difficult to prove that Proposition 2.5(5) and Proposition 2.6 in [RT] still hold for any closed symplectic manifold (V,ω) with the method of [R]. That is,

(i) $\Psi_{0,0,k}([pt];\alpha_1,\cdots,\alpha_k) = \alpha_1 \cap \cdots \cap \alpha_k$(the intersection number);
(ii) for the product manifold $(V,\omega) = (V_1 \times V_2, \omega_1 \oplus \omega_2)$ of any two closed symplectic manifolds (V_1,ω_1) and (V_2,ω_2) it holds that

$$\Psi^V_{A_1\otimes A_2,0,k}([pt];\alpha_1\otimes\beta_1,\cdots,\alpha_k\otimes\beta_k) = \Psi^{V_1}_{A_1,0,k}([pt];\alpha_1,\cdots,\alpha_k)\Psi^{V_2}_{A_2,0,k}([pt];\beta_1,\cdots,\beta_k).$$

Thus if $\Psi^{V_2}_{A_2,0,m+1}([pt];[pt],\beta_1,\cdots,\beta_m) \neq 0$, we get

$$\Psi^V_{A_1\otimes A_2,0,m+1}([pt];[pt],\alpha_1\otimes\beta_1,\cdots,\alpha_m\otimes\beta_m) \neq 0$$

for $A_1 = 0$ and $\alpha_1 = \cdots = \alpha_m = [V_1]$. This leads to

Corollary 3. *Weinstein conjecture holds in the product symplectic manifolds of any closed symplectic manifold and a symplectic manifold satisfying the condition of Theorem 2 for $g=0$.*

Recall that a smooth Kahler manifold (M,ω) is called *uniruled* if it can be covered by rational curves. Y. Miyaoka and S. Mori showed that a smooth complex projective manifold X is uniruled if and only if there exists a non-empty open subset $U \subset X$ such that for every $x \in U$ there is an irreducible curve C with $(K_X, C) < 0$ through x [MiMo]. Specially, any *Fano* manifold is uniruled [Ko]. The complex projective spaces, the complete intersections in it, the Grassmann manifolds and more general flag manifold are the important examples of the Fano manifolds. In [R, Prop. 4.9] it was proved that if a smooth Kahler manifold M is symplectic deformation equivalient to uniruled manifold, M is uniruled. Actually, as mentioned there, Kollar showed that on the uniruled manifold (M,ω) there exists a class $A \in H_2(V;\mathbb{Z})$ such that

$$\Phi_{A,0,3}([pt];[pt],\beta_1,\beta_2) \neq 0 \qquad (1)$$

for some classes β_1 and β_2(see [R] for more general case). Combing these with Corollary 3 we get

Corollary 4. *Every hypersurface S of contact type in the uniruled manifold V or the product of any closed symplectic manifold and an uniruled manifold carries one contractible (in V) closed characteristic.*

The ideas of proof are combing Liu-Tian's Theorem 1 above, the properties of the Gromov-Witten invariants and Viterbro's trick of [V4].

Proof of Theorem 2. Under the assumptions of Theorem 2, the reduction formula of the Gromov-Witten invariants [R, Prop. C] implies that

$$\Psi_{A,g,m+3}(\pi^*(C);[pt],PD([\omega]),\beta_1,\cdots,\beta_{m+1}) = \omega(A)\cdot\Psi_{A,g,m+2}(C;[pt],\beta_1,\cdots,\beta_{m+1})\neq 0, \tag{2}$$

since A contains the nontrivial pseudoholomorphic curves. To use Theorem 1 we need to show that there exists a homology class $\gamma\in H_2(V;\mathbb{R})$ with support $supp(\gamma)\hookrightarrow int(V_+)$ (or $int(V_-)$) such that

$$\Psi_{A,g,m+3}(\pi^*(C);[pt],\gamma,\beta_1,\cdots,\beta_{m+1})\neq 0. \tag{3}$$

To this goal we note S to be a hypersurface of contact type, and thus there exists a Liouville vector field X defined in a neighborhood U of S, which is transverse to S. The flow of X define a diffeomorphism Φ from $S\times(-3\epsilon,3\epsilon)$ onto an open neighborhood of S in U for some $\epsilon>0$. Here we may assume $\Phi(S\times(-3\epsilon,0])\subset V_+$ and $\Phi(S\times[0,3\epsilon))\subset V_-$. For any $0<\delta<3\epsilon$ let us denote by $U_\delta:=\Phi(S\times[-\delta,\delta])$. We also denote by $\alpha=i_X\omega$, then $d\alpha=\omega$ on U. Choose a smooth function $f:V\to\mathbb{R}$ such that $f|_{U_\epsilon}\equiv 1$ and vanishes outside $U_{2\epsilon}$. Define $\beta:=f\alpha$. This is a smooth 1-form on V, and $d\beta=\omega$ on U_ϵ. Denote by $\widehat{\omega}=\omega-d\beta$. Then $\widehat{\omega}|_{U_\epsilon}\equiv 0$ and thus cohomology classes $[\omega]=[\widehat{\omega}]$ is in $H^2(V,U_\epsilon)$. Now from the naturality of Poincare-Lefschetz duality [Sp, p. 296]: $H_{2n-2}(V-U_\epsilon)\cong H^2(V,U_\epsilon)$ it follows that we can choose a cycle representive $\widehat{\gamma}$ of $\gamma:=PD([\omega])$ with support $supp(\widehat{\gamma})\hookrightarrow int(V-U_\epsilon)$. Notice that $V-U_\epsilon\subset V-S=int(V_+)\cup int(V_-)$ and $int(V_+)\cap int(V_-)=\emptyset$. We can denote by $\widehat{\gamma}_+$ and $\widehat{\gamma}_-$ the union of connected components of $\widehat{\gamma}$ lying $int(V_+)$ and $int(V_-)$ respectively. Then the homology classes determined by them in $H_*(V,\mathbb{R})$ satisfy: $[\widehat{\gamma}_+]+[\widehat{\gamma}_-]=\gamma$. Thus $[\widehat{\gamma}_+]$ and $[\widehat{\gamma}_-]$ have at least one nonzero class. By the property of the Gromov-Witten invariants we get

$$\begin{aligned}\Psi_{A,g,m+3}(\pi^*(C);[pt],\gamma,\beta_1,\cdots,\beta_{m+1}) &= \Psi_{A,g,m+3}(\pi^*(C);[pt],[\widehat{\gamma}_+],\beta_1,\cdots,\beta_{m+1})\\ &\quad+\Psi_{A,g,m+3}(\pi^*(C);[pt],[\widehat{\gamma}_-],\beta_1,\cdots,\beta_{m+1})\neq 0.\end{aligned} \tag{4}$$

Hence the right side of (4) has at least one nonzero term. Without loss of generality we assume that

$$\Psi_{A,g,m+3}(\pi^*(C);[pt],[\widehat{\gamma}_+],\beta_1,\cdots,\beta_{m+1})\neq 0.$$

Then Theorem 1 directly leads to the first conclusion.

The second claim is easily obtained by carefully checking the arguments in [LiuT]. □

Remark 5. Actually we believe that Theorem 1 still holds provided the hypersurface S of contact type therein is replaced by the stable hypersurface in the sense of [HV2]. Hence the hypersurface S of contact type in our results above may be replaced by the stable hypersurface for which the symplectic form is exact in some open neighborhood of it.

Remark 6. In [B] it was proved that the system of Gromov-Witten invariants of the product of two varieties is equal to the tensor product of the systems of Gromov-Witten invariants of the two factors. Using the methods developed in [FO][LiT][R][Si] we believe that one can still prove these product formula of Gromov-Witten invariants with any genus for any product of two closed symplectic manifolds, and thus Corollary 3 also holds for any genus g.

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