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EQUIDIMENSIONALITY OF LAGRANGIAN FIBRATIONS ON HOLOMORPHIC SYMPLECTIC MANIFOLDS

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Abstract. We prove that every irreducible component of every fibre of Lagrangian fibrations on holomorphic symplectic manifolds is a Lagrangian subvariety. Especially, Lagrangian fibrations are equidimensional.

1. Introduction

We begin with the definition of *Lagrangian subvarieties* .

Definition 1. *Let X be a complex manifold with a holomorhpic symplectic form ω. A subvariety Y is said to be a Lagrangian subvariety if $\dim Y = (\frac{1}{2})\dim X$ and there exists a resolution $\nu : \tilde{Y} \to Y$ such that $\nu^*\omega$ is identically zero on $\tilde{Y}$.*

Note that this notion does not depend on the choice of ν. We prove the following theorem.

Theorem 1. *Let X be a Kähler manifold and $f : X \to B$ a proper surjective morphism over a normal variety B. Assume that there exists a d-closed holomorphic symplectic form ω on X and a general fiber of f is a Lagrangian subvariety with respect to ω. Then every irreducible component of every fibre of f is a Lagrangian subvariety. Especially f is equidimensional.*

Since every holomorphic form on a compact Kähler manifold is d-closed, we obtain the following result from combining Theorem 1 with [2, Theorem 2] and [3, Theorem 1].

Corollary 1. *Let $f : X \to B$ be a surjective morphism from an irreducible symplectic manifold X to a normal projective variety B. Assume that $0 < \dim B < \dim X$. Then every irreducible component of every fibre of f is a Lagrangian subvariety.*

Remark. If we drop the condition of properness, then f is not necessarily equidimensional. Let f be a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ defined by

$$f(x, y, z, w) := (x, xy),$$

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and $\omega := dx \wedge dz + dy \wedge dw$. Then ω is a d-closed holomorphic symplectic form and a general fibre of f is a Lagrangian subvariety with respect to ω. Since $\dim f^{-1}(0) = 3$, f is not equidimensional.

From Theorem 1, we obtain some information of the singularities of B.

Corollary 2. *For every point p of B, there exists a Stein neighborhood U of p and a finite morphism $\pi : \tilde{U} \to U$ from a smooth Stein manifold $\tilde{U}$.*

Proof. For a point p of B, we choose a point $q \in f^{-1}(p)$ and a smooth Stein neighborhoods W of q. Since f is equidimensional, we obtain a finite morphism $\pi : \tilde{U} \to U$ from a smooth Stein manifold $\tilde{U}$ by cutting W with hypersurfaces. □

Remark. The author does not know whether there exists an example such that B is not smooth.

2. Proof of Theorem 1

We refer the following theorem due to Kollár [1, Theorem 2.2] and Mo. Saito [4, Theorem 2.3, Remark 2.9].

Theorem 2. *Let $f : X \to B$ be a proper surjective morhpism from a smooth Kähler manifold X to a normal variety B. Then $\mathrm{R}^i f_* \omega_X$ is torsion free, where ω_X is the dualizing sheaf of X.*

Proof. Let $\bar{\omega}$ be the complex conjugate of ω. Since ω is d-closed, $\bar{\omega}$ can be considered as an element of $H^2(X, \mathcal{O}_X)$. By Leray spectral sequence, there exists a morphism

$$H^2(X, \mathcal{O}_X) \to H^0(B, \mathrm{R}^2 f_* \mathcal{O}_X).$$

Then $\bar{\omega}$ is a torsion element in $H^0(B, \mathrm{R}^2 f_* \mathcal{O}_X)$ since a general fibre of f is a Lagrangian subvariety. In addition, $\omega_X \cong \mathcal{O}_X$. Hence $\bar{\omega}$ is zero in $H^0(B, \mathrm{R}^2 f_* \mathcal{O}_X)$ by Theorem 2. We derive a contradiction assuming that there exists an irrducible component of a fibre of f which is not a Lagrangian subvariety. The letter V denotes an non Lagrangian component. We take an embedding resolution $\pi : \tilde{X} \to X$ of V. Let $\tilde{V}$ be the proper transform of V. We will show that $\pi^* \omega$ is not zero in $H^0(\tilde{V}, \Omega^2_{\tilde{V}})$. If $\dim V = (1/2) \dim X$, it is obious by the definition. If $\dim V > (1/2) \dim X$, we take a smooth point $q \in V$ such that π is isomorhpic in a neighborhood of q. Since $\dim V > (1/2) \dim X$ and ω is nondegenerate, the restriction of ω on the tangent space of V at q is nonzero. Because π is isomorphic in a neighborhood of q, $\pi^* \omega$ is not zero in $H^0(\tilde{V}, \Omega^2_{\tilde{V}})$. Take the complex conjugate, $\pi^* \bar{\omega}$ is not zero in $H^2(\tilde{V}, \mathcal{O}_{\tilde{V}})$. Therefore $\bar{\omega}$ is not zero in $H^2(V, \mathcal{O}_V)$. Let $p := f(V)$ and $X_p := f^{-1}(p)$. We consider the following morphism:

$$\mathrm{R}^2 f_* \mathcal{O}_X \otimes k(p) \to H^2(X_p, \mathcal{O}_{X_p}) \to H^2(V, \mathcal{O}_V).$$

Then $\bar{\omega}$ is zero in $\mathrm{R}^2 f_* \mathcal{O}_X \otimes k(p)$ and nonzero in $H^2(V, \mathcal{O}_V)$. That is a contradiction. □

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